The algebra of symmetric tensors

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The algebra $S(X)$

Setup: X smooth projective$_{|\mathbb{C}}$, $\dim X = n$.

$T^*X :=$ cotangent bundle, $\mathbb{P}T^*X := (T^*X \smallsetminus 0_X)/\mathbb{C}^*$.

$$S(X) := \bigoplus_{p\geqslant 0} H^0(X, \mathrm{S}^p T_X) = \mathcal{O}(T^*X) = \bigoplus_{p\geqslant 0} H^0(\mathbb{P}T^*X, \mathcal{O}_{\mathbb{P}T^*X}(p))$$

Graded $\mathbb{C}$-algebra $\big($+ Poisson structure: Lie bracket on $H^0(T_X)$ extends to $S(X)$ (*Schouten bracket*)$\big)$.

Much less studied than $\bigoplus H^0(\mathrm{S}^p\Omega^1_X)$. Plan:

1. **Examples**

2. **Bound on $\dim \mathbf{S(X)}$.**

Example 1: $\mathbb{P}^n$

Proposition

$S(\mathbb{P}(V)) = \bigoplus (S^pV \otimes S^pV^*)/(\mathrm{Id}_V), \ \ \mathrm{Id}_V \in \mathrm{End}(V) \cong V \otimes V^*.$

In coordinates: $S(\mathbb{P}^n) = \mathbb{C}[x_0y_0, \ldots, x_iy_j, \ldots, x_ny_n]/(\sum x_iy_i)$.

Proof: $\mathbb{P}T^*\mathbb{P} = \mathcal{I} \subset \mathbb{P} \times \mathbb{P}^*$, $\mathcal{I} = \{(x, H) \,|\, x \in H\}$, defined by

$$\mathrm{Id}_V \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \otimes H^0(\mathbb{P}^*, \mathcal{O}_{\mathbb{P}*}(1)) = V^* \otimes V,$$

with $\mathcal{O}_{\mathbb{P}T*\mathbb{P}}(1) = \left(\mathcal{O}_{\mathbb{P}}(1) \boxtimes \mathcal{O}_{\mathbb{P}*}(1)\right)_{|\mathcal{I}}$.

$$\implies S(\mathbb{P}(V)) = \bigoplus \left(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(p)) \otimes H^0(\mathbb{P}^*, \mathcal{O}_{\mathbb{P}*}(p))\right)/\langle \mathrm{Id}_V \rangle. \quad \blacksquare$$

Example 2: the quadric

$Q \subset \mathbb{P}(V)$ smooth, given by $q = 0$, $q : \mathsf{S}^2 V \to \mathbb{C}$ non-degenerate.

Consider $\mathbb{G}(2, V) \subset \mathbb{P}(\bigwedge^2 V)$, homogeneous ideal $\mathcal{I}_{\mathbb{G}}$.

Proposition

$$S(Q) = \mathrm{Pol}(\bigwedge^2 V)/\langle \mathcal{I}_{\mathbb{G}}, \wedge^2 q\rangle.$$

Proof: $\mathbb{P}T^*Q = \{(x, P) \mid P \text{ hyperplane in } \mathbb{P}T_x(Q), x \in P\}$.

Gauss map: $\gamma : \mathbb{P}T^*Q \to \mathbb{G} := \mathbb{G}(n, V) \cong \mathbb{G}(2, V)$, $(x, P) \mapsto P$.

$T := \mathrm{Im}\,\gamma = \{P \in \mathbb{G} \text{ tangent to } Q\}$ given by $\wedge^2 q = 0$ in $\mathbb{P}(\bigwedge^2 V)$.

For $P \in T$, $\gamma^{-1}(P) = \{(x, P) \mid P \text{ tangent to } Q \text{ at } x\}$. Either :

- $\gamma^{-1}(P) = \{(x, P)\}$ for $P \in T$ general, or
- (ℓ, P) if P tangent along ℓ line: $P \in T_1 \subset T$ smooth of codim 2.

$\implies \gamma =$ blow-up of T along T_1.

The quadric (continued)

To summarize :

$$\gamma : \mathbb{P}T^*Q \xrightarrow{\text{blow-up}} T \subset \mathbb{G} \subset \mathbb{P}(\bigwedge^2 V), \quad T : \wedge^2 q = 0\,.$$

Claim: $\gamma^*\mathcal{O}_T(1) = \mathcal{O}_{\mathbb{P}T*Q}(1)$.

$\Longrightarrow \; H^0(T, \mathcal{O}_T(p)) \xrightarrow{\sim} H^0(\mathcal{O}_{\mathbb{P}T*Q}(p)) = H^0(Q, S^pT_Q)$, hence

$S(Q) =$ homogeneous ideal of $T = \mathrm{Pol}(\bigwedge^2 V)/\langle \mathcal{I}_{\mathbb{G}}, \wedge^2 q\rangle$. ■

Remark : Other proof applies more generally to homogeneous varieties, using the moment map. Then $\bigwedge^2 V$ appears as $\mathfrak{so}(V)$.

Example 3: Two quadrics

$X = Q_1 \cap Q_2 \subset \mathbb{P}^{n+2}$, $n \geqslant 2$.

Theorem (A. Etesse, A. Höring, J. Liu, C. Voisin, —)

1. $S(X) \xrightarrow{\sim} \mathbb{C}[x_1, \ldots, x_n]$, $\deg x_i = 2$.
2. $\Phi : T^*X \xrightarrow{(x_i)} \mathbb{C}^n$ *Lagrangian fibration.*
3. *For X general, λ general in $\mathbb{C}^n$, $\Phi^{-1}(\lambda) \cong A \smallsetminus Z$, A explicit abelian variety,* codim $Z \geqslant 2$.

③ means that Φ defines an **algebraically completely integrable system** (ACIS) — I will say a few words below. This is a rather exceptional situation. There are a handful of such systems that have been extensively studied classically: geodesics of the ellipsoid, Lagrange, Euler, and Kovalevskaya tops.

Interlude: ACIS

In the hamiltonian formulation, a mechanical system on a symplectic manifold M is governed by a function $h \in \mathcal{O}(M)$ (usually the total energy of the system).

The evolution of the system is given by the flow of the vector field V_h on M corresponding to dh via the symplectic form.

Put $\dim M = 2n$. An ACIS on M is given by a map $\Phi : M \to \mathbb{C}^n$ satisfying ② and ③.

Then ② implies that V_h is tangent to the fibers of Φ, and ③ that its restriction to a fiber $A \smallsetminus Z$ extends to a vector field on A. Thus if $A = \mathbb{C}^n/\Lambda$, the flow of V_h is just the projection of a linear flow $t \mapsto t\mathbf{v}$ on $\mathbb{C}^n$.

To describe the evolution of the system, it remains to go back from A to T^*X, usually using theta functions.

Example 4: the Hitchin fibration

To my knowledge, the only other known examples of ACIS on T^*X are given by the **Hitchin fibration**.

$C :=$ curve of genus $g \geqslant 2$.

$\mathcal{M} :=$ moduli space of rank r, degree d stable vector bundles on C, $(r, d) = 1$. $\mathcal{M}$ smooth projective.

$T_E(\mathcal{M}) = H^1(\mathcal{E}nd(E)) \overset{\text{Serre}}{\Longrightarrow} T_E^*(\mathcal{M}) = H^0(\mathcal{E}nd(E)) \otimes K)$.

$a_i : H^0(\mathcal{E}nd(E) \otimes K) \xrightarrow{\wedge^i} H^0(\mathcal{E}nd(\bigwedge^i E) \otimes K^i) \xrightarrow{\text{Tr}} H^0(K^i)$

$$\Phi : T^*\mathcal{M} \xrightarrow{(a_i)} V := \bigoplus_{i=1}^r H^0(K^i)$$

Hitchin fibration

Theorem

$\Phi : T^*\mathcal{M} \xrightarrow{(a_i)} V := \bigoplus_{i=1}^r H^0(K^i)$ *is an ACIS. More precisely:*

1. $\Phi^* : \mathrm{Pol}(V) \xrightarrow{\sim} \mathcal{O}(T^*\mathcal{M}) = S(\mathcal{M})$;
2. Φ *is a Lagrangian fibration;*
3. *For* $\lambda \in V$ *general,* $\Phi^{-1}(\lambda) = J \smallsetminus Z$, J *Jacobian,* $\mathrm{codim} Z \geqslant 2$.

(2) and $\Phi^{-1}(\lambda) \overset{open}{\subset} J$ due to Hitchin.

$T^*\mathcal{M} \overset{open}{\subset} \mathcal{H}$ (stable **Higgs bundles**), Φ extends to $\bar{\Phi} : \mathcal{H} \to V$ proper, $\bar{\Phi}^{-1}(\lambda) = J$ and $\mathrm{codim}(\mathcal{H} \smallsetminus T^*\mathcal{M}) \geqslant 2 \implies$ (3) $\implies$ (1).

Remarks : Same for $\mathcal{M}_L$ $(\det(E) = L)$, or $\mathcal{M}_p$ (parabolic bundles).

- For $g = r = 2$, $d = 1$, $\mathcal{M}_L \cong Q_1 \cap Q_2 \subset \mathbb{P}^5 \rightsquigarrow$ Example 3.

Example 5: Ruled surfaces

$C :=$ curve of genus $g \geqslant 2$.

E stable rank 2 bundle, $\det E = \mathcal{O}_C$, $X = \mathbb{P}(E)$.

Lemma

$$H^0(X, S^p T_X) \cong H^0(C, S^{2p} E)$$

Idea: $T_{X/C} = \mathcal{O}_{\mathbb{P}(E)}(2) \to T_X$ induces isomorphism on $H^0(S^p)$. ■

Corollary

For E general, $S(\mathbb{P}(E)) = \mathbb{C}$.

Example: $\rho : \pi_1(C) \twoheadrightarrow G \subset \mathrm{SU}(2, \mathbb{C})$, G finite $\rightharpoonup \mathbb{C}^2$ irreducibly.

$\rightsquigarrow E_\rho$ stable, $\det E = \mathcal{O}_C$. Then

$$S(X) = \bigoplus H^0(S^{2p} E) = \mathbb{C}[u, v]^G = \mathcal{O}(\mathbb{C}^2/G)\,.$$

e.g. $G = \tilde{\mathfrak{A}}_5 \implies S(X) = \mathbb{C}[x, y, z]/(x^2 + y^3 + z^5)$.

Cases with $S(X) = \mathbb{C}$

Theorem

$S(X) = \mathbb{C}$ when:

1. *$c_1(X) = 0$ and $\pi_1(X)$ finite (Kobayashi, using Yau's theorem);*
2. *X of general type (Höring-Peternell)*
3. *X hypersurface of degree $\geqslant 3$ (Höring-Liu-Shao).*

The Krull dimension of $S(X)$

Recall: for a line bundle L on Y, the **Iitaka dimension** is

$$\kappa(L) := \max_p [\dim \operatorname{Im} \varphi_{L^p}], \text{ where } \varphi_{L^p} : Y \xrightarrow{|L^p|} \mathbb{P}^{N_p}.$$

In particular, $\kappa(K_Y) =: \kappa(Y)$, the **Kodaira dimension** of Y.

Proposition

Put $S(L) := \bigoplus_{p \geqslant 0} H^0(Y, L^p)$. *If* $S(L) \neq \mathbb{C}$, $\dim S(L) = 1 + \kappa(L)$.

In particular: if $S(X) \neq \mathbb{C}$, $\dim S(X) = 1 + \kappa(\mathcal{O}_{\mathbb{P}T^*X}(1))$.

Hence: $0 \leqslant \dim S(X) \leqslant 2n$; $\dim S(X) = 0 \iff S(X) = \mathbb{C}$.

$\dim S(X) = 2n \iff \mathcal{O}_{\mathbb{P}T^*X}(1)$ big $\overset{\text{def}}{\iff} T_X$ big. Holds for X toric, homogeneous,...

$\dim S(X) \leqslant n - \kappa(X)$

Theorem

$$\dim S(X) \leqslant n - \kappa(X)$$

Equality $\iff$ $A \times Y \xrightarrow{\text{étale}} X$, *A abelian, Y of general type.*

Equality case : $\dim S(X) + \kappa(X) = n$ holds for

- A abelian: $S(A) = \mathbb{C}[x_1, \ldots, x_n], \dim S(A) = n, \kappa(A) = 0$;
- Y of general type: $\dim S(Y) = 0$, $\kappa(Y) = n$;
- $X = A \times Y$: $S(A \times Y) = S(A) \otimes S(Y) = S(A) \implies$ $\dim S(A \times Y) = \dim A$, and $\kappa(A \times Y) = \kappa(Y) = \dim Y$.
- For $X \to Y$ étale, $\dim S(X) = \dim S(Y)$, $\kappa(X) = \kappa(Y)$ (Ueno).

$\dim S(X) \leqslant n - \kappa(X)$: sketch of proof

The proof uses a deep result of Höring, Peternell, Pereira, Touzet:

Theorem

Assume $S(X) \neq \mathbb{C}$, $\kappa(X) \geqslant 0$. Fix a polarization. Then

$$T_X = F \oplus G, \text{ where}$$

1. $F = \bigoplus F_i$ *with* F_i *stable,* $c_1(F_i) = 0$*, and* $(\det F)^{\otimes N} = \mathcal{O}_X$.
2. G *is "negative" (in a precise sense).*

Put $S(F) := \bigoplus H^0(X, \mathrm{S}^p F)$. ② implies $S(X) = S(F)$.

F_i stable $\Longrightarrow$ $h^0(F_i) \leqslant 1 \Rightarrow h^0(F) \leqslant \operatorname{rk} F =: r$. Apply to $\mathrm{S}^p F$:

$h^0(\mathrm{S}^p F) \leqslant \operatorname{rk} \mathrm{S}^p F = h^0(\mathrm{S}^p \mathcal{O}_X^r) \rightsquigarrow \dim S(F) \leqslant \dim S(\mathcal{O}_X^r) = r$.

Passing to étale cover, may assume $\det F = \mathcal{O}_X \rightsquigarrow \det G^* = K_X$.

$$G^* \hookrightarrow \Omega_X^1 \implies K_X = \det G^* \hookrightarrow \Omega_X^{n-r}.$$

$\kappa(X) \leqslant n - r$ (Bogomolov), hence $\dim S(X) + \kappa(X) \leqslant n$. ■

Some questions

Q1. Is the $\mathbb{C}$-algebra $S(X)$ finitely generated?

(No reason, but no counter-example.)

- $S(X) \neq \mathbb{C} \iff c_1(\mathcal{O}_{\mathbb{P}T^*X}(1)) \in \mathrm{Eff} \subset H^2(X, \mathbb{R})$.
- T_X **pseudo-effective** if $c_1(\mathcal{O}_{\mathbb{P}T^*X}(1)) \in \overline{\mathrm{Eff}} \subset H^2(X, \mathbb{R})$.

Q2. Does T_X pseudo-effective imply $S(X) \neq \mathbb{C}$?

Q3. Does T_X pseudo-effective and $\pi_1(X) = 0$ imply X uniruled?

We have a proof (of a more general statement) for $\dim X \leqslant 5$.

THE END