The Lüroth problem and the Cremona group

Arnaud Beauville

Université de Nice

Tokyo, January 2013

The Lüroth problem

Definitions

- A variety V is unirational if $\exists$ generically surjective rational map $\mathbb{P}^n \dashrightarrow V$. Equivalently, $\mathbb{C}(V) \hookrightarrow \mathbb{C}(t_1, \ldots, t_n)$.
- V is rational if $\exists$ birational map $\mathbb{P}^n \overset{\sim}{\dashrightarrow} V$.

  Equivalently, $\mathbb{C}(V) \xrightarrow{\sim} \mathbb{C}(t_1, \ldots, t_n)$.
- Lüroth problem: unirational $\implies$ rational?

Lüroth (1875): yes for curves.

(Quite easy with Riemann surface theory; but Lüroth's proof is algebraic.)

Higher dimension

Castelnuovo (1894): a unirational surface is rational.

Enriques (1912): proposed counter-example : $V_{2,3} \subset \mathbb{P}^5$.

Actually Enriques proves unirationality, and relies on an earlier paper of Fano (1908) for the non-rationality.

But Fano's analysis is incomplete.

Fano made further attempts (1915, 1947), but not acceptable by modern standards.

Around 1971 three "modern" counter-examples appeared:

The counter-examples

Authors	Example	Method
Clemens-Griffiths	$V_3 \subset \mathbb{P}^4$	$J(V)$
Iskovskikh-Manin	some $V_4 \subset \mathbb{P}^4$	$\mathrm{Bir}(V)$
Artin-Mumford	specific	$\mathrm{Tors}\ H^3(V,\mathbb{Z})$

Comments

- The 3 papers have been very influential: many other examples worked out.

  They are still (essentially) the only methods known to prove non-rationality.
- Each method has its advantages and its drawbacks.
- The 3 methods use in an essential way Hironaka's results (elimination of indeterminacies).

Let us test them on the threefolds studied by Fano:

Threefolds V with $-K_V$ very ample, $\mathrm{Pic}(V) = \mathbb{Z}\,[K_V]$.

(*Fano threefolds of the first species* : modern classification due to Iskovskikh).

Rationality of Fano threefolds

variety	unirational	rational	method
$V_4 \subset \mathbb{P}^4$	some	no	Bir(V)
$V_{2,3} \subset \mathbb{P}^5$	yes	gen. no	J(V) , Bir(V)
$V_{2,2,2} \subset \mathbb{P}^6$	"	no	J(V)
$V_{10} \subset \mathbb{P}^7$	"	gen. no	J(V)
$V_{12}, V_{16}, V_{18}, V_{22}$	"	yes	
$V_{14} \subset \mathbb{P}^9$	"	no	J(V)

The main result

So the situation is quite satisfactory, except for $V_{2,3}$ and V_{10}.

Note that in both cases, "generic" means "in an (unspecified) Zariski open subset of the moduli space". So this does not say anything for a particular variety of this type.

Theorem

The threefold $\sum X_i = \sum X_i^2 = \sum X_i^3 = 0$ in $\mathbb{P}^6$ is not rational.

What is the point of giving one more counter-example?

- This gives one specific example of a non-rational $V_{2,3}$.
- The proof is very simple – maybe the simplest non-rationality proof available.
- Real motivation: it completes the work of Prokhorov on the finite simple subgroups of Cr_3.

The intermediate Jacobian

Recall the definition of the Jacobian of a curve C:

$$H^1(C,\mathbb{Z}) \subset H^1(C,\mathbb{C}) = H^{1,0} \oplus H^{0,1}$$

The image of $H^1(C,\mathbb{Z})$ in $H^{0,1}$ is a lattice, so get complex torus

$$JC := H^{0,1}/H^1(C,\mathbb{Z}) \ .$$

The cup-product defines a unimodular skew-symmetric form

$$E : H^1(C,\mathbb{Z}) \times H^1(C,\mathbb{Z}) \to \mathbb{Z}$$

such that $E_{\mathbb{R}}(ix,iy) = E_{\mathbb{R}}(x,y)$, $E_{\mathbb{R}}(x,ix) > 0$ for $x \neq 0$.

$\rightsquigarrow$ $\theta \in H^2(JC,\mathbb{Z}) \cap H^{1,1}$, hence $\theta = c_1(L)$, L ample, $h^0(L) = 1$:

This is a principal polarization on JC: we say that JC is a p.p.a.v.

Defines unique divisor on JC (up to translation), the theta divisor.

The Clemens-Griffiths criterion

V Fano threefold, completely analogous Hodge decomposition

$$H^3(V,\mathbb{Z}) \subset H^3(V,\mathbb{C}) = H^{2,1} \oplus H^{1,2}$$

$JV = H^{1,2}/H^3(V,\mathbb{Z})$ is a p.p.a.v., the intermediate Jacobian of V.

The Clemens-Griffiths criterion

If V is rational, JV is a Jacobian or a product of Jacobians.

Sketch of proof : Assume $\exists\ u : \mathbb{P}^3 \overset{\sim}{\dashrightarrow} V$. Hironaka gives

$$\begin{array}{ccc} & P & \\ {\scriptstyle b}\swarrow & & \searrow{\scriptstyle v} \\ \mathbb{P}^3 & \overset{u}{\dashrightarrow} & V \end{array}$$

b: composition of blow-ups of points and smooth curves $C_1, \ldots C_p$;

v birational morphism. Then:

The Clemens-Griffiths criterion (continued)

$b : P \to \mathbb{P}^3$ blow up $\Rightarrow$ $JP = J_1 \times \ldots \times J_p$, with $J_i := JC_i$;

$v : P \to V$ morphism $\Rightarrow H^*(P,\mathbb{Z}) \underset{v^*}{\overset{v_*}{\rightleftarrows}} H^*(V,\mathbb{Z})$ with $v_* v^* = \mathrm{Id}$,

so $H^*(P,\mathbb{Z}) = H^*(V,\mathbb{Z}) \oplus M \Rightarrow JP \cong JV \times A$ for some p.p.a.v. A.

Miracle

The decomposition $JP = J_1 \times \ldots \times J_p$ is unique (up to permutation).

This is because

$$\Theta_{JP} = \Theta_{J_1} \times J_2 \times \ldots \times J_p + \ldots + J_1 \times \ldots \times J_{p-1} \times \Theta_{J_p}$$

and the theta divisor of a Jacobian is irreducible.

So $JP \cong J_1 \times \ldots \times J_p \cong JV \times A \implies JV \cong J_{k_1} \times \ldots \times J_{k_m}$. ∎

Proof of the theorem

How can one prove that $JV \not\cong J_1 \times \ldots \times J_p$?

Usually by studying the geometry of the theta divisor (singular locus, Gauss map, ...). I will use instead the action of $\mathfrak{A}_7$.

Proof of the theorem :

$$V \text{ defined by } \sum X_i = \sum X_i^2 = \sum X_i^3 = 0 \text{ in } \mathbb{P}^6 \ :$$

action of $\mathfrak{S}_7$, hence of $\mathfrak{A}_7$.

Thus $\mathfrak{A}_7$ acts on JV. Non-trivially?

Lemma

JV contains no abelian subvariety fixed by $\mathfrak{A}_7$.

Proof : analyze the action of $\mathfrak{A}_7$ on $T_0(JV) = H^{1,2} \cong H^2(V, \Omega^1_V)$.

Find: $T_0(JV) = V_6 \oplus V_{14}$, both faithful. ∎

In particular, $\mathfrak{A}_7 \subset \mathrm{Aut}(JV)$. Note: $\dim JV = 20$.

Step 1: If $\mathfrak{A}_7 \subset \mathrm{Aut}(JC)$, $g(C) \geq 31$ (hence $JV \neq JC$).

$$\text{Torelli:} \quad \mathrm{Aut}(JC) = \begin{cases} \mathrm{Aut}(C) & \text{if } C \text{ hyperelliptic} \\ \mathrm{Aut}(C) \times \mathbb{Z}/2 & \text{otherwise.} \end{cases}$$

Thus $\mathfrak{A}_7 \hookrightarrow \mathrm{Aut}(C) \implies \frac{1}{2}7! \leq 84(g-1)$, gives $g \geq 31$.

Step 2: Assume $JV = J_1 \times \ldots \times J_n$.

(more subtle: e.g. $\mathrm{Aut}(E^{20}) \supset \mathfrak{S}_{20}$).

Assume $JV \cong J_1 \times \ldots \times J_n$

Unicity of the decomposition $\Rightarrow$ $\mathfrak{A}_7$ permutes the J_i's:

$\rightsquigarrow$ action of $\mathfrak{A}_7$ on $[1,n]$. Reorder $[1,n]$:

$$JV \cong \underbrace{J_1 \times \ldots \times J_p}_{\text{orbit } [1,p]} \times \underbrace{J_{p+1} \times \ldots \times J_{p+q}}_{\text{orbit } [p+1,p+q]} \times \ldots$$

that is, $JV \cong J_1^p \times J_{p+1}^q \times \ldots$ Hence

$$20 = \dim JV = p \dim J_1 + q \dim J_{p+1} + \cdots$$

Lemma (classical)

If $\mathfrak{A}_7$ acts transitively on a set S, then $\#S = 1, 7, 15$ or ≥ 21.

But $p = 1 \implies \mathfrak{A}_7$ acts on J_1: either trivially, (no by lemma)

or $\mathfrak{A}_7 \subset \mathrm{Aut}(J_1) \implies \dim J_1 \geq 31$: impossible.

Thus $p, q, \cdots = 7$ or 15; contradiction! ■

The method applies to other threefolds :

- $V_{2,3} : \sum X_i^2 = \sum X_i^3 = 0$ in $\mathbb{P}^5$, with group $\mathfrak{S}_6$; more difficult.
- Klein cubic $\displaystyle\sum_{i\in\mathbb{Z}/5} X_i^2 X_{i+1} = 0$ in $\mathbb{P}^4$, with group $PSL(2,\mathbb{F}_{11})$.
- The $\mathfrak{S}_6$-invariant quartic threefolds

$$X_t : \sum x_i = 0 \quad , \quad t\sum x_i^4 - (\sum x_i^2)^2 = 0 \quad \text{in } \mathbb{P}^5 \ , \quad t\in\mathbb{P}^1 \ .$$

X_2 is the Burkhardt quartic, X_4 the Igusa quartic.

For $t\neq 0,2,4,6,\frac{10}{7}$, X_t has 30 nodes :

$\mathrm{Sing}(X_t) \ = \ \mathfrak{S}_6$-orbit of $(1,1,\rho,\rho,\rho^2,\rho^2)$, $\rho = e^{\frac{2\pi i}{3}}$.

$\dim J\hat{X}_t = 5$, action of $\mathfrak{S}_6$ nontrivial $\Rightarrow$ X_t not rational.

Is it unirational?

$Cr_n := \{\text{birational automorphisms of } \mathbb{P}^n\}$.

The finite subgroups of Cr_2 are known (Kantor, Wiman, Dolgachev-Iskovskikh); very long list.

The simple (non-cyclic) finite subgroups of Cr_2 are much easier to classify: $\mathfrak{A}_5$, $\mathfrak{A}_6$ and $PSL(2, \mathbb{F}_7)$.

Theorem (Prokhorov)

The simple finite subgroups of Cr_3 not contained in Cr_2 are $\mathfrak{A}_7$, $SL(2, \mathbb{F}_8)$ and $PSp(4, \mathbb{F}_3)$.

Up to conjugacy, $SL(2, \mathbb{F}_8)$ admits only one embedding in Cr_3, and $PSp(4, \mathbb{F}_3)$ two.

A complement

Proposition

Up to conjugacy, $\mathfrak{A}_7$ admits only one embedding in Cr_3.

It is given by $\mathfrak{A}_7 \hookrightarrow SO(6,\mathbb{C})$ (standard representation), plus double covering $SO(6,\mathbb{C}) \to PGL(4,\mathbb{C})$.

Proof : Prokhorov classifies (up to birational equivalence) all

$G \subset \mathrm{Aut}(V)$, G finite simple, V rationally connected 3-fold.

Embeddings $G \hookrightarrow Cr_3$ are obtained when V is rational.

$\mathfrak{A}_7$ appears twice: action on $\mathbb{P}^3$ above, and action on V:

$$\sum X_i = \sum X_i^2 = \sum X_i^3 = 0 \text{ in } \mathbb{P}^6 .$$

Since V is not rational, only one embedding $\mathfrak{A}_7 \subset Cr_3$. ∎

Another corollary

Proposition

The group $\mathfrak{S}_7$ does not embed in Cr_3.

Idea of the proof : extend Prokhorov's method to $\mathfrak{S}_7$ $\rightsquigarrow$ any rationally connected 3-fold with an action of $\mathfrak{S}_7$ is birational to V, hence not rational. ∎

Definition : $\mathrm{crdim}(G) := \min\{n \mid \exists\ G \hookrightarrow Cr_n\}$.

Proposition

For $n \geq 4$, $\mathrm{crdim}(\mathfrak{S}_n) \leq n-3$, with equality for $4 \leq n \leq 7$.

Proof : $\mathfrak{S}_n$ acts on the quadric Q^{n-3} : $\sum X_i = \sum X_i^2 = 0$ in $\mathbb{P}^{n-1}$.
$\mathfrak{S}_5 \not\subset Cr_1$, $\mathfrak{S}_6 \not\subset Cr_2$, $\mathfrak{S}_7 \not\subset Cr_3$. ∎

Question : Is it true that $\mathrm{crdim}(\mathfrak{S}_n) = n-3$?

THE END