IV. Birational hyperkähler manifolds

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Atiyah's example

$f : \mathcal{X} \to D$ family of K3 surfaces, smooth over D^*; $\mathcal{X}$ smooth, $\mathcal{X}_0$ has one node s.

local coordinates (x, y, z) at s, $f(x, y, z) = x^2 + y^2 + z^2$.

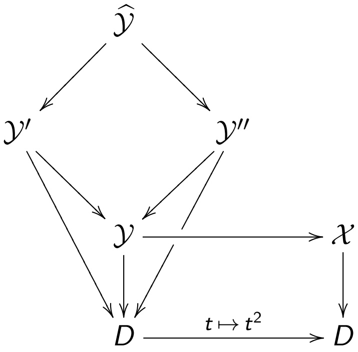
Pull back by $t \mapsto t^2$:

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ D & \xrightarrow{t \mapsto t^2} & D \end{array}$$

$\mathcal{Y}$ at s : $x^2 + y^2 + z^2 = t^2$.

Blow up s in $\mathcal{Y}$ $\rightsquigarrow$ $\widehat{\mathcal{Y}}$ smooth, exceptional divisor = quadric. Can blow down along each ruling:

Atiyah's example, II



$\mathcal{Y}'$ and $\mathcal{Y}''$ smooth over D, fibre at 0 $=$ resolution of $\mathcal{X}_0$,

isomorphic over D^*, but not over D.

Atiyah's example, III

Choosing trivializations of $H^2(\mathcal{Y}'_t,\mathbb{Z})_{t\in D}$ and $H^2(\mathcal{Y}''_t,\mathbb{Z})_{t\in D}$ which coincide over D^*, get

$\wp'$ and $\wp'' : D \to \mathcal{M}_L$ which coincide on D^* but not on $D \ \Rightarrow$

$\mathcal{Y}'_0$ and $\mathcal{Y}''_0$ give non-separated points in $\mathcal{M}_L$.

Mukai's elementary transformations

X hyperkähler, $\dim X = 2r$, contains $P \cong \mathbb{P}^r$.

Then $\sigma_{|P} = 0$ (P Lagrangian) $\Rightarrow$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_P & \longrightarrow & T_{X\,|P} & \longrightarrow & N_{P/X} & \longrightarrow & 0 \\
 & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\
0 & \longrightarrow & N^*_{P/X} & \longrightarrow & \Omega^1_{X\,|P} & \longrightarrow & \Omega^1_P & \longrightarrow & 0
\end{array}$$

Blow-up P in X:

$$\begin{array}{ccc}
E & \hookrightarrow & \widehat{X} \\
\downarrow & & \downarrow \\
P & \hookrightarrow & X
\end{array}$$

Mukai's elementary transformations, II

$$E = \mathbb{P}_P(N_{P/X}) = \mathbb{P}_P(\Omega^1_P) = \{(p,h) \in P \times P^* \mid p \in h\} = \mathbb{P}_{P^*}(\Omega^1_{P^*})$$

Thus can blow down E to $P^* \hookrightarrow X^*$:

$$\begin{array}{ccccccc}
 & & E & \hookrightarrow & \widehat{X} & & \\
 & \swarrow & & \searrow\!\!\swarrow & & \searrow & \\
P & \hookrightarrow & X & & P^* & \hookrightarrow & X^*
\end{array}$$

X^* symplectic, not necessarily Kähler. If it is, hyperkähler.

Atiyah's construction in higher dimension

Suppose $X = \mathcal{X}_0$, $\mathcal{X} \to D$ family of hyperkähler manifolds

$\rightsquigarrow$ deformation vector $v \in H^1(X, T_X) \cong H^1(X, \Omega^1_X)$.

$$P \hookrightarrow X \hookrightarrow \mathcal{X} \quad \text{gives exact sequence}$$

$$0 \to N_{P/X} \cong \Omega^1_P \longrightarrow N_{P/\mathcal{X}} \longrightarrow (N_{X/\mathcal{X}})_{|P} \cong \mathcal{O}_P \to 0 \qquad (*)$$

Lemma

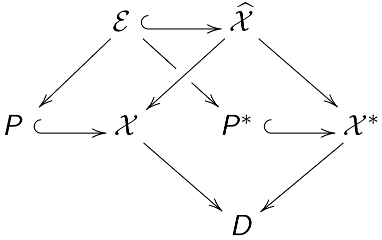
Extension class $e \in H^1(P, \Omega^1_P) =$ *pull back of* $v \in H^1(X, \Omega^1_X)$. ■

Suppose $e \neq 0$ (e.g. v Kähler). Then $(*) =$ Euler exact sequence:

$$0 \to \Omega^1_P \longrightarrow N_{P/\mathcal{X}} \cong \mathcal{O}_P^{r+1}(-1) \longrightarrow \mathcal{O}_P \to 0$$

Blow up P in $\mathcal{X}$. Exceptional divisor $\mathcal{E} \cong P \times P^*$. As above, can blow down $\mathcal{E}$ onto P^*:

Atiyah's construction in higher dimension, II



$\mathcal{X}, \mathcal{X}^*$ isomorphic over D^* $\rightsquigarrow$ non-separated points in $\mathcal{M}_L$.

Theorem (Huybrechts)

X, X' birational hyperkähler. $\exists\ \mathcal{X} \to D$ and $\mathcal{X}' \to D$ isomorphic over D^* with $\mathcal{X}_0 \cong X$, $\mathcal{X}'_0 \cong X'$.

Corollary

Two birational hyperkähler manifolds are diffeomorphic.

Compare:

1. If X, X' birational Calabi-Yau, $b_i(X) = b_i(X')$ and $h^{p,q}(X) = h^{p,q}(X')$ (Batyrev, Kontsevich);
2. there exists X, X' birational Calabi-Yau threefolds s.t. $H^*(X,\mathbb{Z}) \ncong H^*(X',\mathbb{Z})$ as algebras (Friedman).

   ($\Rightarrow$ X and X' not diffeomorphic).