Holomorphic symplectic geometry

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I. Symplectic structure

Definition

A symplectic form on a manifold X is a 2-form φ such that:

- $d\varphi = 0$ and $\varphi(x) \in \mathrm{Alt}(T_x(X))$ non-degenerate $\forall x \in X$.
- $\iff$ locally $\varphi = dp_1 \wedge dq_1 + \ldots + dp_r \wedge dq_r$ (Darboux)

Then (X, φ) is a symplectic manifold.

(In mechanics, typically $q_i \leftrightarrow$ positions, $p_i \leftrightarrow$ velocities)

$\rightsquigarrow$ Unlike Riemannian geometry, symplectic geometry is locally trivial; the interesting problems are global.

All this makes sense with X complex manifold, φ holomorphic.

global $\rightsquigarrow$ X compact, usually projective or Kähler.

Holomorphic symplectic manifolds

Definition: holomorphic symplectic manifold

- X compact, Kähler, simply-connected;
- X admits a (holomorphic) symplectic form, unique up to $\mathbb{C}^*$.

Consequences : $\dim_{\mathbb{C}} X = 2r$; the *canonical bundle* $K_X := \Omega_X^{2r}$ is trivial, generated by $\varphi \wedge \ldots \wedge \varphi$ (r times).

(*Note* : on X compact Kähler, holomorphic forms are closed)

Why is it interesting?

The Decomposition theorem

Decomposition theorem

X compact Kähler with $K_X = \mathcal{O}_X$. $\exists\ \tilde{X} \to X$ étale finite and

$$\tilde{X} = T \times \prod_i Y_i \times \prod_j Z_j$$

- T complex torus ($= \mathbb{C}^g/\text{lattice}$);
- Y_i holomorphic symplectic manifolds;
- Z_j simply-connected, projective, $\dim \geq 3$,

  $H^0(Z_j, \Omega^*) = \mathbb{C} \oplus \mathbb{C}\omega$, where ω is a generator of K_{Z_j}.

  (these are the Calabi-Yau manifolds)

Thus holomorphic symplectic manifolds (also called hyperkähler) are building blocks for manifolds with K trivial, which are themselves building blocks in the classification of projective (or compact Kähler) manifolds.

Examples?

Many examples of Calabi-Yau manifolds, very few of holomorphic symplectic.

- dim 2: X simply-connected, $K_X = \mathcal{O}_X \overset{\text{def}}{\Longleftrightarrow} X$ K3 surface. (Example: $X \subset \mathbb{P}^3$ of degree 4, etc.)
- dim > 2? Idea: take S^r for S K3. Many symplectic forms:

$$\varphi = \lambda_1\, p_1^*\varphi_S + \ldots + \lambda_r\, p_r^*\varphi_S \ , \quad \text{with} \ \ \lambda_1, \ldots, \lambda_r \in \mathbb{C}^* \ .$$

  Try to get unicity by imposing $\lambda_1 = \ldots = \lambda_r$, i.e. φ invariant under $\mathfrak{S}_r$, i.e. φ comes from $S^{(r)} := S^r/\mathfrak{S}_r =$ $\{$subsets of r points of S, counted with multiplicities$\}$

- $S^{(r)}$ is singular, but admits a natural desingularization $S^{[r]} :=$ $\{$finite analytic subspaces of S of length $r\}$ (Hilbert scheme)

Examples

Theorem

For S K3, $S^{[r]}$ is holomorphic symplectic, of dimension $2r$.

Other examples

1. Analogous construction with $S =$ complex torus (dim. 2); gives generalized Kummer manifold K_r of dimension $2r$.
2. Two isolated examples by O'Grady, of dimension 6 and 10.

All other known examples belong to one of the above families!

Example: $V \subset \mathbb{P}^5$ cubic fourfold. $F(V) := \{\text{lines contained in } V\}$ is holomorphic symplectic, deformation of $S^{[2]}$ with S K3.

The period map

A fundamental tool to study holomorphic symplectic manifolds is the period map, which describes the position of $[\varphi]$ in $H^2(X,\mathbb{C})$.

Proposition

1. $\exists\ q : H^2(X,\mathbb{Z}) \to \mathbb{Z}$ quadratic and $f \in \mathbb{Z}$ such that
$$\int_X \alpha^{2r} = f\, q(\alpha)^r \quad \text{for} \quad \alpha \in H^2(X,\mathbb{Z}) \ .$$
2. For L lattice, there exists a complex manifold $\mathcal{M}_L$ parametrizing isomorphism classes of pairs (X,λ), where $\lambda : (H^2(X,\mathbb{Z}), q) \xrightarrow{\sim} L$.

(Beware that $\mathcal{M}_L$ is non Hausdorff in general.)

The period package

$$(X,\lambda) \in \mathcal{M}_L,\ \lambda_{\mathbb{C}} : H^2(X,\mathbb{C}) \xrightarrow{\sim} L_{\mathbb{C}};\ \text{put}\ \wp(X,\lambda) := \lambda_{\mathbb{C}}(\mathbb{C}\varphi).$$

$$\wp : \mathcal{M}_L \longrightarrow \mathbb{P}(L_{\mathbb{C}})\ \text{is the period map.}$$

Theorem

Let $\Omega := \{x \in \mathbb{P}(L_{\mathbb{C}}) \mid q(x) = 0\ ,\ q(x,\bar{x}) > 0\}$.

1. *(AB)* $\wp$ *is a local isomorphism* $\mathcal{M}_L \to \Omega$.
2. *(Huybrechts)* $\wp$ *is surjective.*
3. *(Verbitsky) The restriction of* $\wp$ *to any connected component of* $\mathcal{M}_L$ *is generically injective.*

Gives very precise information on the structure of $\mathcal{M}_L$ and the geometry of X.

Completely integrable systems

Symplectic geometry provides a set-up for the differential equations of classical mechanics:

M real symplectic manifold; φ defines $\varphi^\sharp : T^*(M) \xrightarrow{\sim} T(M)$.

For h function on M, $X_h := \varphi^\sharp(dh)$: hamiltonian vector field of h.

$X_h \cdot h = 0$, i.e. h constant along trajectories of X_h

("integral of motion")

$\dim(M) = 2r$. $h : M \to \mathbb{R}^r$, $h = (h_1, \ldots, h_r)$. Suppose:

$h^{-1}(s)$ connected, smooth, compact, Lagrangian ($\varphi_{|h^{-1}(s)} = 0$).

Arnold-Liouville theorem

$h^{-1}(s) \cong \mathbb{R}^r/\text{lattice}$; X_{h_i} tangent to $h^{-1}(s)$, constant on $h^{-1}(s)$.

$\rightsquigarrow$ explicit solutions of the ODE X_{h_i} (e.g. in terms of θ functions): "algebraically completely integrable system". Classical examples: geodesics of the ellipsoid, Lagrange and Kovalevskaya tops, etc.

Holomorphic set-up

No global functions $\rightsquigarrow$ replace $\mathbb{R}^r$ by $\mathbb{P}^r$.

Definition

X holomorphic symplectic, $\dim(X) = 2r$. Lagrangian fibration: $h : X \to \mathbb{P}^r$, general fiber connected Lagrangian.

$\Rightarrow$ on $h^{-1}(\mathbb{C}^r) \to \mathbb{C}^r$, Arnold-Liouville situation.

Theorem

$f : X \to B$ surjective with connected fibers $\Rightarrow$

1. h is a Lagrangian fibration (Matsushita);
2. If X projective, $B \cong \mathbb{P}^r$ (Hwang).

Is there a simple characterization of Lagrangian fibration?

Conjecture

$\exists\, X \dashrightarrow \mathbb{P}^r$ Lagrangian $\iff \exists\, L$ on X, $q(c_1(L)) = 0$.

An example

- Many examples of such systems. Here is one:

$S \subset \mathbb{P}^5$ given by $P = Q = R = 0$, P, Q, R quadratic $\Rightarrow$ S K3.

$$\Pi = \{\text{quadrics } \supset S\} = \{\lambda P + \mu Q + \nu R\} \cong \mathbb{P}^2$$

$$\Pi^* = \text{ dual projective plane } = \{\text{pencils of quadrics } \supset S\}\,.$$

$$h : S^{[2]} \to \Pi^* : \quad h(x,y) = \{\text{quadrics of } \Pi \supset \langle x,y\rangle\}\,.$$

By the theorem, h Lagrangian fibration $\Rightarrow$

$h^{-1}(\langle P, Q\rangle) = \{\text{lines} \subset \{P = Q = 0\} \subset \mathbb{P}^5\} \cong$ 2-dim'l complex torus ,

a classical result of Kummer.

What about odd dimensions?

Definition

A contact form on a manifold X is a 1-form η such that:

- $\operatorname{Ker}\eta(x) = H_x \subsetneq T_x(X)$ and $d\eta_{|H_x}$ non-degenerate $\forall x \in X$;
- $\iff$ locally $\eta = dt + p_1 dq_1 + \ldots + p_r dq_r$.

- A contact structure on X is a family $H_x \subsetneq T_x(X) \quad \forall x \in X$, defined locally by a contact form.

Again the definition makes sense in the holomorphic set-up $\rightsquigarrow$ holomorphic contact manifold. We will be looking for *projective* contact manifolds.

Examples

Examples of contact projective manifolds

1. $\mathbb{P}T^*(M)$ for every projective manifold M

   $\big(= \{(m,H) \mid H \subset T_m(M)\}$: "contact elements"$\big)$;

2. $\mathfrak{g}$ simple Lie algebra; $\mathcal{O}_{min} \subset \mathbb{P}(\mathfrak{g})$ unique closed adjoint orbit.

   (example: rank 1 matrices in $\mathbb{P}(\mathfrak{sl}_r)$.)

Conjecture

These are the only contact projective manifolds.

$\big($ $\Rightarrow$ classical conjecture in Riemannian geometry: classification of compact quaternion-Kähler manifolds (LeBrun, Salamon).$\big)$

Partial results

Definition : A projective manifold X is Fano if K_X negative, i.e. K_X^{-N} has "enough sections" for $N \gg 0$.

X contact manifold; $L := T(X)/H$ line bundle; then $K_X \cong L^{-k}$ with $k = \frac{1}{2}(\dim(X) + 1)$. Thus X Fano $\iff$ L^N has enough sections for $N \gg 0$.

Theorem

1. If X is not Fano, $X \cong \mathbb{P}T^*(M)$ (Kebekus, Peternell, Sommese, Wiśniewski + Demailly)
2. X Fano and L has "enough sections" $\Rightarrow$ $Z \cong \mathcal{O}_{min} \subset \mathbb{P}(\mathfrak{g})$ (AB)

III. Poisson manifolds

Few symplectic or contact manifolds $\rightsquigarrow$ look for weaker structure.

φ symplectic $\rightsquigarrow$ $\varphi^\sharp : T(X) \xrightarrow{\sim} T^*(X)$ $\rightsquigarrow$ $\tau \in \wedge^2 T(X)$ $\rightsquigarrow$

$$(f,g) \mapsto \{f,g\} := \langle \tau, df \wedge dg \rangle \text{ for } f,g \text{ functions on } U \subset X .$$

Fact: $d\varphi = 0 \iff$ Lie algebra structure (Jacobi identity).

Definition

Poisson structure on X: bivector field $\tau : x \mapsto \tau(x) \in \wedge^2 T_x(X)$, such that $(f,g) \mapsto \{f,g\}$ Lie algebra structure.

Again this makes sense for X complex manifold, τ holomorphic.

Examples

1. $\dim(X) = 2$: any global section of $\wedge^2 T(X) = K_X^{-1}$ is Poisson.
2. $\dim(X) = 3$; wedge product $\wedge^2 T(X) \otimes T(X) \to K_X^{-1}$ gives $\wedge^2 T(X) \xrightarrow{\sim} \Omega^1_X \otimes K_X^{-1}$. Then $\alpha \in H^0(\Omega^1_X \otimes K_X^{-1})$ is Poisson $\iff \alpha \wedge d\alpha = 0 \iff$ locally $\alpha = fdg$.
3. On $\mathbb{P}^3$, P, Q quadratic $\rightsquigarrow \alpha = PdQ - QdP \in \Omega^1_{\mathbb{P}^3}(4) = \Omega^1_{\mathbb{P}^3} \otimes K_{\mathbb{P}^3}^{-1}$ Poisson.
4. A holomorphic symplectic manifold is Poisson.
5. If X is Poisson, any $X \times Y$ is Poisson.

The Bondal conjecture

τ Poisson, $x \in X$. $\tau_x : T_x^*(X) \to T_x(X)$ skew-symmetric, rk even.

$$X_r := \{x \in X \mid \mathrm{rk}(\tau_x) = r\} \quad (r \text{ even}) \qquad X = \coprod X_r$$

Proposition

If $X_r \neq \varnothing$, $\dim X_r \geq r$.

Proof : X_r is a Poisson submanifold, i.e. at a smooth $x \in X_r$

$\tau_x \in \wedge^2 T_x(X_r) \subset \wedge^2 T_x(X) \implies \mathrm{rk}(\tau_x) \leq \dim X_r$. ■

Conjecture (Bondal)

X compact Poisson manifold, $X_r \neq \varnothing \Rightarrow \dim X_r > r$.

Example: $X_0 = \{x \in X \mid \tau_x = 0\}$ contains a curve.

(e.g.: on $\mathbb{P}^3$, $PdQ - QdP$ vanishes on the curve $P = Q = 0$.)

The Bondal conjecture 2

Some evidence

1. True for X projective threefold (Druel: $X_0 = \varnothing$ or $\dim \geq 1$).
2. $\mathrm{rk}(\tau_x) = r$ for x general $\Rightarrow$ true for X_{r-2} if $c_1(X)^q \neq 0$, $q = \dim X - r + 1$.

Proposition (Polishchuk)

τ Poisson on $\mathbb{P}^3$, vanishes along smooth curve C. Then C elliptic, $\deg(C) = 3$ or 4; if $= 4$, $\tau = PdQ - QdP$ and $C : P = Q = 0$.

THE END