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# Limits of the trivial bundle on a curve

**Arnaud Beauville**

**Abstract.** We attempt to describe the vector bundles on a curve $C$ which are specializations of $\mathcal{O}_C^2$. We get a complete classification when $C$ is Brill-Noether-Petri general, or when it is hyperelliptic; in both cases all limit vector bundles are decomposable. We give examples of indecomposable limit bundles for some special curves.

**Keywords.** Vector bundles; limits; Brill-Noether theory; hyperelliptic curves

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**[Français]**

**Titre. Limites du fibré trivial sur une courbe**

**Résumé**. Nous essayons de décrire les fibrés vectoriels qui sont des spécialisations de $\mathcal{O}_C^2$. Nous obtenons une classification complète lorsque $C$ est générale au sens de Brill-Noether-Petri, ou lorsque $C$ est hyperelliptique; les fibrés limites sont décomposables dans chacune des deux situations. Nous donnons également des exemples de fibrés limites indécomposables sur certaines courbe spéciales.

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## Contents



# 1. Introduction

Let $C$ be a smooth complex projective curve, and $E$ a vector bundle on $C$, of rank $r$. We will say that $E$ *is a limit of* $\mathcal{O}_C^r$ if there exists an algebraic family $(E_b)_{b\in B}$ of vector bundles on $C$, parametrized by an algebraic curve $B$, and a point $o \in B$, such that $E_o = E$ and $E_b \cong \mathcal{O}_C^r$ for $b \neq o$. Can we classify all these vector bundles? If $E$ is a limit of $\mathcal{O}_C^2$ clearly $E \oplus \mathcal{O}_C^{r-2}$ is a limit of $\mathcal{O}_C^r$, so it seems reasonable to start in rank 2.

We get a complete classification in two extreme cases: when $C$ is generic (in the sense of Brill-Noether theory), and when it is hyperelliptic. In both cases the limit vector bundles are of the form $L \oplus L^{-1}$, with some precise conditions on $L$. However for large families of curves, for instance for plane curves, some limits of $\mathcal{O}_C^2$ are indecomposable, and those seem hard to classify.

# 2. Generic curves

Throughout the paper we denote by $C$ a smooth connected projective curve of genus $g$ over $\mathbb{C}$.

**Proposition 1.** *Let $L$ be a line bundle on $C$ which is a limit of globally generated line bundles (in particular, any line bundle of degree $\geq g+1$). Then $L \oplus L^{-1}$ is a limit of $\mathcal{O}_C^2$.*

*Proof.* By hypothesis there exist a curve $B$, a point $o \in B$ and a line bundle $\mathcal{L}$ on $C \times B$ such that $\mathcal{L}_{|C\times\{o\}} \cong L$ and $\mathcal{L}_{|C\times\{b\}}$ is globally generated for $b \neq o$. We may assume that $B$ is affine and that $o$ is defined by $f = 0$ for a global function $f$ on $B$; we put $B^* := B \smallsetminus \{o\}$.

We choose two general sections $s, t$ of $\mathcal{L}$ on $C \times B^*$; reducing $B^*$ if necessary, we may assume that they generate $\mathcal{L}$. Thus we have an exact sequence on $C \times B^*$

$$0 \to \mathcal{L}^{-1} \xrightarrow{(t,-s)} \mathcal{O}_{C\times B^*}^2 \xrightarrow{(s,t)} \mathcal{L} \to 0$$

which corresponds to an extension class $e \in H^1(C \times B^*, \mathcal{L}^{-2})$. For $n$ large enough, $f^n e$ comes from a class in $H^1(C \times B, \mathcal{L}^{-2})$ which vanishes along $C \times \{o\}$; this class gives rise to an extension

$$0 \to \mathcal{L}^{-1} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \to 0$$

with $\mathcal{E}_{|C\times\{b\}} \cong \mathcal{O}_C^2$ for $b \neq o$, and $\mathcal{E}_{|C\times\{o\}} \cong L \oplus L^{-1}$. □

**Remark 1.** Let $E$ be a vector bundle limit of $\mathcal{O}_C^2$. We have $\det E = \mathcal{O}_C$, and $h^0(E) \geq 2$ by semi-continuity. If $E$ is semi-stable this implies $E \cong \mathcal{O}_C^2$; otherwise $E$ is unstable. Let $L$ be the maximal destabilizing sub-line bundle of $E$; we have an extension $0 \to L \to E \to L^{-1} \to 0$, with $h^0(L) \geq 2$. Note that this extension is trivial (so that $E = L \oplus L^{-1}$) if $H^1(L^2) = 0$, in particular if $\deg(L) \geq g$.

**Proposition 2.** *Assume that $C$ is Brill-Noether-Petri general. The following conditions are equivalent:*

(i) *$E$ is a limit of $\mathcal{O}_C^2$;*

(ii) $h^0(E) \geq 2$ *and* $\det E = \mathcal{O}_C$ ;

(iii) $E = L \oplus L^{-1}$ *for some line bundle* $L$ *on* $C$ *with* $h^0(L) \geq 2$ *or* $L = \mathcal{O}_C$ .

*Proof.* We have seen that (i) implies (ii) (Remark 1). Assume (ii) holds, with $E \not\cong \mathcal{O}_C^2$. Then $E$ is unstable, and we have an extension $0 \to L \to E \to L^{-1} \to 0$ with $h^0(L) \geq 2$. Since $C$ is Brill-Noether-Petri general we have $H^0(C, K_C \otimes L^{-2}) = 0$ [ACG, Ch. 21, Proposition 6.7], hence $H^1(C, L^2) = 0$. Therefore the above extension is trivial, and we get (iii).

Assume that (iii) holds. Brill-Noether theory implies that any line bundle $L$ with $h^0(L) \geq 2$ is a limit of globally generated ones [1]. So (i) follows from Proposition 1. □

## 3. Hyperelliptic curves

**Proposition 3**. *Assume that* $C$ *is hyperelliptic, and let* $H$ *be the line bundle on* $C$ *with* $h^0(H) = \deg(H) = 2$. *The limits of* $\mathcal{O}_C^2$ *are the decomposable bundles* $L \oplus L^{-1}$, *with* $\deg(L) \geq g+1$ *or* $L = H^k$ *for* $k \geq 0$.

*Proof.* Let $\pi : C \to \mathbb{P}^1$ be the two-sheeted covering defined by $|H|$. Let us say that an effective divisor $D$ on $C$ is *simple* if it does not contain a divisor of the form $\pi^* p$ for $p \in \mathbb{P}^1$. We will need the following well-known lemma:

**Lemma 1**. *Let* $L$ *be a line bundle on* $C$.

1) *If* $L = H^k(D)$ *with* $D$ *simple and* $\deg(D) + k \leq g$, *we have* $h^0(L) = h^0(H^k) = k+1$.

2) *If* $\deg(L) \leq g$, $L$ *can be written in a unique way* $H^k(D)$ *with* $D$ *simple. If* $L$ *is globally generated, it is a power of* $H$.

*Proof of Lemma 1.* 1) Put $\ell := g-1-k$ and $d := \deg(D)$. Recall that $K_C \cong H^{g-1}$. Thus by Riemann-Roch, the first assertion is equivalent to $h^0(H^\ell(-D)) = h^0(H^\ell) - d$. We have $H^0(C, H^\ell) = \pi^* H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\ell))$; since $D$ is simple of degree $\leq \ell + 1$, it imposes $d$ independent conditions on $H^0(C, H^\ell)$, hence our claim.

2) Let $k$ be the greatest integer such that $h^0(L \otimes H^{-k}) > 0$; then $L = H^k(D)$ for some effective divisor $D$, which is simple since $k$ is maximal. By 1) $D$ is the fixed part of $|L|$, hence is uniquely determined, and so is $k$. In particular the only globally generated line bundles on $C$ of degree $\leq g$ are the powers of $H$. □

*Proof of the Proposition* : Let $E$ be a vector bundle on $C$ limit of $\mathcal{O}_C^2$. Consider the exact sequence

$$0 \to L \to E \to L^{-1} \to 0, \tag{1}$$

where we can assume $\deg(L) \leq g$ (Remark 1). By Lemma 1 we have $L = H^k(D)$ with $D$ simple of degree $\leq g - 2k$. After tensor product with $H^k$, the corresponding cohomology exact sequence reads

$$0 \to H^0(C, H^{2k}(D)) \to H^0(C, E \otimes H^k) \to H^0(C, \mathcal{O}_C(-D)) \xrightarrow{\partial} H^1(C, H^{2k}(D))$$

which implies $h^0(E \otimes H^k) = h^0(H^{2k}(D)) + \dim \operatorname{Ker} \partial = 2k + 1 + \dim \operatorname{Ker} \partial$ by Lemma 1.

By semi-continuity we have $h^0(E \otimes H^k) \geq 2h^0(H^k) = 2k+2$; the only possibility is $D = 0$ and $\partial = 0$. But $\partial(1)$ is the class of the extension (1), which must therefore be trivial; hence $E = H^k \oplus H^{-k}$. □

[1] ↑ Indeed, the subvariety $W_d^r$ of $\operatorname{Pic}^d(C)$ parametrizing line bundles $L$ with $h^0(L) \geq r+1$ is equidimensional, of dimension $g-(r+1)(r+g-d)$; the line bundles which are not globally generated belong to the subvariety $W_{d-1}^r + C$, which has codimension $r$.

# 4. Examples of indecomposable limits

To prove that some limits of $\mathcal{O}_C^2$ are indecomposable we will need the following easy lemma:

**Lemma 2**. *Let $L$ be a line bundle of positive degree on $C$, and let*

$$0 \to L \to E \to L^{-1} \to 0 \tag{2}$$

*be an exact sequence. The following conditions are equivalent:*

(i) *$E$ is indecomposable;*

(ii) *The extension (2) is nontrivial;*

(iii) $h^0(E \otimes L) = h^0(L^2)$.

*Proof.* The implication (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii) : After tensor product with $L$, the cohomology exact sequence associated to (2) gives

$$0 \to H^0(L^2) \xrightarrow{\ i\ } H^0(E \otimes L) \longrightarrow H^0(\mathcal{O}_C) \xrightarrow{\ \partial\ } H^1(L^2),$$

where $\partial$ maps $1 \in H^0(\mathcal{O}_C)$ to the extension class of (2). Thus (ii) implies that $i$ is an isomorphism, hence (iii).

(iii) $\Rightarrow$ (i): If $E$ is decomposable, it must be equal to $L \oplus L^{-1}$ by unicity of the destabilizing bundle. But this implies $h^0(E \otimes L) = h^0(L^2) + 1$. ☐

The following construction was suggested by N. Mohan Kumar:

**Proposition 4**. *Let $C \subset \mathbb{P}^2$ be a smooth plane curve, of degree $d$. For $0 < k < \dfrac{d}{4}$, there exist extensions*

$$0 \to \mathcal{O}_C(k) \to E \to \mathcal{O}_C(-k) \to 0$$

*such that $E$ is indecomposable and is a limit of $\mathcal{O}_C^2$.*

*Proof.* Let $Z$ be a finite subset of $\mathbb{P}^2$ which is the complete intersection of two curves of degree $k$, and such that $C \cap Z = \varnothing$. By [S, Remark 4.6], for a general extension

$$0 \to \mathcal{O}_{\mathbb{P}^2}(k) \to E \to \mathcal{I}_Z(-k) \to 0, \tag{3}$$

the vector bundle $E$ is a limit of $\mathcal{O}_{\mathbb{P}^2}^2$; therefore $E_{|C}$ is a limit of $\mathcal{O}_C^2$.

The extension (3) restricts to an exact sequence

$$0 \to \mathcal{O}_C(k) \to E_{|C} \to \mathcal{O}_C(-k) \to 0.$$

To prove that $E_{|C}$ is indecomposable, it suffices by Lemma 2 to prove that $h^0(E_{|C}(k)) = h^0(\mathcal{O}_C(2k))$. Since $2k < d$ we have $h^0(\mathcal{O}_C(2k)) = h^0(\mathcal{O}_{\mathbb{P}^2}(2k)) = h^0(E(k))$, so in view of the exact sequence

$$0 \to E(k-d) \longrightarrow E(k) \longrightarrow E_{|C}(k) \to 0$$

it suffices to prove $H^1(E(k-d)) = 0$, or by Serre duality $H^1(E(d-k-3)) = 0$.

The exact sequence (3) gives an injective map $H^1(E(d-k-3)) \hookrightarrow H^1(\mathcal{I}_Z(d-2k-3))$. Now since $Z$ is a complete intersection we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2k) \to \mathcal{O}_{\mathbb{P}^2}(-k)^2 \to \mathcal{I}_Z \to 0;$$

since $4k < d$ we have $H^2(\mathcal{O}_{\mathbb{P}^2}(d-4k-3)) = 0$, hence $H^1(\mathcal{I}_Z(d-2k-3)) = 0$, and finally $H^1(E(d-k-3)) = 0$ as asserted. ☐

We can also perform the Strømme construction directly on the curve $C$, as follows. Let $L$ be a base point free line bundle on $C$. We choose sections $s,t\in H^0(L)$ with no common zero. This gives rise to a Koszul extension

$$0\to L^{-1}\xrightarrow{\ i\ }\mathcal{O}_C^2\xrightarrow{\ p\ }L\to 0\quad\text{with}\ \ i=(-t,s)\,,\ p=(s,t)\,. \tag{4}$$

We fix a nonzero section $u\in H^0(L^2)$. Let $\mathcal{L}$ be the pull-back of $L$ on $C\times\mathbb{A}^1$. We consider the complex ("monad")

$$\mathcal{L}^{-1}\xrightarrow{\ \alpha\ }\mathcal{L}^{-1}\oplus\mathcal{O}^2\oplus\mathcal{L}\xrightarrow{\ \beta\ }\mathcal{L}\,,\qquad \alpha=(\lambda,i,u)\,,\ \beta=(u,p,-\lambda),$$

where $\lambda$ is the coordinate on $\mathbb{A}^1$. Let $\mathcal{E}:=\operatorname{Ker}\beta/\operatorname{Im}\alpha$, and let $E:=\mathcal{E}_{|C\times\{0\}}$.

**Lemma 3**. *$E$ is a rank 2 vector bundle, limit of $\mathcal{O}_C^2$. There is an exact sequence $0\to L\to E\to L^{-1}\to 0$; the corresponding extension class in $H^1(L^2)$ is the product by $u^2\in H^0(L^4)$ of the class $e\in H^1(L^{-2})$ of the Koszul extension* (4).

*Proof.* The proof is essentially the same as in [S]; we give the details for completeness.

For $\lambda\neq 0$, we get easily $\mathcal{E}_{|C\times\{\lambda\}}\cong\mathcal{O}_C^2$; we will show that $E$ is a rank 2 vector bundle. This implies that $\mathcal{E}$ is a vector bundle on $C\times\mathbb{A}^1$, and therefore that $E$ is a limit of $\mathcal{O}_C^2$.

Let us denote by $\alpha_0,\beta_0$ the restrictions of $\alpha$ and $\beta$ to $C\times\{0\}$. We have $\operatorname{Ker}\beta_0=L\oplus N$, where $N$ is the kernel of $(u,p):L^{-1}\oplus\mathcal{O}_C^2\to L$. Applying the snake lemma to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^{-1} & \xrightarrow{\ i\ } & \mathcal{O}^2 & \xrightarrow{\ p\ } & L & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & L^{-1}\oplus\mathcal{O}^2 & \longrightarrow & L & \longrightarrow & 0
\end{array}$$

we get an exact sequence

$$0\to L^{-1}\to N\to L^{-1}\to 0\,, \tag{5}$$

which fits into a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^{-1} & \longrightarrow & N & \longrightarrow & L^{-1} & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow{\scriptstyle\times u} & & \\
0 & \longrightarrow & L^{-1} & \longrightarrow & \mathcal{O}^2 & \longrightarrow & L & \longrightarrow & 0\,;
\end{array}$$

this means that the extension (5) is the pull-back by $\times u:L^{-1}\to L$ of the Koszul extension (4).

Now since $E$ is the cokernel of the map $L^{-1}\to L\oplus N$ induced by $\alpha_0$, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^{-1} & \longrightarrow & N & \longrightarrow & L^{-1} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\times u} & & \downarrow & & \| & & \\
0 & \longrightarrow & L & \longrightarrow & E & \longrightarrow & L^{-1} & \longrightarrow & 0
\end{array}$$

so that the extension $L\to E\to L^{-1}$ is the push-forward by $\times u$ of (5). This implies the Lemma. □

Unfortunately it seems difficult in general to decide whether the extension $L\to E\to L^{-1}$ nontrivial. Here is a case where we can conclude:

**Proposition 5**. *Assume that $C$ is non-hyperelliptic. Let $L$ be a globally generated line bundle on $C$ such that $L^2\cong K_C$. Let $0\to L\to E\to L^{-1}\to 0$ be the unique nontrivial extension of $L^{-1}$ by $L$. Then $E$ is indecomposable, and is a limit of $\mathcal{O}_C^2$.*

*Proof.* We choose $s, t$ in $H^0(L)$ without common zero, and use the previous construction. It suffices to prove that we can choose $u \in H^0(K_C)$ so that $u^2 e \neq 0$: since $H^1(K_C) \cong \mathbb{C}$, the vector bundle $E$ will be the unique nontrivial extension of $L^{-1}$ by $L$, and indecomposable by Lemma 2.

Suppose that $u^2 e = 0$ for all $u$ in $H^0(K_C)$; by bilinearity this implies $uve = 0$ for all $u, v$ in $H^0(K_C)$. Since $C$ is not hyperelliptic, the multiplication map $\mathrm{S}^2 H^0(K_C) \to H^0(K_C^2)$ is surjective, so we have $we = 0$ for all $w \in H^0(K^2)$. But the pairing

$$H^1(K_C^{-1}) \otimes H^0(K_C^2) \to H^1(K_C) \cong \mathbb{C}$$

is perfect by Serre duality, hence our hypothesis implies $e = 0$, a contradiction. □

**Remark 2.** In the moduli space $\mathcal{M}_g$ of curves of genus $g \geq 3$, the curves $C$ admitting a line bundle $L$ with $L^2 \cong K_C$ and $h^0(L)$ even $\geq 2$ form an irreducible divisor [T2]; for a general curve $C$ in this divisor, the line bundle $L$ is unique, globally generated, and satisfies $h^0(L) = 2$ [T1]. Thus Proposition 5 provides for $g \geq 4$ a codimension 1 family of curves in $\mathcal{M}_g$ admitting an indecomposable vector bundle limit of $\mathcal{O}_C^2$.

**Remark 3.** Let $\pi : C \to B$ be a finite morphism of smooth projective curves. If $E$ is a vector bundle limit of $\mathcal{O}_B^2$, then clearly $\pi^* E$ is a limit of $\mathcal{O}_C^2$. Now if $E$ is indecomposable, $\pi^* E$ is also indecomposable. Consider indeed the nontrivial extension $0 \to L \to E \to L^{-1} \to 0$ (Remark 1); by Lemma 2 it suffices to show that the class $e \in H^1(B, L^2)$ of this extension remains nonzero in $H^1(C, \pi^* L^2)$. But the pull-back homomorphism $\pi^* : H^1(B, L^2) \to H^1(C, \pi^* L^2)$ can be identified with the homomorphism $H^1(B, L^2) \to H^1(B, \pi_* \pi^* L^2)$ deduced from the linear map $L^2 \to \pi_* \pi^* L^2$, and the latter is an isomorphism onto a direct factor; hence $\pi^*$ is injective and $\pi^* e \neq 0$, so $E$ is indecomposable.

Thus any curve dominating one of the curves considered in Propositions 4 and 5 carries an indecomposable vector bundle which is a limit of $\mathcal{O}_C^2$.

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