AN AMPLENESS CRITERION FOR RANK 2 VECTOR BUNDLES ON SURFACES

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1. Introduction

We observe in this note that the proof of the Bogomolov stable restriction theorem [B] can be adapted to give an ampleness criterion for globally generated rank 2 vector bundles on certain surfaces. This applies to the Lazarsfeld-Mukai bundles, to congruences of lines in $\mathbb{P}^3$, and possibly to the construction of surfaces with ample cotangent bundle.

2. Main result

Throughout the note, S will be a smooth projective surface over $\mathbb{C}$. We denote by $N^1(S)$ the group of divisors on S modulo numerical equivalence; this is a free, finitely generated abelian group, quotient of $\mathrm{NS}(S) = H^2(S,\mathbb{Z})_{\mathrm{alg}}$ by its torsion subgroup.

Proposition 1. *Let E be a globally generated rank 2 vector bundle on S, with $h^0(E) \geq 4$. Assume that $N^1(S) = \mathbb{Z}\cdot c_1(E)$. Then either E is ample, or $E = \mathscr{O}_S \oplus \det(E)$.*

We will need the following lemma:

Lemma. *Let S be a smooth projective surface, and let E be a globally generated rank 2 vector bundle on S, with $h^0(E) \geq 4$ and $H^1(S, \det(E)^{-1}) = 0$. Then $c_1^2(E) > c_2(E)$.*

Proof : Let V be a general 4-dimensional subspace of $H^0(S,E)$. Then V generates E globally, giving rise to an exact sequence

$$0 \to N \to V \otimes_{\mathbb{C}} \mathscr{O}_S \to E \to 0\,. \qquad (1)$$

Since N^* is globally generated, the zero locus of a general section s of N^* is finite, of length $c_2(N^*) = c_1^2(E) - c_2(E)$. Thus this number is ≥ 0; if it is zero, s does not vanish, so we have an exact sequence

$$0 \to \mathscr{O}_S \xrightarrow{s} N^* \to \det(E) \to 0\,.$$

Since $H^1(S, \det(E)^{-1}) = 0$, this sequence splits, so that $N \cong \mathscr{O}_S \oplus \det(E)^{-1}$. Thus the exact sequence (1) reduces to

$$0 \to \det(E)^{-1} \to \mathscr{O}_S^3 \to E \to 0\,;$$

but using again $H^1(S, \det(E)^{-1}) = 0$ this implies $h^0(E) \leq 3$, contradicting the hypothesis. ∎

Proof of the Proposition : We denote by c_1 and c_2 the Chern classes of E in $H^*(S,\mathbb{Z})$, and by $\Delta_E := 4c_2 - c_1^2$ its discriminant. Assume that E is not ample. By Gieseker's lemma [L, Proposition 6.1.7], there exists an irreducible curve C in S and a surjective homomorphism $u : E \twoheadrightarrow \mathscr{O}_C$. The kernel F of u is a vector bundle, with total Chern class $c(F) = c(E)c(\mathscr{O}_C)^{-1} = (1 + c_1 + c_2)(1 - [C])$, hence

$$c_1(F) = c_1 - [C]\ ,\ c_2(F) = c_2 - c_1\cdot[C]\ ,\ \text{and}\ \ \Delta_F = \Delta_E - 2c_1\cdot[C] - [C]^2\,.$$

The curve C is numerically equivalent to rc_1 for some integer $r \geq 1$. Therefore

$$\Delta_F = 4c_2 - (r+1)^2 c_1^2 \leq 4(c_2 - c_1^2)\,.$$

Because of our hypotheses $\det(E)$ is ample, so $H^1(S,\det(E)^{-1})=0$ and we can apply the Lemma, which gives $\Delta_F<0$. By Bogomolov's theorem (see [Ra, Théorème 6.1]), we have an exact sequence

$$0\to L\to F\to \mathscr{I}_Z M\to 0$$

where Z is a finite subscheme of S, L and M are line bundles on S, with $c_1(L)=ac_1$, $c_1(M)=bc_1$ for some integers a,b such that $a\geq b$.

From that exact sequence we get $c_1(F)=(a+b)c_1$, hence $a+b=1-r$, and $c_2(F)=\deg(Z)+abc_1^2$, hence $\Delta_F=4\deg(Z)-(a-b)^2c_1^2$. Comparing with the previous expression for Δ_F and using the Lemma again we find

$$(a-b)^2c_1^2\geq -\Delta_F=(r+1)^2c_1^2-4c_2>(r^2+2r-3)c_1^2\geq (r^2-1)c_1^2\,,$$

hence $a-b\geq r$, and $a\geq 1$.

We have $H^0(E\otimes L^{-1})=H^0(E^*\otimes\det(E)\otimes L^{-1})\neq 0$. Since E is globally generated, the natural map $E^*\to H^0(E)^*\otimes_{\mathbb{C}}\mathscr{O}_S$ is injective, hence $H^0(\det(E)\otimes L^{-1})\neq 0$. Since $c_1(L)=ac_1$ with $a\geq 1$, the only possibility is $L\cong\det(E)$, and therefore $H^0(E^*)\neq 0$. Using again that E is globally generated, we obtain $E=\mathscr{O}_S\oplus\det(E)$. ■

Remark. The condition $h^0(E)\geq 4$ is necessary: if E is ample and globally generated, the rational map $\mathbb{P}(E)\to\mathbb{P}(H^0(E))$ associated to the linear system $|\mathscr{O}_{\mathbb{P}(E)}(1)|$ is a finite morphism, hence $\dim\mathbb{P}(H^0(E))\geq 3$. On the other hand, the condition $N^1(S)=\mathbb{Z}\cdot c_1$ is quite restrictive, but it is not clear how it could be weakened. For instance, we will exhibit in Example 1 of §4 a globally generated rank 2 vector bundle E on $\mathbb{P}^2$ with $h^0(E)\geq 4$, $\det E=\mathscr{O}_{\mathbb{P}^2}(2)$, which is not ample.

3. APPLICATION 1: LAZARSFELD-MUKAI BUNDLES

Let C be an irreducible curve in S, L a line bundle on C, and V a 2-dimensional subspace of $H^0(L)$ which generates L. The *Lazarsfeld-Mukai bundle* $E_{C,V}$ is defined by the exact sequence

$$0\to E_{C,V}^*\to V\otimes_{\mathbb{C}}\mathscr{O}_S\to L\to 0\,.$$

Let $N_C:=\mathscr{O}_S(C)_{|C}$ be the normal of C in S. By duality we get an exact sequence

$$0\to V^*\otimes_{\mathbb{C}}\mathscr{O}_S\to E_{C,V}\to N_C\otimes L^{-1}\to 0\,.$$

Proposition 2. *Assume $H^1(S,\mathscr{O}_S)=0$, $N^1(S)=\mathbb{Z}\cdot[C]$, and that the line bundle $N_C\otimes L^{-1}$ on C is globally generated and nontrivial. Then $E_{C,V}$ is globally generated and ample.*

Proof : We put $E:=E_{C,V}$. Since $H^1(S,\mathscr{O}_S)=0$, we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V^*\otimes_{\mathbb{C}}\mathscr{O}_S & \longrightarrow & H^0(S,E)\otimes_{\mathbb{C}}\mathscr{O}_S & \longrightarrow & H^0(C,N_C\otimes L^{-1})\otimes_{\mathbb{C}}\mathscr{O}_S & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V^*\otimes_{\mathbb{C}}\mathscr{O}_S & \longrightarrow & E & \longrightarrow & N_C\otimes L^{-1} & \longrightarrow & 0\,.
\end{array}$$

This implies that E is globally generated, with $h^0(E)=2+h^0(N_C\otimes L^{-1})\geq 4$. From the bottom exact sequence we get $c_1(E)=[C]$ and $c_2(E)=\deg(L)>0$. The conclusion follows from Proposition 1. ■

4. APPLICATION 2: CONGRUENCES OF LINES

Let $\mathbb{G}$ be the Grassmannian of lines in $\mathbb{P}^3$, which we view as a smooth quadric in $\mathbb{P}^5$; let $S \subset \mathbb{G}$ be a smooth surface. This defines a 2-dimensional family of lines in $\mathbb{P}^3$, classically called a *congruence*. A point $p \in \mathbb{P}^3$ through which pass infinitely many lines of the congruence is called a *fundamental point* (or, more classically, a singular point) of the congruence.

Proposition 3. *Assume that S has degree > 1 and that $N^1(S)$ is generated by the restriction of $\mathscr{O}_{\mathbb{G}}(1)$. Then S has no fundamental point.*

Proof : Let E be the restriction to S of the universal quotient bundle Q on $\mathbb{G}$. The projective bundle $\mathbb{P}(E)$ on S parametrizes pairs (ℓ, p) in $S \times \mathbb{P}^3$ with $p \in \ell$, and the second projection $q : \mathbb{P}_S(E) \to \mathbb{P}^3$ satisfies $q^*\mathscr{O}_{\mathbb{P}^3}(1) = \mathscr{O}_{\mathbb{P}(E)}(1)$. Thus q is finite (that is, S has no fundamental point) if and only if E is ample.

We have $h^0(Q) = 4$, and a nonzero section of Q vanishes along a linear plane; therefore $h^0(E) \geq 4$, and we can apply Proposition 1. If $E = \mathscr{O}_S \oplus \mathscr{O}_S(1)$, we have $c_2(E) = 0$, that is, $c_2(Q) \cdot [S] = 0$; this can only happen if S is a linear plane, which we have excluded. Therefore E is ample. ■

Corollary. *Let d, e be two integers with $d, e > 1$, or $d = 1$ and $e \geq 3$; let $S \subset \mathbb{G}$ be the complete intersection of two general hypersurfaces of degree d and e. Then S has no fundamental point.*

Indeed $\mathrm{Pic}(S)$ is generated by $\mathscr{O}_S(1)$ [D, Théorème 1.2].

Examples.— 1) Perhaps the simplest example of a nontrivial congruence is the surface S of lines bisecant to a twisted cubic $T \subset \mathbb{P}^3$; it is isomorphic to $\mathrm{Sym}^2 T \cong \mathbb{P}^2$, embedded in $\mathbb{G} \subset \mathbb{P}^5$ by the Veronese map. In that case $N^1 = \mathbb{Z} \cdot [\mathscr{O}_{\mathbb{P}^2}(1)]$ but $\det E = \mathscr{O}_{\mathbb{P}^2}(2)$, and indeed the fundamental locus of S is T, so E is not ample.

2) Let A be an abelian surface such that $\mathrm{NS}(A) = \mathbb{Z} \cdot [L]$, where L is a line bundle with $L^2 = 10$. The linear system $|L|$ embeds A into $\mathbb{P}^4$ [R], giving the famous Horrocks-Mumford abelian surface. The projection $\pi : \mathbb{G} \to \mathbb{P}^4$ from a general point of $\mathbb{P}^5$ is a double covering, and the surface $S := \pi^{-1}(A) \subset \mathbb{G}$ is smooth. The line bundle π^*L is not divisible in $N^1(S)$: since $(\pi^*L)^2 = 20$, this could happen only if π^*L is divisible by 2; but $\pi^*L = K_S$, so this would imply that K_S^2 is divisible by 8, a contradiction. It then follows from [Bu] that $N^1(S)$ is generated by $\pi^*L = \mathscr{O}_S(1)$, so Proposition 2 applies and S has no fundamental point.

5. APPLICATION 3 (VIRTUAL): SURFACES WITH AMPLE COTANGENT BUNDLE

The original motivation of this work was to obtain new examples of surfaces with ample cotangent bundle – these surfaces have very interesting properties, but there are few concrete examples known. Applying Proposition 1 to Ω^1_S we get the following result; unfortunately we do not know any example of a surface satisfying the hypotheses (help welcome!).

Proposition 4. *Assume that Ω^1_S is globally generated (for instance that S is a subvariety of an abelian variety), $q(S) \geq 4$, and $N^1(S) = \mathbb{Z} \cdot [K_S]$. Then Ω^1_S is ample.*

Proof : The hypotheses imply that K_S is ample, hence $c_2(S) > 0$; therefore Ω^1_S is not isomorphic to $\mathscr{O}_S \oplus K_S$. The conclusion follows from Proposition 1. ■

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