II. The classical examples

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Reminder

X complex manifold, compact and simply-connected.

X hyperkähler = Kähler, with holonomy $Sp(r)$.

$\Leftrightarrow\ H^0(X,\Omega^2_X)=\mathbb{C}\sigma,\ \ \sigma$ everywhere non-degenerate.

Examples?

- K3 surface S: $\omega \in H^0(S, \Omega_S^2)$. Note that $Sp(2) = SU(2)$.
- Idea: S^r has (too) many symplectic structures:

$$\sigma = \lambda_1\, p_1^*\omega + \ldots + \lambda_r\, p_r^*\omega \ , \quad \text{with} \ \ \lambda_1, \ldots, \lambda_r \in \mathbb{C} \ .$$

- Try to get unicity by imposing $\lambda_1 = \ldots = \lambda_r$, i.e. σ invariant under $\mathfrak{S}_r$, i.e. σ comes from $S^{(r)} := S^r/\mathfrak{S}_r = \{$subsets of r points of S, counted with multiplicities$\}$
- $S^{(r)}$ is singular, but admits a natural desingularization

$$S^{[r]} := \{\text{finite analytic subspaces of } S \text{ of length } r\,\}$$

(Hilbert scheme or "Douady space")

Understanding the Hilbert scheme

- "Hilbert-Chow morphism" $h : S^{[r]} \to S^{(r)}$; induces $S_0^{[r]} \xrightarrow{\sim} S_0^{(r)}$,

  where $S_0^{[r]} \cong S_0^{(r)} = \{p_1, \ldots, p_r\}$ *distinct*.

  $\Delta_r := S^{[r]} - S_0^{[r]}$ is an irreducible divisor, $\operatorname{codim} h(\Delta_r) = 2$.

- subscheme of length 2 at $p \iff$ tangent direction $v \in \mathbb{P}(T_p(S))$.

- *Exercise*: length 3 subscheme at p: ideal of the form (x, y^3) or (x^2, xy, y^2).

- $S^{[r]}$ is smooth (Fogarty, infinitesimal calculation).

Theorem

For S K3, $S^{[r]}$ is a hyperkähler manifold.

Idea of proof: $r = 2$.

1. Cartesian diagram

$$\begin{array}{ccc} B_\Delta(S^2) & \xrightarrow{\ \rho\ } & S^{[2]} \\ {\scriptstyle b}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ S^2 & \xrightarrow[\ \pi\ ]{} & S^{(2)} \end{array}$$

2. $b^*(p_1^*\omega + p_2^*\omega)$ invariant by the involution, hence $= \rho^*\sigma$, σ holomorphic 2-form on $S^{[2]}$.

3. $\mathrm{div}(b^*(p_1^*\omega + p_2^*\omega)^2) = E := b^{-1}(\Delta)$; $\mathrm{div}(\rho^*\sigma^2) = \rho^*\,\mathrm{div}(\sigma^2) + E$

$\Longrightarrow \quad \mathrm{div}(\sigma^2) = 0$, hence σ symplectic. ∎

Sketch of proof (r arbitrary).

- $S^{[r]} \supset S^{[r]}_* := \{$subschemes with at most one double point$\}$

  then $\operatorname{codim}(S^{[r]} - S^{[r]}_*) = 2 \implies$

  enough to find σ symplectic on $S^{[r]}_*$ (Hartogs), and to prove $\pi_1(S^{[r]}_*) = 0$.

- Same cartesian diagram

$$\begin{array}{ccc} B_\Delta(S^r_*) & \xrightarrow{\ \rho\ } & S^{[r]}_* \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle h} \\ S^r_* & \xrightarrow[\ \pi\ ]{} & S^{(r)}_* \end{array}$$

  and same argument works. ∎

The second series

- The same construction gives a symplectic form on $A^{[r]}$, A complex torus of dimension 2; but $\pi_1(A^{[r]}) \neq 0$.

- Consider $A^{[r+1]} \xrightarrow{h} A^{(r+1)} \xrightarrow{S} A$, where $S =$ addition map $\Big(S(a_1, \ldots, a_{r+1}) = \sum a_i\ .\Big)$ Define $K_r(A) := S^{-1}(0)$.

Theorem

$K_r(A)$ is a hyperkähler manifold.

- ("generalized Kummer varieties": $r = 1$ gives usual Kummer)

Deformations

X compact manifold. *Deformation* of X over pointed space (B, o):

$f : \mathcal{X} \to B$ proper smooth, with $\mathcal{X}_o \xrightarrow{\sim} X$.

If $H^0(X, T_X) = 0$, there exists a universal local deformation, parametrized by $B \subset H^1(X, T_X)$, with $T_0(B) = H^1(X, T_X)$.

If B smooth at 0 ($\Leftrightarrow$ $B = H^1(X, T_X)$ locally around 0), we say that X is unobstructed.

Theorem (Bogomolov, Tian, Todorov, Ran, Deligne, Kawamata...)

If $K_X = \mathcal{O}_X$, X is unobstructed.

Back to hyperkähler manifolds:

Theorem

Any Kähler deformation of a hyperkähler manifold is hyperkähler.

Proof.

- $f : \mathcal{X} \to B$ smooth, proper, $\mathcal{X}_b$ Kähler $\forall b$, $\mathcal{X}_o$ hyperkähler. Then
- $c_1(\mathcal{X}_b) = 0$ in $H^2(\mathcal{X}_b, \mathbb{Z})$ and $\pi_1(\mathcal{X}_b) = 0$ for all b ;
- $h^{p,q}(\mathcal{X}_b)$ independent of b, thus $h^{2p,0} = 1$ for all $p \leq r$.
- Decomposition theorem $\Rightarrow$ $\mathcal{X}_b \cong \prod_{i=1}^{s} Y_i \times \prod_{j=1}^{t} Z_j$,

  with Y_i hyperkähler and Z_j Calabi-Yau.
- Define $P_X(t) := \sum_p h^{p,0}(X) t^p$; then $P_{\mathcal{X}_b} = \prod_i P_{Y_i} \prod_j P_{Z_j}$.

  Exercise : $\Rightarrow$ $s = 1,\ t = 0$.

Deformations of $S^{[r]}$?

For X hyperkähler, $H^1(X, T_X) \cong H^{1,1}$. Compute $H^2(S^{[r]}, \mathbb{C})$?

Proposition

Canonical isomorphism $H^2(S^{[r]}, \mathbb{C}) \cong H^2(S, \mathbb{C}) \oplus \mathbb{C}\,[\Delta_r]$, *compatible with Hodge structures.*

Sketch of proof.

1. $\pi : S^r \to S^{(r)}$ induces $\pi^* : H^2(S^{(r)}) \xrightarrow{\sim} H^2(S^r)^{\mathfrak{S}_r} \cong H^2(S)$

   (can replace $S^{(r)}$ by $S_{\mathrm{o}}^{(r)}$).

2. $h : S^{[r]} \to S^{(r)}$ induces $h^* : H^2(S^{(r)}) \to H^2(S^{[r]})$ injective,

   hence $i : H^2(S) \to H^2(S^{[r]})$ injective.

3. Gysin: $\quad 0 \to H^0(\Delta_r) \longrightarrow H^2(S^{[r]}) \longrightarrow H^2(S_{\mathrm{o}}^{[r]})$ and

   $$H^2(S_{\mathrm{o}}^{[r]}) \cong H^2(S_{\mathrm{o}}^{(r)}) \cong H^2(S^{(r)}) \cong H^2(S)$$

4. $$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\Delta_r) & \longrightarrow & H^2(S^{[r]}) & \xrightarrow{u} & H^2(S) \\ & & & & \uparrow{\scriptstyle i} & \nearrow{\scriptstyle \mathrm{Id}} & \\ & & & & H^2(S) & & \end{array}$$

- Thus $H^1(S^{[r]}, T_{S^{[r]}}) \cong H^1(S, T_S) \oplus \mathbb{C} \cong \mathbb{C}^{21}$:

  deformations of $S^{[r]}$ obtained by deforming S form a hypersurface in the space of all deformations ($r \geq 2$).

- Same result for $K_r(A)$: $H^1(K_r(A), T_{K_r(A)}) \cong H^1(A, T_A) \oplus \mathbb{C}$.

Some history

- Examples: $S^{[r]}$ and $K_r(A)$ (1983)
- Mukai (1984): The moduli of stable sheaves on S or A have a symplectic structure, hence hyperkähler when compact. But
- They are deformations of $S^{[r]}$ or $K_r(A)$ (O'Grady, Yoshioka).
- Two new examples by O'Grady, dim. 6 and 10.

That's all what is known!