Ulrich bundles : to be or not to be

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A classical problem

$X \subset \mathbb{P}$ hypersurface of degree d, defined by $F = 0$.

Can we write $F = \det(L_{ij})$ for a $d \times d$-matrix (L_{ij}) of linear forms?

Yes for cubic surfaces (Schröter, 1863: used to find the 27 lines), for special quartic surfaces (Jessop, Dickson)...

But $\Rightarrow$ X singular for $\dim(X) \geqslant 3$.

Let us settle for a weaker property: can we write $F^r = \det(L_{ij})$, that is, $X = V(\det(L_{ij}))$ as sets?

Proposition (almost tautological)

$X \subset \mathbb{P}$ smooth hypersurface of degree d, defined by $F = 0$; L $(rd \times rd)$-matrix of linear forms.

$\Updownarrow$

1) $F^r = \det L$;
2) $\exists$ *E rank r vector bundle on X with a resolution*

$$0 \to \mathcal{O}_{\mathbb{P}}(-1)^{rd} \xrightarrow{L} \mathcal{O}_{\mathbb{P}}^{rd} \to E \to 0.$$

Modern formulation

So the problem is reduced to find such a vector bundle E.

Proposition

X smooth hypersurface of degree d in $\mathbb{P}^{n+1}$, E rank r vector bundle on X.

$\Updownarrow$

1) $\exists$ *resolution* $0 \to \mathcal{O}_{\mathbb{P}}(-1)^{rd} \xrightarrow{L} \mathcal{O}_{\mathbb{P}}^{rd} \to E \to 0$;

2) $H^{\bullet}(X, E(-1)) = \ldots = H^{\bullet}(X, E(-n)) = 0$;

3) *If* $\pi : X \to \mathbb{P}^n$ *projection from* $p \notin X$, $\pi_* E = \mathcal{O}_{\mathbb{P}^n}^{rd}$.

It turns out that this is a particular case of a general result for any smooth projective variety:

Definition of Ulrich bundle

Theorem (Eisenbud-Schreyer, 2003)

$X^n \subset \mathbb{P}^{n+c}$ smooth, E rank r vector bundle on X.

$\Updownarrow$

1) *E admits a linear resolution*

$$0 \to \mathcal{O}_{\mathbb{P}}(-c)^{\bullet} \to \cdots \to \mathcal{O}_{\mathbb{P}}(-1)^{\bullet} \to \mathcal{O}_{\mathbb{P}}^{\bullet} \to E \to 0\,;$$

2) $H^{\bullet}(X, E(-1)) = \ldots = H^{\bullet}(X, E(-n)) = 0\,;$

3) *If $\pi : X \to \mathbb{P}^n$ projection, $\pi_* E = \mathcal{O}_{\mathbb{P}^n}^{rd}$.*

If this holds, we say that E is an **Ulrich bundle**.

We'll say also that E is an Ulrich bundle for $(X, \mathcal{O}_X(1))$.

Sketch of proof

To be proved: equivalence of

1. $0 \to \mathcal{O}_{\mathbb{P}^{n+c}}(-c)^{\bullet} \to \cdots \to \mathcal{O}_{\mathbb{P}^{n+c}}(-1)^{\bullet} \to \mathcal{O}_{\mathbb{P}^{n+c}}^{\bullet} \to E \to 0\,;$
2. $H^{\bullet}(X, E(-1)) = \ldots = H^{\bullet}(X, E(-n)) = 0\,;$
3. If $\pi : X \to \mathbb{P}^n$ projection, $\pi_* E = \mathcal{O}_{\mathbb{P}^n}^{rd}$

1) $\Rightarrow$ **2)** : $\mathcal{O}_{\mathbb{P}}(-1), \ldots, \mathcal{O}_{\mathbb{P}}(-n-c)$ have zero cohomology.

3) $\Rightarrow$ **2)** : $H^i(X, E(-p)) = H^i(\mathbb{P}^n, (\pi_* E)(-p))$.

Assume 2). Then $H^i(X, E(-i)) = 0$ for $i > 0 \;\Rightarrow\; E$ is 0-*regular* (Mumford) $\Rightarrow\; E$ globally generated and $H^i(X, E) = 0$ for $i > 0$.
$\chi(E(t)) = 0$ for $t = -1, \ldots, -n \Rightarrow \chi(E(t)) = \frac{rd}{n!}(t+1)\ldots(t+n)$
$\Rightarrow\; h^0(E) = \chi(E) = rd$.

Proof of 3) : $F = \pi_* E$ satisfies 2) $\Rightarrow\; \mathcal{O}_{\mathbb{P}^n}^{rd} \twoheadrightarrow F \;\Rightarrow\; \mathcal{O}_{\mathbb{P}^n}^{rd} \xrightarrow{\sim} F$.

Proof of 1) : $0 \to K_0 \to \mathcal{O}_X^\bullet \to E \to 0$; then $K_0(-1)$ is 0-regular, hence $0 \to K_1 \to \mathcal{O}_X(-1)^\bullet \to \mathcal{O}_X^\bullet \to E \to 0$ with $K_1(-2)$ 0-regular, then 1) by induction. ■

Some consequences of the proof : E Ulrich $\Rightarrow$

- E globally generated, $h^0(E) = rd$, $h^0(E(-1)) = 0$;
- $\chi(E(t)) = rd\chi(\mathcal{O}_{\mathbb{P}^n}(t)) = \frac{rd}{n!}(t+1)\ldots(t+n)$.
- E semi-stable (by 3)).

Main problem: Does every smooth $X \subset \mathbb{P}$ carry an Ulrich bundle?

Some history

• Introduced and studied in 1985-95 in commutative algebra (Ulrich, Herzog, ...) under the name "maximally generated maximal Cohen-Macaulay modules".

• Revived geometrically by Eisenbud-Schreyer (2003), then Casanellas-Hartshorne (2011), and many others.

Examples

- On $\mathbb{P}$, Ulrich bundle $= \mathcal{O}_{\mathbb{P}}^r$.
- Curves: E general vector bundle of slope $g-1 \ \Rightarrow \ E(1)$ Ulrich.
- Grassmannians (Costa-Miró-Roig), some flag varieties.

Theorem (Herzog-Ulrich-Backelin (1991))

Any smooth complete intersection $X \subset \mathbb{P}$ carries an Ulrich bundle.

Proof involves *matrix factorization* and *generalized Clifford algebra*.

Example : for a smooth quadric $Q \subset \mathbb{P}^{n+1}$, the indecomposable Ulrich bundles are:

- for $n = 2k+1$, the *spinor bundle*, of rank 2^k;
- for $n = 2k$, the two *half-spinor bundles*, of rank 2^{k-1}.

- If $(X, \mathcal{O}_X(1))$ admits an Ulrich bundle, so does $(X, \mathcal{O}_X(d))$.

Ulrich line bundles

In some (rare) cases, there exist Ulrich line bundles:

- $S \subset \mathbb{P}$ del Pezzo surface, $L \in \operatorname{Pic}(S)$, $L' := K_S \otimes L$. Then
$$L \text{ Ulrich} \iff L' \cdot K = 0 \text{ and } (L')^2 = -2.$$
(always exists if $\deg(S) \leqslant 7$.)
- For $X \subset \mathbb{P}$ scroll (i.e. $X \xrightarrow{p} C$, fibers are linear subspaces): if $M \in \operatorname{Pic}(C)$ with $H^\bullet(C, M) = 0$, $p^*M(1)$ is Ulrich.
- Many Enriques surfaces (Borisov-Nuer).

But : for $X \subset \mathbb{P}$ with $\operatorname{Pic}(X) = \mathbb{Z}[\mathcal{O}_X(1)]$ and $\deg(X) > 1$, no Ulrich line bundle. (must be $\mathcal{O}_X \Rightarrow d = h^0(\mathcal{O}_X) = 1$.)

In particular: a general surface of degree $d \geqslant 4$ cannot be defined by a $(d \times d)$ linear determinant.

Rank 2: surfaces

We want E of rank 2 with $H^\bullet(E(-1)) = H^\bullet(E(-2)) = 0$.

Easy case: $E(-1)$ and $E(-2)$ are Serre dual, i.e. $\det E = K_S(3)$.

Definition : E *special* if $\det E = K_S(3)$.

($\Rightarrow$ the Chow form of $S \subset \mathbb{P}$ can be written as a pfaffian.)

Theorem (Aprodu-Farkas-Ortega)

Most K3 surfaces admit a special rank 2 Ulrich bundle.

"Most" := for each g, the possible exceptions $\subset Z \subsetneq \mathcal{F}_g$.

The construction uses the Lazarsfeld-Mukai bundle.

Note : For $g = 3$, *every* smooth quartic surface admits a special rank 2 Ulrich bundle (Coskun-Kulkarni-Mustopa).

Surfaces with $\kappa = 0$

Theorem (AB)

Every minimal surface $S \subset \mathbb{P}$ of Kodaira dimension 0 which is not a K3 admits a special rank 2 Ulrich bundle.

Remaining surfaces with $\kappa = 0$: Enriques, abelian, bielliptic.

Proof (essentially) uniform, using Serre's construction. Recall:

$\underset{\text{finite}}{Z} \subset S \subset \mathbb{P}$ has the **Cayley-Bacharach property** if

$$H \supset Z \smallsetminus \{pt\} \ \Rightarrow \ H \supset Z \ .$$

$\Rightarrow$ extension $0 \to K_S \to E \to \mathcal{I}_Z(1) \to 0$ with E rank 2 vector bundle, $\det E = K_S(1)$.

Existence for $\kappa = 0$

Lemma

E rank 2 bundle on $S \subset \mathbb{P}$, $\det E = K_S(1)$, $h^0(E) = \chi(E) = 0 \;\Rightarrow$ $E(1)$ is a special Ulrich bundle.

Proof : $K_S \otimes E^* \cong E(-1) \;\Rightarrow\; h^2(E) = h^0(E(-1)) = 0$, hence $H^\bullet(E) = 0$. Then $H^\bullet(E(-1)) = H^\bullet(K_S \otimes E^*) = 0$. ■

For Enriques surfaces, existence follows from:

Proposition (Casnati)

$S \subset \mathbb{P}^n$ with $q = p_g = 0$ and $H^1(S, \mathcal{O}_S(1)) = 0 \;\Rightarrow\; S$ admits a special rank 2 Ulrich bundle.

Existence for $\kappa = 0$

Proposition (Casnati)

$S \subset \mathbb{P}^n$ with $q = p_g = 0$ and $H^1(S, \mathcal{O}_S(1)) = 0 \Rightarrow S$ admits a special rank 2 Ulrich bundle.

Proof : Choose $Z \subset S$ general with $\#Z = n+2$. C-B holds $\rightsquigarrow$ $0 \to K_S \to E \to \mathcal{I}_Z(1) \to 0$ with $\det E = K_S(1)$. Then $h^0(E) = 0$,

$$\begin{aligned} \chi(E) &= \chi(K_S) + \chi(\mathcal{I}_Z(1)) \\ &= 1 + \chi(\mathcal{O}_S(1)) - (n+2) = 0 \,, \end{aligned}$$

hence $E(1)$ special Ulrich bundle by the Lemma. ∎

For the other cases, choose C smooth hyperplane section of S and $Z \subset C$ general, $\#Z = n+1$; twist by a 2-torsion line bundle.

Fano threefolds of index 2

X Fano threefold, $K_X^{-1} = L^2$. Assume $d := (L^3) \geqslant 3$.

Then $|L|$ embeds X in $\mathbb{P}^{d+1}$; 7 families, with $3 \leqslant d \leqslant 8$:

$V_3 \subset \mathbb{P}^4$, $V_{2,2} \subset \mathbb{P}^5$, etc.

Proposition

$X \subset \mathbb{P}^{d+1}$ admits a special rank 2 Ulrich bundle E.

("special" $:= \det E = K_X(4)$.)

Serre's construction: $Z \subset X$ smooth codimension 2. Suppose:

$L \in \mathrm{Pic}(X)$ with $K_Z = (K_X \otimes L)_{|Z}$, and $H^2(X, L^{-1}) = 0$. Then $\exists$

$$0 \to \mathcal{O}_X \to E \to \mathcal{I}_Z L \to 0 \text{ with } E \text{ rank 2 vector bundle.}$$

Lemma

X contains a normal elliptic curve $\Gamma \subset X \subset \mathbb{P}^{d+1}$ (of degree $d+2$).

Proof of the Proposition

Idea of proof of the Lemma : A smooth hyperplane section $S_d \subset \mathbb{P}^d$ of X contains a normal elliptic curve $\Gamma_0 \subset \mathbb{P}^d$. Take a line $\ell \subset X$ such that $\Gamma_0 \pitchfork \ell = \{p\}$, and deform $\Gamma_0 \cup \ell$ in X. ■

Proof of the Proposition :

$L = \mathcal{O}_X(2)$ satisfies $(K_X \otimes L)_{|\Gamma} = K_\Gamma$ and $H^2(X, L^{-1}) = 0$

$\rightsquigarrow \quad 0 \to \mathcal{O}_X \to E \to \mathcal{I}_\Gamma(2) \to 0$ with $\det E = \mathcal{O}_X(2) = K_X(4)$.

Claim : E is Ulrich.

Proof : $E(-2) \cong K_X \otimes E(-2)^*$ and $E(-3) \cong K_X \otimes E(-1)^* \ \Rightarrow$ suffices to prove $H^\bullet(E(-1)) = 0$ and $H^i(E(-2)) = 0$ for $i = 0, 1$.

- $H^\bullet(\mathcal{O}_X(-1)) = H^\bullet(\mathcal{I}_\Gamma(1)) = 0 \ \Rightarrow \ H^\bullet(E(-1)) = 0$;
- For $i = 0, 1$, $H^i(\mathcal{O}_X(-2)) = H^i(\mathcal{I}_\Gamma) = 0 \Rightarrow H^i(E(-2)) = 0$. ■

Moduli space

Proposition

The moduli space $\mathcal{M}$ of rank 2 special Ulrich bundles on X is smooth of dimension 5.

Sketch of proof : $\Gamma \longleftrightarrow E+[s]\subset \mathbb{P}(H^0(E))$ with $Z(s)$ smooth.

$\mathcal{H}:=$ Hilbert scheme of $\Gamma\subset X$; $p:\mathcal{H}\to\mathcal{M}$, $p(\Gamma)=E$.

For $E\in\mathcal{M}$, $p^{-1}(E)$ open in $\mathbb{P}(H^0(E))$, has dimension $2d-1$.

Using $N_{\Gamma/X}\cong E_{|\Gamma}$, get $H^1(N_{\Gamma/X})=0$, $h^0=2d+4\ \Rightarrow$

$\mathcal{H}$ smooth of dimension $2d+4\ \Rightarrow\ \mathcal{M}$ smooth of dimension 5. ■

Examples

① The rank 2 Ulrich bundles on $X_3 \subset \mathbb{P}^4$ have been studied by Iliev-Markushevich-Tikhomirov and Druel. The 2nd Chern class defines an isomorphism of $\mathcal{M}$ onto an (explicit) open subset of JX.

② $X_{2,2} \subset \mathbb{P}^5 \longleftrightarrow$ genus 2 curve C, such that $JX \cong JC$. Then $\mathcal{M}$ is isomorphic to an open subspace of the moduli space of stable bundles on C of rank 2 and degree 0 (Cho-Kim-Lee, 2017).

③ For $d = 8$, $X = \mathbb{P}^3$ embedded in $\mathbb{P}^9$ by $|\mathcal{O}_{\mathbb{P}}(2)|$. Any rank 2 Ulrich bundle E on X appears in an exact sequence

$$0 \to E \to T_{\mathbb{P}^3}(1) \xrightarrow{\eta} \mathcal{O}_{\mathbb{P}^3}(3) \to 0$$

for a *contact form* $\eta \in H^0(\mathbb{P}^3, \Omega^1(2))$.

Thus $\mathcal{M}$ = open subset of contact forms in $\mathbb{P}(H^0(\mathbb{P}^3, \Omega^1(2)))$

$= \{$bilinear symplectic forms on $\mathbb{C}^4\}/\mathbb{C}^*$.

An inequality for surfaces

Proposition

$S \subset \mathbb{P}$ surface with $\operatorname{rk} \mathrm{NS}(S) = 1$*, E Ulrich bundle of rank r. Then*

$$\deg(S) \geqslant \operatorname{sign}(S)\ , \ \textit{with}\ \operatorname{sign}(S) = K_S^2 - 8\chi(\mathcal{O}_S) = \frac{1}{3}(c_1^2 - 2c_2)\ .$$

Proof : Put $H :=$ hyperplane class in $H^2(S, \mathbb{Q})$. Recall $\chi(E(t)) = \frac{rd}{2}(t+1)(t+2)$. Comparing with Riemann-Roch gives $c_1(E)\cdot H = \frac{r}{2}(K+3H)\cdot H\ ,\ \mathrm{ch}_2(E) = \frac{1}{2}K\cdot c_1(E)) + r(H^2 - \chi(\mathcal{O}_S))\,.$ Since $\operatorname{rk} \mathrm{NS}(S) = 1$, $c_1(E) = \frac{r}{2}(K+3H)$. We compute the *discriminant* $\Delta_E := 2rc_2(E) - (r-1)c_1(E)^2 = c_1(E)^2 - 2r\,\mathrm{ch}_2(E)$:

$$\begin{aligned}\Delta_E &= \frac{r^2}{4}\Big((K+3H)^2 - 2K\cdot(K+3H) - 8(H^2 - \chi(\mathcal{O}_S)\Big)\\ &= \frac{r^2}{4}(H^2 - (K^2 - 8\chi(\mathcal{O}_S))) \ = \ \frac{r^2}{4}(\deg(S) - \operatorname{sign}(S)).\end{aligned}$$

Surfaces without Ulrich bundles?

Thus we get $$\Delta_E = \frac{r^2}{4}(\deg(S) - \mathrm{sign}(S))\,.$$

Since E is semi-stable, $\Delta_E \geqslant 0$ (Bogomolov) $\Rightarrow$ ■.

Corollary

A surface $S \subset \mathbb{P}$ with $\mathrm{rk}\,\mathrm{NS}(S) = 1$ and $\deg(S) < \mathrm{sign}(S)$ does not carry any Ulrich bundle.

Question : Does such a surface exist?

There are many examples of surfaces with $\mathrm{sign}(S) > 0$, but most of them have $\mathrm{rk}\,\mathrm{NS}(S) > 1$. The only exceptions I know are the Blasius-Rogawski surfaces, with $K_S^2 = 9\chi(\mathcal{O}_S)$ (see below).

Question : Does there exist a surface S with $\mathrm{rk}\,\mathrm{NS}(S) = 1$ and $8\chi(\mathcal{O}_S) < K_S^2 < 9\chi(\mathcal{O}_S)$?

The Blasius-Rogawski family

$S=\mathbb{B}/\Gamma$, $\mathbb{B}$ unit ball in $\mathbb{C}^2$, Γ arithmetic subgroup of $\mathrm{PU}(2,1)$ associated to a degree 3 division algebra satisfying particular arithmetic conditions.

Then $\operatorname{rk}\operatorname{Pic}(S)=1$; if Γ lifts to $\mathrm{SU}(2,1)$, $K=3L$.

$K^2=9\chi(\mathcal{O}_S) \;\Rightarrow\; L^2=\chi(\mathcal{O}_S)=\operatorname{sign}(S)$.

According to the experts, L should be very ample for Γ small enough, so $S\subset\mathbb{P}$ would satisfy $\deg(S)=\operatorname{sign}(S)$.

Since $\pi_1(SU(2,1))=\mathbb{Z}$, there exists subgroups Γ for which $L=kL'$ with $k>1$; if L' were very ample, this would give the required example. Unfortunately this seems out of reach at the moment.

Conclusion

Conclusion : It seems hard to get a counter-example out of this. On the other hand, proving existence in general looks even worse: we understand very poorly vector bundles on projective varieties, even on $\mathbb{P}^n$ (**recall :** for $n \geqslant 6$, no indecomposable E known on $\mathbb{P}^n$ with $2 \leqslant \mathrm{rk}(E) \leqslant n-2$). The problem remains wide open...