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On the Splitting of the Bloch-Beilinson Filtration

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Introduction

This paper deals with the *Chow ring* $\mathsf{CH}(X)$ (with rational coefficients) of a smooth projective variety X – that is, the $\mathbb{Q}$-algebra of algebraic cycles on X, modulo rational equivalence. This is a basic invariant of the variety X, which may be thought of as an algebraic counterpart of the cohomology ring of a compact manifold; in fact there is a $\mathbb{Q}$-algebra homomorphism $c_X : \mathsf{CH}(X) \to \mathsf{H}(X, \mathbb{Q})$, the *cycle class map*. But unlike the cohomology ring, the Chow ring, and in particular the kernel of c_X, is poorly understood.

Still some insight into the structure of this ring is provided by the deep conjectures of Bloch and Beilinson. They predict the existence of a functorial ring filtration $(F^j)_{j\geq 0}$ of $\mathsf{CH}(X)$, with $\mathsf{CH}^p(X) = F^0\mathsf{CH}^p(X) \supset \ldots \supset F^{p+1}(X) = 0$ and $F^1\mathsf{CH}(X) = \operatorname{Ker} c_X$. We refer to [J] for a discussion of the various candidates for such a filtration and the consequences of its existence.

The existence of that filtration is not even known for an abelian variety A. In that case, however, there is a canonical *ring graduation* given by $\mathsf{CH}^p(A) = \bigoplus_s \mathsf{CH}^p_s(A)$, where $\mathsf{CH}^p_s(A)$ is the subspace of elements $\alpha \in \mathsf{CH}^p(A)$ with $k_A^*\alpha = k^{2p-s}\alpha$ for all $k \in \mathbb{Z}$ (k_A denotes the endomorphism $a \mapsto ka$ of A) [B2]. Unfortunately this does not define the required filtration because the vanishing of the terms $\mathsf{CH}^p_s(A)$ for $s < 0$ is not known in general – in fact, this vanishing is essentially equivalent to the existence of the Bloch-Beilinson filtration (the precise relationship is thoroughly analyzed in [Mu]). So if the Bloch-Beilinson filtration indeed exists, it *splits* in the sense that it is the filtration associated to a graduation of $\mathsf{CH}(A)$.

In [B-V] we observed that this also happens for a K3 surface S. Here the filtration is essentially trivial; the fact that it splits means that the image of the intersection product $\mathsf{CH}^1(S) \otimes \mathsf{CH}^1(S) \to \mathsf{CH}^2(S)$ is always one-dimensional – an easy but somewhat surprising property.

The motivation for this paper was to understand whether the splitting of the Bloch-Beilinson filtration for abelian varieties and K3 surfaces is accidental or part of a more general framework. Now asking for a conjectural splitting of a conjectural filtration may look like a rather idle occupation. The point we want to make is that the mere existence of such a splitting has quite concrete consequences, which at least in some cases can be tested. We will restrict for simplicity to the case of regular varieties, that is, varieties X for which $F^1\mathsf{CH}^1(X) = 0$. Then if the filtration comes from a graduation, any product of divisors must have degree 0; therefore, if we denote by $\mathsf{DCH}(X)$ the sub-algebra of $\mathsf{CH}(X)$ spanned by divisor classes, *the cycle class map*

$$c_X : \mathsf{DCH}(X) \longrightarrow \mathsf{H}(X)$$

is injective. In other words, any polynomial relation $P(D_1, \dots, D_s) = 0$ between divisor classes which hold in cohomology must hold in $\mathsf{CH}(X)$. We will call this property the *weak splitting property.* Despite its name it is rather restrictive: it implies for instance the existence of a class $\xi_X \in \mathsf{CH}^n(X)$, with $n = \dim X$, such that

$$D_1 \cdot \ldots \cdot D_n = \deg(D_1 \cdot \ldots \cdot D_n) \cdot \xi_X \quad \text{in } \mathsf{CH}^n(X)$$

for any divisor classes $D_1, \dots, D_n$ in $\mathsf{CH}^1(X)$.

What kind of varieties can we expect to have the weak splitting property? A natural class containing abelian varieties and K3 surfaces is that of Calabi-Yau varieties, but that turns out to be too optimistic – it is quite easy to give counter-examples (Example 9.1.5. b)). A more restricted class is that of holomorphic symplectic manifolds – projective manifolds admitting an everywhere non-degenerate holomorphic 2-form. We want to propose the following conjecture:

Conjecture. — A symplectic (*projective*) *manifold satisfies the weak splitting property.*

We have to admit that the evidence we are able to provide is not overwhelming. We will prove that the weak splitting property is invariant under some simple birational transformations called Mukai flops (Proposition 9.2.4). We will also prove that the conjecture holds for the simplest examples of symplectic manifolds, the Hilbert schemes $S^{[2]}$ and $S^{[3]}$ associated to a K3 surface S (Proposition 9.3.1). Already for $S^{[3]}$ the proof is intricate, and makes use of some nontrivial relations in the Chow rings of S^2 and S^3 established in [B-V]. We hope that this might indicate a deep connection between the symplectic structure and the Bloch-Beilinson filtration, but we have not even a conjectural formulation of what this connection could be.

9.1 Intersection of divisors

Let X be a projective (complex) manifold. We denote by $\mathsf{CH}(X)$ and $\mathsf{H}(X)$ the Chow and cohomology rings with rational coefficients, and by $\mathsf{CH}(X,\mathbb{C})$ and $\mathsf{H}(X,\mathbb{C})$ the corresponding rings with complex coefficients. We denote by $\mathsf{DCH}(X)$ the sub-algebra of $\mathsf{CH}(X)$ spanned by divisor classes. We will say that X has the *weak splitting property* if the cycle class map $c_X : \mathsf{DCH}(X) \to \mathsf{H}(X)$ is injective.

Remark 9.1.1. The property as stated implies that $\mathsf{CH}^1(X)$ is finite-dimensional, that is, X is regular in the sense that $H^1(X,\mathcal{O}_X)=0$. For irregular varieties the definition should be adapted, either by considering cycles modulo algebraic equivalence, or by picking up an appropriate subspace of $\mathsf{CH}^1(X)$. We will restrict ourselves to regular varieties in what follows.

Examples 9.1.2. a) A regular surface S satisfies the weak splitting property if and only if the image of the intersection map $\mathsf{CH}^1(S)\otimes \mathsf{CH}^1(S) \to \mathsf{CH}^2(S)$ has rank 1; in other words, there exists a class $\xi_S \in \mathsf{CH}^2(S)$, of degree 1, such that $C\cdot D = \deg(C.D)\,\xi_S$ for all curves C, D on S. This is the case when S is a K3 surface, or also an elliptic surface over $\mathbb{P}^1$ with a section [B-V].

b) Let S be a K3 surface, p a point of S with$[p]\neq \xi_S$ in $\mathsf{CH}^2(S)$. Let $\varepsilon : \widehat{S} \to S$ be the blowing-up of S at p. The space $\mathsf{DCH}^2(\widehat{S})$ is spanned by $\varepsilon^*\xi_S$ and $[q]$, where q is any point of $\widehat{S}$ above p. Since the pushforward map $\varepsilon_* : \mathsf{CH}^2(\widehat{S}) \to \mathsf{CH}^2(S)$ is an isomorphism, theses classes are linearly independent in $\mathsf{CH}^2(\widehat{S})$, so the map $c^2_{\widehat{S}} : \mathsf{DCH}^2(\widehat{S}) \to \mathsf{CH}^2(\widehat{S})$ is not injective.

Observe that we get a family of surfaces parameterized by $p\in S$, for which the weak splitting property fails generically, but holds when p lies in the union of countably many subvarieties of the parameter space.

c) We will give later (9.1.5) examples of Fano and Calabi-Yau threefolds which do not satisfy the weak splitting property.

Proposition 9.1.3. *Let X, Y be two smooth projective* regular *varieties.*

a) *We have* $\mathsf{DCH}^p(X\times Y) = \bigoplus_{r+s=p} \mathrm{pr}_1^*\,\mathsf{DCH}^r(X)\otimes \mathrm{pr}_2^*\,\mathsf{DCH}^s(Y)$. *In particular, $X\times Y$ satisfies the weak splitting property if and only if X and Y do.*

b) *Let $f : X\to Y$ be a surjective map. If $c^p_X : \mathsf{DCH}^p(X) \to \mathsf{H}^{2p}(X)$ is injective, then so is $c^p_Y : \mathsf{DCH}^p(Y) \to \mathsf{H}^{2p}(Y)$.*

Proof a) We have $\mathsf{CH}^1(X\times Y)=\mathrm{pr}_1^*\,\mathsf{CH}^1(X)\oplus \mathrm{pr}_2^*\,\mathsf{CH}^1(Y)$ since X and Y are regular; the assertion a) follows at once.
b) Follows from the commutative diagram

$$\begin{array}{ccc} \mathsf{DCH}^p(X) & \xrightarrow{c_X^p} & \mathsf{H}^{2p}(X) \\ {\scriptstyle f^*}\big\uparrow & & {\scriptstyle f^*}\big\uparrow \\ \mathsf{DCH}^p(Y) & \xrightarrow{c_Y^p} & \mathsf{H}^{2p}(Y) \end{array}$$

and the injectivity of $f^*:\mathsf{CH}^p(Y)\to\mathsf{CH}^p(X)$ (if h is an ample class in $\mathsf{CH}^1(X)$ and $d=\dim X-\dim Y$, we have $f_*(h^d)=r\cdot 1_Y$, with $r\in\mathbb{Q}^*$, and $f_*(h^d\cdot f^*\xi)=r\,\xi$ for ξ in $\mathsf{CH}(Y)$). $\square$

We now consider the behaviour of the weak splitting property when the variety X is blown up along a smooth subvariety B. We will use the notation summarized in the following diagram:

$$\begin{array}{ccc} E & \overset{i}{\hookrightarrow} & \widehat{X} \\ {\scriptstyle \eta}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon} \\ B & \overset{j}{\hookrightarrow} & X\ . \end{array} \tag{9.1}$$

We denote by c the codimension of B in X and by N its normal bundle.

Lemma 9.1.4. *Let p be an integer. Assume*

i) *The cycle class map $c_B^q:\mathsf{DCH}^q(B)\to\mathsf{H}^{2q}(B)$ is injective for $p-c<q<p$;*
ii) *The Chern classes $c_i(N)$ belong to $\mathsf{DCH}(B)$;*
iii) *The map $c_X^p:\mathsf{CH}^p(X)\to\mathsf{H}^{2p}(X)$ restricted to $\mathsf{DCH}^p(X)+j_*\mathsf{DCH}^{p-c}(B)$ is injective.*

Then the cycle class map $c_{\widehat{X}}^p:\mathsf{DCH}^p(\widehat{X})\to\mathsf{H}^{2p}(\widehat{X})$ is injective.

Proof The projection $p:E\to B$ identifies E to $\mathbb{P}_B(N^\vee)$. Let $h\in\mathsf{CH}^1(E)$ be the class of the tautological bundle $\mathcal{O}_E(1)$; we have $i^*[E]=-h$, and therefore, for $\xi\in\mathsf{CH}(X)$, $[E]^p\cdot\varepsilon^*\xi=i_*(i^*[E]^{p-1}\cdot i^*\varepsilon^*\xi)=(-1)^{p-1}i_*(h^{p-1}\cdot\eta^*j^*\xi)$.

Since $\mathsf{CH}^1(\widehat{X})=\varepsilon^*\mathsf{CH}^1(X)\oplus\mathbb{Q}[E]$, we get

$$\begin{aligned}\mathsf{DCH}^p(\widehat{X}) &= \varepsilon^*\mathsf{DCH}^p(X)+[E]\cdot\varepsilon^*\mathsf{DCH}^{p-1}(X)+\ldots+\mathbb{Q}[E]^p\\ &\subset \varepsilon^*\mathsf{DCH}^p(X)+i_*\eta^*\mathsf{DCH}^{p-1}(B)+i_*(h\cdot\eta^*\mathsf{DCH}^{p-2}(B))+\ldots+\mathbb{Q}i_*h^{p-1}\ .\end{aligned}$$

For $q\geq c$ we have a relation $h^q=h^{c-1}\cdot\eta^*c_{q,c-1}+\ldots+\eta^*c_{q,0}$, where the

$c_{i,j}$ are polynomial in the Chern classes of N; by our hypothesis (ii) these classes lie in $\mathsf{DCH}(B)$. Moreover the "key formula" [F, 6.7]

$$i_*(\gamma\cdot\eta^*\xi)=\varepsilon^*j_*\xi \qquad \text{for } \xi\in\mathsf{CH}(B)\ ,$$

with $\gamma=h^{c-1}+h^{c-2}\cdot\eta^*c_1(N)+\ldots+\eta^*c_{c-1}(N)$, implies

$$i_*(h^{c-1}\cdot\eta^*\mathsf{DCH}^{p-c}(B))\subset\varepsilon^*j_*\mathsf{DCH}^{p-c}(B)+\sum_{k=0}^{c-2}i_*(h^k\cdot\eta^*\mathsf{DCH}^{p-k-1}(B))\ ,$$

so that we finally get

$$\mathsf{DCH}^p(\widehat{X})\ \subset\ \varepsilon^*\big(\mathsf{DCH}^p(X)+j_*\mathsf{DCH}^{p-c}(B)\big)+\sum_{k=0}^{c-2}i_*(h^k\cdot\eta^*\mathsf{DCH}^{p-k-1}(B))$$

Since the map

$$\begin{array}{ccc}\mathsf{H}^{2p}(X)\oplus\displaystyle\sum_{k=0}^{c-2}\mathsf{H}^{2(p-k-1)}(B) & \longrightarrow & \mathsf{H}^{2p}(\widehat{X})\\ (\alpha\ ;\ \beta_0,\ldots,\beta_{c-2}) & \longmapsto & \varepsilon^*\alpha+\displaystyle\sum_k i_*(h^k\cdot\eta^*\beta_k)\end{array}$$

is an isomorphism (see for instance [Jo]), our hypotheses (i) and (iii) ensure that $c^p_{\widehat{X}}$ is injective. □

Examples 9.1.5. a) Take $X=\mathbb{P}^3$, and let B be a smooth curve, of degree d and genus g. Let ℓ be the class of a hyperplane in $\mathbb{P}^3$, ℓ_B its pull back to B. The space $\mathsf{DCH}^2(\widehat{X})$ is generated by

$$\varepsilon^*\ell^2\quad,\quad\varepsilon^*\ell\cdot[E]=i_*p^*\ell_B\quad,\quad[E]^2=-i_*h=i_*\eta^*c_1(N)-\varepsilon^*[B]$$

We have $c_1(N)=4\ell_B+K_B$, so $\mathsf{DCH}^2(\widehat{X})$ contains the elements $i_*\eta^*\ell_B$ and $i_*\eta^*K_B$.

The map $i_*\eta^*\ :\ \mathsf{CH}^1(B)\to\mathsf{CH}^2(X)$ induces an isomorphism of the subspace of degree 0 divisor classes on B onto the subspace of homologically trivial classes in $\mathsf{CH}^2(X)$. If we choose ℓ_B non proportional to K_B in $\mathsf{CH}^1(B)$, the class $i_*\eta^*(d\,K_B-(2g-2)\ell_B)$ in $\mathsf{DCH}^2(\widehat{X})$ is homologically trivial, but non-trivial. Thus the map $c^2_{\widehat{X}}:\mathsf{DCH}^2(\widehat{X})\to\mathsf{H}^4(\widehat{X})$ is not injective.

If B is a scheme-theoretical intersection of cubics, $\widehat{X}$ is a Fano variety [M-M] – we can take for instance B of genus 2 and ℓ_B a general divisor class of degree 5 (or B of genus 3 and ℓ_B general of degree 6, or B of genus 5 and $\ell_B\equiv K_B-p$ for p a general point of B). Note that by making the linear system vary we get again families where the general member does not satisfy the weak splitting property, while countably many special members of the family do satisfy it.

b) Going on with the Fano case, let D be a smooth divisor in $|-2K_X|$, and let $V \to X$ be the double covering of X ramified along D. Then by the above example and Proposition 9.1.3. b), *V is a Calabi-Yau threefold which does not satisfy the weak splitting property.*

9.2 The weak splitting property for symplectic manifolds

By a symplectic manifold we mean here a simply-connected projective manifold which admits a holomorphic, everywhere non-degenerate 2-form. The manifold is said to be *irreducible* if the 2-form is unique up to a scalar; any symplectic manifold admits a canonical decomposition as a product of irreducible ones. In view of Proposition 9.1.3. a), we may restrict ourselves to irreducible symplectic manifolds.

Let X be an irreducible symplectic manifold, of dimension $2r$. Recall that the space $H^2(X)$ admits a canonical quadratic form q ([B1], [H]) with the following properties:

- every class $x \in H^2(X,\mathbb{C})$ with $q(x) = 0$ satisfies $x^{r+1} = 0$;
- there exists $\lambda \in \mathbb{Q}$ such that $\int_X x^{2r} = \lambda\, q(x)^r$ for all $x \in H^2(X,\mathbb{C})$, where $\int_X$ is the canonical isomorphism $H^{2r}(X,\mathbb{C}) \xrightarrow{\sim} \mathbb{C}$.

In fact the following more precise statement has been proved by Bogomolov:

Proposition 9.2.1. *Let V be a subspace of $\mathsf{H}^2(X,\mathbb{C})$ such that the restriction of q to V is non-degenerate (for instance $V = \mathsf{H}^2(X,\mathbb{C})$ or $V = \mathsf{CH}^1(X,\mathbb{C})$). The kernel of the map* $\operatorname{Sym} V \to \mathsf{H}(X,\mathbb{C})$ *is the ideal of* $\operatorname{Sym} V$ *spanned by the elements x^{r+1} for $x \in V,\ q(x) = 0$.*

Proof The case $V = \mathsf{H}^2(X,\mathbb{C})$ is the main result of [Bo], but the proof given there implies the slightly more general statement 9.2.1. Namely, define $A(V)$ as the quotient of $\operatorname{Sym} V$ by the ideal spanned by the elements x^{r+1} for $x \in V,\ q(x) = 0$. Then Lemma 2.5 in [Bo] says that $A(V)$ is a finite-dimensional graded Gorenstein $\mathbb{C}$- algebra, with socle in degree $2r$ – in other words, $A_{2r}(V)$ is one-dimensional, and the multiplication pairing $A_d(V) \times A_{2r-d}(V) \to A_{2r}(V) \cong \mathbb{C}$ is a perfect duality.

Since any element x of $\mathsf{H}^2(X,\mathbb{C})$ with $q(x) = 0$ satisfies $x^{r+1} = 0$, we get a $\mathbb{C}$- algebra homomorphism $u : A(V) \to \mathsf{H}(X,\mathbb{C})$. The kernel of u is an ideal of $A(V)$; if it is non-zero, it contains the minimal ideal $A_{2r}(V)$ of $A(V)$. But this is impossible because V contains an element h with $q(h) \neq 0$, hence with $h^{2r} \neq 0$. $\square$

Corollary 9.2.2. *The following conditions are equivalent:*

i) *The cycle class map* $c_X : DCH(X) \to \mathsf{H}(X)$ *is injective* (*that is,* X *satisfies the weak splitting property*);
ii) *The map* $c_X^{r+1} : \mathsf{DCH}^{r+1}(X) \to H^{2r+2}(X)$ *is injective;*
iii) *Every element* x *of* $\mathsf{CH}^1(X, \mathbb{C})$ *with* $q(x) = 0$ *satisfies* $x^{r+1} = 0$ (*in* $\mathsf{CH}^{r+1}(X, \mathbb{C})$).

Proof Consider the diagram

$$\begin{tikzcd} \mathrm{Sym}\,\mathsf{CH}^1(X,\mathbb{C}) \arrow[d, "u"'] \arrow[dr, "v"] & \\ \mathsf{DCH}(X,\mathbb{C}) \arrow[r, "c_X"'] & \mathsf{H}(X,\mathbb{C}) \end{tikzcd} .$$

The injectivity of c is equivalent to $\operatorname{Ker} v \subset \operatorname{Ker} u$. In view of the Proposition, this is exactly condition (iii), and it is equivalent to $\operatorname{Ker} v^{r+1} \subset \operatorname{Ker} u^{r+1}$. $\square$

Remark 9.2.3. Assume that there is an element $\alpha \in \mathsf{CH}^1(X)$ with $q(\alpha) = 0$ – this is the case for instance if $\dim_{\mathbb{Q}} \mathsf{CH}^1(X) \geq 5$. Then the set of such elements is Zariski dense in the quadric $q = 0$ of $\mathsf{CH}^1(X, \mathbb{C})$. Thus the conditions of the Corollary are also equivalent to:
(iii′) *Every element* x *of* $\mathsf{CH}^1(X)$ *with* $q(x) = 0$ *satisfies* $x^{r+1} = 0$.

A possible proof of (iii′) could be as follows. It seems plausible that the subset of *nef* classes $x \in \mathsf{CH}^1(X)$ with $q(x) = 0$ is Zariski dense in the quadric $q = 0$ (this holds at least when X is a K3 surface). If this is the case, it would be enough to prove (iii′) for nef classes. Now it is a standard conjecture (see [S]) that a nef class $x \in \mathsf{CH}^1(X)$ with $q(x) = 0$ should be the pull back of the class of a hyperplane in $\mathbb{P}^r$ under a Lagrangian fibration $f : X \to \mathbb{P}^r$, so that $x^{r+1} = f^*(h^{r+1}) = 0$.

We will now consider the behaviour of the weak splitting property under a Mukai flop. Let X be an irreducible symplectic manifold, of dimension $2r$; assume that X contains a subvariety P isomorphic to $\mathbb{P}^r$. Then P is a Lagrangian subvariety, and its normal bundle in X is isomorphic to Ω^1_P. We blow up P in X, getting our standard diagram

$$\begin{tikzcd} E \arrow[r, hook, "i"] \arrow[d, "\eta"'] & \widehat{X} \arrow[d, "\varepsilon"] \\ P \arrow[r, hook, "j"] & X \end{tikzcd} .$$

The exceptional divisor E is the cotangent bundle $\mathbb{P}(T_P)$, which can be identified with the incidence divisor in $P \times P^\vee$, where $P^\vee$ is the projective space dual to P. The projection $\eta^\vee : E \to P^\vee$ identifies E to $\mathbb{P}(T_{P^\vee})$, and E can be blown down to $P^\vee$ by a map $\varphi : \widehat{X} \to X'$, where X' is a smooth algebraic space. To remain in our previous framework we will assume that X' is projective, so that X' is again an irreducible symplectic manifold. The diagram

$$\begin{tikzcd} & \widehat{X} \arrow[dl, "\varepsilon"'] \arrow[dr, "\varphi"] & \\ X & & X' \end{tikzcd}$$

is called a *Mukai flop.* There are many concrete examples of such flops, see [M].

Proposition 9.2.4. *If X satisfies the weak splitting property, so does X'.*

Proof Consider the $\mathbb{Q}$- linear map $\varphi_*\varepsilon^* : \mathsf{CH}^1(X) \to \mathsf{CH}^1(X')$. It is bijective and preserves the canonical quadratic forms (see e.g. [H, Lemma 2.6]. In view of Corollary 9.2.2, the Proposition will follow from

Lemma 9.2.5. *Let $\alpha \in \mathsf{CH}^1(X)$, and $\alpha' := \varphi_*\varepsilon^*\alpha$. Then ${\alpha'}^{r+1} = \varphi_*\varepsilon^*(\alpha^{r+1})$.*

Proof We have $\varphi^*\alpha' = \varepsilon^*\alpha + m[E]$ for some $m \in \mathbb{Q}$. Let $\ell \in \mathsf{CH}^{2r-1}(\widehat{X})$ be the class of a line contained in a fibre of $\eta^\vee$; we have $\deg([E]\cdot\ell) = -1$, and $\varepsilon_*\ell$ is the class of a line in P. Intersecting the above equality with ℓ gives $m = \deg(\alpha_{|P})$, or equivalently $\alpha_{|P} = mk$ in $\mathsf{CH}^1(P)$, where k is the class of a hyperplane in P. Then

$$\varphi^*{\alpha'}^{r+1} = (\varepsilon^*\alpha + m[E])^{r+1} = \sum_{p=0}^{r+1} \binom{r+1}{p} m^{r+1-p}\, \varepsilon^*\alpha^p \cdot [E]^{r+1-p} \; .$$

As in (9.1.4), let $h \in \mathsf{CH}^1(E)$ be the class of $\mathcal{O}_E(1)$. For $p \leq r$ we have

$$\varepsilon^*\alpha^p \cdot [E]^{r+1-p} = (-1)^{r-p} i_*(h^{r-p} \cdot i^*\varepsilon^*\alpha^p) = (-1)^{r-p} i_*(h^{r-p} \cdot \eta^*\alpha^p_{|P}) \; ,$$

Thus $$\varphi^*\alpha'^{r+1} = \varepsilon^*\alpha^{r+1} + m^{r+1} i_*\Bigl(\sum_{p=0}^{r}\binom{r+1}{p}(-1)^{r-p}h^{r-p}\,\eta^*k^p\Bigr)\ .$$

Now since the total Chern class of T_P is $(1+k)^{r+1}$ we have in $\mathsf{CH}^r(E)$

$$\sum_{p=0}^{r}\binom{r+1}{p}(-1)^p h^{r-p}\,\eta^*k^p = \sum_{p=0}^{r}(-1)^p h^{r-p}\,\eta^* c_p(T_P) = 0\ ,$$

hence $\varphi^*\alpha'^{r+1} = \varepsilon^*\alpha^{r+1}$. Applying φ_* gives the lemma, hence the Proposition. □

Corollary 9.2.6. *Let X, X' be birationally equivalent projective symplectic fourfolds. Then X satisfies the weak splitting property if and only if X' does.*

Indeed any birational map between projective symplectic fourfolds is a composition of Mukai flops [W].

9.3 The weak splitting property for $\mathbf{S^{[2]}}$ and $\mathbf{S^{[3]}}$.

The simplest symplectic manifolds are K3 surfaces, for which we have already seen that the weak splitting property holds (Example 9.1.2). More precisely [B-V], let S be a K3 surface and o a point of S lying on a (singular) rational curve R. The class of o in $\mathsf{CH}^2(S)$ is independent of the choice of R, and we have, for every $\alpha, \beta \in \mathsf{CH}^1(S)$,

$$\alpha\cdot\beta = \deg(\alpha\cdot\beta)\,[o] \quad \text{in} \quad \mathsf{CH}^2(S)\ .$$

Let $\Delta : S \hookrightarrow S\times S$ be the diagonal embedding. For $\alpha \in \mathsf{CH}^1(S)$, we have in $\mathsf{CH}^3(S\times S)$ ([B-V, Prop. 1.6],)

$$\Delta_*\alpha = \mathrm{pr}_1^*\,\alpha\cdot\mathrm{pr}_2^*[o] + \mathrm{pr}_1^*[o]\cdot\mathrm{pr}_2^*\,\alpha\ . \tag{9.2}$$

K3 surfaces are the first instance of a famous series of symplectic manifolds, the *Hilbert schemes* $S^{[r]}$ parameterizing finite subschemes of length r on the K3 surface S.

Proposition 9.3.1. *Let S be a $K3$ surface. The symplectic varieties $S^{[2]}$ and $S^{[3]}$ satisfy the weak splitting property.*

Proof

— Let us warm up with the easy case of $S^{[2]}$. Let $S^{\{2\}}$ be the variety obtained by blowing up the diagonal of $S\times S$. The Hilbert scheme $S^{[2]}$ is the quotient of $S^{\{2\}}$ by the involution which exchanges the factors. In view of Corollary 9.2.1 and Proposition 9.1.3. b) it suffices to prove that the cycle class map $c^3_{S^{\{2\}}} : \mathsf{DCH}^3(S^{\{2\}}) \to \mathsf{H}^6(S^{\{2\}})$ is injective. We will check that the hypotheses of Lemma 9.1.4 are satisfied. Condition (i) is the weak splitting property for S. The normal bundle to the diagonal in $S \times S$ is T_S, so (ii) means that the class $c_2(T_S)$ belongs to $\mathsf{DCH}^2(S)$; this is proved in ([B-V, thm. 1 c]). Formula (9.2) implies $\Delta_*\mathsf{CH}^1(S) \subset \mathsf{DCH}^3(S \times S)$, so condition (iii) reduces to the injectivity of $c^3_{S\times S}$, which follows from Proposition 9.1.3. a) and the corresponding result for S. $\square$

— Let us pass to the more difficult case of $S^{[3]}$. The Hilbert scheme $S^{[3]}$ is dominated by the *nested Hilbert scheme* $S^{[2,3]}$ which parameterizes pairs $(Z, Z') \in S^{[2]} \times S^{[3]}$ with $Z \subset Z'$; it is isomorphic to the blow-up of $S \times S^{[2]}$ along the incidence subvariety $\mathcal{I} = \{(x, Z) \mid x \in Z\}$. Let $\pi : S^{\{2\}} \to S^{[2]}$ be the quotient map, and $p : S^{\{2\}} \to S$ the first projection. Then the map $j = (p, \pi) : S^{\{2\}} \hookrightarrow S \times S^{[2]}$ induces an isomorphism of $S^{\{2\}}$ onto $\mathcal{I}$ (see for instance [L, 1.2]).

To prove the theorem, it suffices, by Corollary 9.2.2 and Proposition 9.1.3. b), to prove that the cycle class map $\mathsf{DCH}^4(S^{[2,3]}) \to \mathsf{H}^8(S^{[2,3]})$ is injective. We will again check that the hypotheses of Lemma 9.1.4 are satisfied. Condition (i) is the injectivity of the cycle class map $c^3_{S^{\{2\}}} : \mathsf{DCH}^3(S^{\{2\}}) \to \mathsf{H}^6(S^{\{2\}})$, which has just been proved. Let N be the normal bundle to the embedding $j : S^{\{2\}} \hookrightarrow S \times S^{[2]}$, and $E \subset S^{\{2\}}$ the exceptional divisor, which is the ramification locus of π. From the exact sequences

$$\begin{gathered} 0 \to N^\vee \longrightarrow p^*\Omega^1_S \oplus \pi^*\Omega^1_{S^{[2]}} \longrightarrow \Omega^1_{S^{\{2\}}} \to 0 \\ 0 \to \pi^*\Omega^1_{S^{[2]}} \longrightarrow \Omega^1_{S^{\{2\}}} \longrightarrow \mathcal{O}_E(-E) \to 0 \\ 0 \to \mathcal{O}_{S^{\{2\}}}(-2E) \longrightarrow \mathcal{O}_{S^{\{2\}}}(-E) \longrightarrow \mathcal{O}_E(-E) \to 0 \end{gathered}$$

we obtain the equality in K-theory $[N^\vee] = [p^*\Omega^1_S] + [\mathcal{O}_{S^{\{2\}}}(-2E)] - [\mathcal{O}_{S^{\{2\}}}(-E)]$. We conclude that $c_2(N) = c_2(N^\vee)$ belongs to $\mathsf{DCH}^2(S^{\{2\}})$, so that condition (ii) holds.

— The rest of the proof will be devoted to check condition (iii), namely the injectivity of

$$\mathsf{DCH}^4(S \times S^{[2]}) + j_*\mathsf{DCH}^2(S^{\{2\}}) \longrightarrow \mathsf{H}^8(S \times S^{[2]}) \ .$$

Let us fix some notation. We will use our standard diagram (9.1)

$$\begin{tikzcd} E \arrow[r, hook, "i"] \arrow[d, "\eta"'] & S^{\{2\}} \arrow[d, "\varepsilon"] \\ S \arrow[r, hook, "\Delta"] & S^2 \ . \end{tikzcd}$$

We denote by p and q the two projections of $S^{\{2\}}$ onto S. We define an injective $\mathbb{Q}$-linear map $\iota : \mathsf{CH}(S) \to \mathsf{CH}(S^{[2]})$ by $\iota(\xi) := \pi_* p^*\xi$; we will use the same notation for cohomology classes. We have $\pi^*\iota(\xi) = p^*\xi + q^*\xi$ for ξ in $\mathsf{CH}(S)$ or $H(S)$. Finally if $\alpha \in \mathsf{CH}(S)$ and $\xi \in \mathsf{CH}(S^{[2]})$ we put $\alpha \boxtimes \xi := \mathrm{pr}_1^*\,\alpha \otimes \mathrm{pr}_2^*\,\xi$.

We have $\mathsf{CH}^1(S^{\{2\}}) = p^*\mathsf{CH}^1(S) \oplus q^*\mathsf{CH}^1(S) \oplus \mathbb{Q}[E]$. In $\mathsf{CH}^2(S^{\{2\}})$ we have $[E]\cdot \varepsilon^*\alpha = i_*\eta^*\Delta^*\alpha$ for $\alpha \in \mathsf{CH}^1(S^2)$, and $[E]^2 = -\varepsilon^*[\Delta(S)]$. Therefore:

$$\mathsf{DCH}^2(S^{\{2\}}) = \mathbb{Q}\,p^*[o] + \mathbb{Q}\,q^*[o] + p^*\mathsf{CH}^1(S)\otimes q^*\mathsf{CH}^1(S) + i_*\eta^*\mathsf{CH}^1(S) + \mathbb{Q}\,\varepsilon^*[\Delta(S)]\,.$$

We want to describe the space $j_*\mathsf{DCH}^2(S) + \mathsf{DCH}^4(S\times S^{\{2\}})$.

Lemma 9.3.2. *Let $\alpha, \beta \in \mathsf{CH}^1(S)$. The classes $j_*p^*[o]$, $j_*(\,p^*\alpha\cdot q^*\beta)$, and $j_*i_*\eta^*\alpha$ belong to $\mathsf{DCH}^4(S\times S^{\{2\}}) + \mathbb{Q}\,([o]\boxtimes \iota([o]))$.*

Proof Let $j' : S^{\{2\}} \hookrightarrow S\times S^{\{2\}}$ be the embedding given by $j'(z) = (p(z), z)$, so that $j = (1,\pi)\circ j'$. From the cartesian diagram

(9.3.2)
$$\begin{tikzcd} S^{\{2\}} \arrow[r, hook, "j'"] \arrow[d, "p"'] & S\times S^{\{2\}} \arrow[d, "(1,p)"] \\ S \arrow[r, hook, "\Delta"] & S\times S \end{tikzcd}$$

we obtain $j'_*p^*[o] = (1,p)^*\Delta_*[o] = [o]\boxtimes p^*[o]$, hence $j_*p^*[o] = [o]\boxtimes\iota([o])$. In the same way we have $j'_*p^*\alpha = (1,p)^*\Delta_*\alpha$, hence, using (9.2) ,

$$j'_*p^*\alpha = \alpha\boxtimes p^*[o] + [o]\boxtimes p^*\alpha\ .$$

Multiplying by $\mathrm{pr}_2^*\,q^*\beta$ and using $\mathrm{pr}_2\circ j' = \mathrm{Id}$ we obtain

$$j'_*(p^*\alpha\cdot q^*\beta) = \alpha\boxtimes(p^*[o]\cdot q^*\beta) + [o]\boxtimes(p^*\alpha\cdot q^*\beta)\ ,$$

hence $$j_*(p^*\alpha\cdot q^*\beta) = \alpha\boxtimes\pi_*(p^*[o]\cdot q^*\beta) + [o]\boxtimes\pi_*(p^*\alpha\cdot q^*\beta)\ .$$

For $\alpha,\beta\in\mathsf{CH}^1(S)$, put $\langle\alpha,\beta\rangle := \deg(\alpha\cdot\beta)$. Then

$$\begin{aligned}\pi^*\pi_*(p^*\alpha\cdot q^*\beta) &= p^*\alpha\cdot q^*\beta + p^*\beta\cdot q^*\alpha\\ &= (p^*\alpha+q^*\alpha)(p^*\beta+q^*\beta) - \langle\alpha,\beta\rangle(p^*[o]+q^*[o])\\ &= \pi^*\big(\iota(\alpha)\iota(\beta) - \langle\alpha,\beta\rangle\iota([o])\big)\ ;\end{aligned}$$

we find similarly $\pi^*\pi_*(p^*[o]\cdot q^*\beta) = \pi^*\iota([o])\iota(\beta)$, and finally

$$j_*(p^*\alpha\cdot q^*\beta) = \alpha\boxtimes\iota([o])\iota([\beta]) \ +\ [o]\boxtimes\big(\iota(\alpha)\iota(\beta) - \langle\alpha,\beta\rangle\,\iota([o])\big)\ .$$

Let $\gamma\in\mathsf{CH}^1(S)$. We have $\iota(\beta)^2\cdot\iota(\gamma) = \langle\beta^2\rangle\,\iota([o])\iota(\gamma) + 2\langle\beta\cdot\gamma\rangle\,\iota([o])\iota(\beta)$ (this is easily checked by applying π^* as above). If $\langle\beta^2\rangle\neq 0$ we conclude by taking $\gamma = \beta$ that $\iota([o])\iota(\beta)$ is proportional to $\iota(\beta)^3$. If $\langle \beta^2 \rangle = 0$ we can choose γ so that $(\beta \cdot \gamma) \neq 0$; then $\iota([o])\iota(\beta)$ is proportional to $\iota(\beta)^2\iota(\gamma)$. In each case we see that $\iota([o])\iota(\beta)$ belongs to $\mathsf{DCH}^3(S^{[2]})$, hence the assertion of the lemma about $j_*(\, p^*\alpha \cdot q^*\beta)$.

Consider finally the cartesian diagram

$$\begin{tikzcd} E \arrow[r, hook, "k"] \arrow[d, "\eta"'] & S \times E \arrow[d, "{(1,\eta)}"] \\ S \arrow[r, hook, "\Delta"] & S \times S \end{tikzcd}$$

with $k(e) = (\eta(e), e)$. Using again (9.2) we get

$$k_*\eta^*\alpha = (1,\eta)^*\Delta_*\alpha = \alpha \boxtimes \eta^*[o] + [o] \boxtimes \eta^*\alpha \ .$$

Pushing forward in $S \times S^{[2]}$ we obtain $j_*i_*\eta^*\alpha = \alpha \boxtimes i'_*\eta^*[o] + [o] \boxtimes i'_*\eta^*\alpha$, where $i' = \pi \circ i$ is the embedding of E in $S^{[2]}$.

To avoid confusion let us denote by $\bar{E}$ the image of E in $S^{[2]}$, so that $\pi^*[\bar{E}] = 2[E]$. We have $i'_*\eta^*\alpha = \pi_*([E] \cdot p^*\alpha) = \frac{1}{2}[\bar{E}] \cdot \iota(\alpha) \in \mathsf{DCH}^2(S^{[2]})$. Likewise $[E]^3 = i_*h^2 = -24 i_*\eta^*[o]$, hence $i'_*\eta^*[o] = -\frac{1}{96}\,[\bar{E}]^3 \in \mathsf{DCH}^2(S^{[2]})$. This finishes the proof of the lemma. $\square$

The lemma and the formula for $\mathsf{DCH}^2(S^{\{2\}})$ show that $j_*\mathsf{DCH}^2(S^{\{2\}})$ is spanned modulo $\mathsf{DCH}^4(S \times S^{[2]})$ by the classes

$$[o] \boxtimes \iota([o]) \ , \quad j_*q^*[o] \ , \quad j_*\varepsilon^*[\Delta(S)] \ .$$

In fact there is one more relation, much more subtle, between these classes modulo $\mathsf{DCH}^4(S \times S^{[2]})$.

Lemma 9.3.3. *We have*

$$2[o] \boxtimes \iota([o]) - 2j_*q^*[o] + j_*\varepsilon^*[\Delta(S)] \ \in\ \mathsf{DCH}^4(S \times S^{[2]}) \ .$$

Proof We start from a relation in $\mathsf{CH}^4(S^3)$, proved in [B-V, Prop. 3.2]. For $1 \leq i < j \leq 3$, let us denote by $p_{ij} : S^3 \to S^2$ the projection onto the i- th and j- th factors, and by $p_i : S^3 \to S$ the projection onto the i- th factor. We will write simply Δ for the diagonal $\Delta(S) \subset S^2$, and $\delta \subset S^3$ for the small diagonal, that is, the subvariety of triples (x,x,x) for $x \in S$. Then:

$$[\delta] - \sum_{i<j, k\neq i,j} p_{ij}^*[\Delta] \cdot p_k^*[o] + \sum_{i<j} p_i^*[o] \cdot p_j^*[o] = 0 \ .$$

Pull back this relation by the map $\varepsilon_S = (1_S, \varepsilon) : S \times S^{\{2\}} \to S \times S^2$. Since

$$\begin{gathered} p_1 \circ \varepsilon_S = \mathrm{pr}_1 \ , \quad p_2 \circ \varepsilon_S = p \circ \mathrm{pr}_2 \ , \quad p_3 \circ \varepsilon_S = q \circ \mathrm{pr}_2 \ , \quad p_{23} \circ \varepsilon_S = \varepsilon \ , \\ \text{we obtain} \quad \varepsilon_S^*[\delta] = j'_* \varepsilon^*[\Delta] \ , \qquad \varepsilon_S^*(p_1^*[o] \cdot p_{23}^*[\Delta]) = [o] \boxtimes \varepsilon^*[\Delta] \ , \\ \varepsilon_S^*(p_2^*[o] \cdot p_3^*[o]) = 1 \boxtimes p^*[o] \cdot q^*[o] \ , \quad \varepsilon_S^*(p_1^*[o] \cdot p_2^*[o]) = [o] \boxtimes p^*[o] \ , \\ \varepsilon_S^*(p_1^*[o] \cdot p_3^*[o]) = [o] \boxtimes q^*[o] \ . \end{gathered}$$

We have $p_{12} \circ \varepsilon_S = (1, p)$, hence $\varepsilon_S^* p_{12}^*[\Delta] = j'_* 1$ (see diagram 9.3.2) and $\varepsilon_S^*(p_3^*[o] \cdot p_{12}^*[\Delta]) = j'_* q^*[o]$. Let $j'' = (q, 1) : S^{\{2\}} \to S \times S^{\{2\}}$; the same argument gives $\varepsilon_S^* p_{13}^*[\Delta] = j''_* 1$ and $\varepsilon_S^*(p_2^*[o] \cdot p_{13}^*[\Delta]) = j''_* p^*[o]$. Finally we have $\mathrm{k} j'_* q^*[o] + j''_* p^*[o] = \pi_S^* j_* q^*[o]$. Pushing forward by π_S we obtain in $\mathsf{CH}^4(S \times S^{[2]})$:

$$j_* \varepsilon^*[\Delta] - 2 j_* q^*[o] - [o] \boxtimes \pi_* \varepsilon^*[\Delta] + 2[o] \boxtimes \iota([o]) + 1 \boxtimes \iota([o])^2 = 0$$

It remains to observe that $[o] \boxtimes \pi_* \varepsilon^*[\Delta]$ and $1 \boxtimes \iota([o])^2$ belong to $\mathsf{DCH}^4(S \times S^{[2]})$. Indeed from $[E]^2 = -\varepsilon^*[\Delta]$ we deduce $\pi_* \varepsilon^*[\Delta] = -\frac{1}{2}[\bar{E}]^2 \in \mathsf{DCH}^2(S^{[2]})$. And if h is any element of $\mathsf{CH}^1(S)$ with $h^2 = d \neq 0$, we have

$$\pi^* \iota(h)^4 = 6 d^2 p^*[o] \cdot q^*[o] = 3 d^2 \pi^* \iota([o])^2 \ , \quad \text{hence} \ \ \iota([o])^2 \in \mathsf{DCH}^4(S^{[2]}) \ . \square$$

For a smooth projective variety X, let us denote by $\mathsf{DH}(X)$ the (graded) subspace of $\mathsf{H}(X)$ spanned by intersection of divisor classes – that is, the image of $\mathsf{DCH}(X)$ by the cycle class map. It remains to prove that the cycle class map $c^4_{S \times S^{[2]}}$ is injective on $\mathsf{DCH}^4(S \times S^{[2]}) + \mathbb{Q}\,([o] \boxtimes \iota([o])) + \mathbb{Q}\, j_* q^*[o]$. Since we know by (9.3) and (9.1.3. a)) that it is injective on $\mathsf{DCH}^4(S \times S^{[2]})$, this amounts to :

Lemma 9.3.4. *There is no non-trivial relation*

$$a\,[o] \boxtimes \iota([o]) + b\, j_* q^*[o] \in \mathsf{DH}^8(S \times S^{[2]}) \ ,$$

with $a, b \in \mathbb{Q}$.

To prove this, suppose that such a relation holds. Let ω be a non-zero class in $H^{2,0}(S)$; for any class ξ in $\mathsf{H}^8(S \times S^{[2]}, \mathbb{C})$ put $h(\xi) := (\mathrm{pr}_2)_*(\mathrm{pr}_1^* \omega \cdot \xi)$. Since the product of ω with any algebraic class in $H^2(S)$ is zero, h is zero on $\mathsf{DH}^8(S \times S^{[2]})$. Clearly $h([o] \boxtimes \iota([o])) = 0$, while $h(j_* q^*[o]) = \pi_*(p^* \omega \cdot q^*[o]) = \iota(\omega)\iota([o])$. This class is nonzero, for instance because $\langle \iota(\omega)\iota([o]), \iota(\bar{\omega}) \rangle = \langle \omega, \bar{\omega} \rangle > 0$.

Thus $b = 0$, and our relation reduces to $[o] \boxtimes \iota([o]) \in \mathsf{DH}^8(S \times S^{[2]})$. Since $\mathsf{DH}^8(S \times S^{[2]}) = \bigoplus_{i+j=8} \mathsf{DH}^i(S) \boxtimes \mathsf{DH}^j(S^{[2]})$ (see Proposition 9.1.3. a)), this is equivalent to $\iota([o]) \in \mathsf{DH}^4(S^{[2]})$. Thus the proof reduces to the following assertion:

Lemma 9.3.5. *The class $\iota([o])$ does not belong to $\mathsf{DH}^4(S^{[2]})$.*

Proof We have

$$\begin{aligned}\mathsf{H}^4(S^{\{2\}}) &= \varepsilon^*\mathsf{H}^4(S^2)\oplus i_*\eta^*\mathsf{H}^2(S)\\ &= \mathbb{Q}\,p^*[o]\oplus\mathbb{Q}\,q^*[o]\oplus(p^*\mathsf{H}^2(S)\otimes q^*\mathsf{H}^2(S))\oplus i_*\eta^*\mathsf{H}^2(S)\ .\end{aligned}$$

Taking the invariants under the involution of $S^{\{2\}}$ which exchanges the factors, we find

$$\mathsf{H}^4(S^{[2]})=\mathbb{Q}\,\iota([o])\oplus\mathrm{Sym}^2\,\mathsf{H}^2(S)\oplus i'_*\eta^*\mathsf{H}^2(S)\ ,$$

where $\mathrm{Sym}^2\,\mathsf{H}^2(S)$ is identified with a subspace of $\mathsf{H}^4(S^{[2]})$ by $\alpha\cdot\beta\mapsto\pi_*(p^*\alpha\cdot q^*\beta)$, and $i':=\pi\circ i$ is the natural embedding of E in $S^{[2]}$. Since $\mathsf{CH}^1(S^{[2]})=\iota(\mathsf{CH}^1(S))\oplus\mathbb{Q}\cdot[E]$, the subspace $\mathsf{DH}^4(S^{[2]})$ is spanned by the classes

$$\iota(\alpha)\iota(\beta)\ ,\quad \iota(\alpha)\cdot[E]=2i'_*\eta^*\alpha\ ,\quad [E]^2=-2\pi_*\varepsilon^*[\Delta]\quad\text{for}\quad \alpha,\beta\in\mathsf{CH}^1(S)\ .$$

Suppose that we have a relation

$$\iota([o])=\sum_{i<j}m_{ij}\,\iota(\alpha_i)\iota(\alpha_j)+i'_*\eta^*\gamma+m\,\pi_*\varepsilon^*[\Delta]\quad\text{in}\quad\mathsf{H}^4(S^{[2]})\ ,$$

where (α_i) is a basis of $\mathsf{CH}^1(S)$. This gives in $\mathsf{H}^4(S^{\{2\}})$:

$$p^*[o]+q^*[o]=\sum_{i<j}m_{ij}\,(p^*\alpha_i+q^*\alpha_i)(p^*\alpha_j+q^*\alpha_j)+2i_*\eta^*\gamma+2m\,\varepsilon^*[\Delta]\ .$$

Projecting onto the direct summand $i_*\eta^*\mathsf{H}^2(S)$ of $\mathsf{H}^4(S^{\{2\}})$ we find $i_*\eta^*\gamma=0$. Multiplying by $p^*\omega$ and pushing forward by q we find as in the proof of (9.3.4) that all terms but $\varepsilon^*[\Delta]$ give 0, so $m=0$. Finally the equality

$$p^*[o]+q^*[o]=\sum_{i<j}m_{ij}\,(p^*\alpha_i+q^*\alpha_i)(p^*\alpha_j+q^*\alpha_j)$$

projected onto $\mathrm{Sym}^2\,\mathsf{H}^2(S)$ gives $m_{ij}=0$ for all i,j. This achieves the proof of the lemma and therefore of the Proposition. □

Comments. A variation of this method can be used to prove that the *generalized Kummer variety* K_2 associated to an abelian surface A [B1] has the weak splitting property; one must replace $\mathsf{CH}^1(A)$ by the subspace of *symmetric* divisor classes. We leave the details to the reader.

We should point out, however, that even among symplectic fourfolds these examples are quite particular. Indeed for each integer $g\geq 2$, the projective K3 surfaces of genus g (that is, embedded in $\mathbb{P}^g$ with degree $2g-2$) form an irreducible 19-dimensional family; the corresponding family of Hilbert schemes $S^{[2]}$ is contained in a 20-dimensional irreducible family of projective symplectic manifolds (see [B1]). Since the weak splitting property is not invariant under deformation, we do not know whether it holds for the general member of such a family. It would be interesting, in particular, to check whether the property holds for the variety of lines contained in a smooth cubic hypersurface in $\mathbb{P}^5$.

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