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SOME SURFACES WITH MAXIMAL PICARD NUMBER

by Arnaud Beauville

Abstract. — For a smooth complex projective variety, the rank ρ of the Néron-Severi group is bounded by the Hodge number $h^{1,1}$. Varieties with $\rho = h^{1,1}$ have interesting properties, but are rather sparse, particularly in dimension 2. We discuss in this note a number of examples, in particular those constructed from curves with special Jacobians.

Résumé (Quelques surfaces dont le nombre de Picard est maximal). — Le rang ρ du groupe de Néron-Severi d'une variété projective lisse complexe est borné par le nombre de Hodge $h^{1,1}$. Les variétés satisfaisant à $\rho = h^{1,1}$ ont des propriétés intéressantes, mais sont assez rares, particulièrement en dimension 2. Dans cette note nous analysons un certain nombre d'exemples, notamment ceux construits à partir de courbes à jacobienne spéciale.

Contents

1. Introduction

The *Picard number* of a smooth projective variety X is the rank ρ of the Néron-Severi group – that is, the group of classes of divisors in $H^2(X,\mathbb{Z})$. It is bounded by the Hodge number $h^{1,1} := \dim H^1(X,\Omega^1_X)$. We are interested here in varieties with maximal Picard number $\rho = h^{1,1}$. As we will see in §2, there are many examples of such varieties in dimension $\geqslant 3$, so we will focus on the case of surfaces.

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Apart from the well understood case of K3 and abelian surfaces, the quantity of known examples is remarkably small. In [Per82] Persson showed that some families of double coverings of rational surfaces contain surfaces with maximal Picard number (see Section 6.4 below); some scattered examples have appeared since then. We will review them in this note and examine in particular when the product of two curves has maximal Picard number – this provides some examples, unfortunately also quite sparse.

2. Generalities

Let X be a smooth projective variety over $\mathbb{C}$. The Néron-Severi group $\mathrm{NS}(X)$ is the subgroup of algebraic classes in $H^2(X,\mathbb{Z})$; its rank ρ is the *Picard number* of X. The natural map $\mathrm{NS}(X)\otimes\mathbb{C}\to H^2(X,\mathbb{C})$ is injective and its image is contained in $H^{1,1}$, hence $\rho\leqslant h^{1,1}$.

Proposition 1. — *The following conditions are equivalent:*

(i) $\rho = h^{1,1}$*;*
(ii) *The map* $\mathrm{NS}(X)\otimes\mathbb{C}\to H^{1,1}$ *is bijective;*
(iii) *The subspace* $H^{1,1}$ *of* $H^2(X,\mathbb{C})$ *is defined over* $\mathbb{Q}$*.*
(iv) *The subspace* $H^{2,0}\oplus H^{0,2}$ *of* $H^2(X,\mathbb{C})$ *is defined over* $\mathbb{Q}$*.*

Proof. — The equivalence of (iii) and (iv) follows from the fact that $H^{2,0}\oplus H^{0,2}$ is the orthogonal of $H^{1,1}$ for the scalar product on $H^2(X,\mathbb{C})$ associated to an ample class. The rest is clear. □

When X satisfies these equivalent properties we will say for short that X is *ρ-maximal* (one finds the terms singular, exceptional or extremal in the literature).

Remarks

(1) A variety with $H^{2,0}=0$ is ρ-maximal. We will implicitly exclude this trivial case in the discussion below.

(2) Let X, Y be two ρ-maximal varieties, with $H^1(Y,\mathbb{C})=0$. Then $X\times Y$ is ρ-maximal. For instance $X\times\mathbb{P}^n$ is ρ-maximal, and $Y\times C$ is ρ-maximal for any curve C.

(3) Let Y be a submanifold of X; if X is ρ-maximal and the restriction map $H^2(X,\mathbb{C})\to H^2(Y,\mathbb{C})$ is bijective, Y is ρ-maximal. By the Lefschetz theorem, the latter condition is realized if Y is a complete intersection of smooth ample divisors in X, of dimension $\geqslant 3$. Together with Remark 2, this gives many examples of ρ-maximal varieties of dimension $\geqslant 3$; thus we will focus on finding ρ-maximal *surfaces*.

Proposition 2. — *Let* $\pi : X \dashrightarrow Y$ *be a rational map of smooth projective varieties.*

(a) *If* $\pi^*: H^{2,0}(Y)\to H^{2,0}(X)$ *is injective (in particular if* π *is dominant), and* X *is* ρ*-maximal, so is* Y*.*

(b) *If* $\pi^*: H^{2,0}(Y)\to H^{2,0}(X)$ *is surjective and* Y *is* ρ*-maximal, so is* X*.*

Note that since π is defined on an open subset $U \subset X$ with $\operatorname{codim}(X \smallsetminus U) \geqslant 2$, the pull back map $\pi^* : H^2(Y,\mathbb{C}) \to H^2(U,\mathbb{C}) \cong H^2(X,\mathbb{C})$ is well defined.

Proof. — Hironaka's theorem provides a diagram

$$\begin{array}{ccc} & \widehat{X} & \\ {}^{b}\swarrow & & \searrow^{\widehat{\pi}} \\ X & \overset{\pi}{\dashrightarrow} & Y \end{array}$$

where $\widehat{\pi}$ is a morphism, and b is a composition of blowing-ups with smooth centers. Then $b^* : H^{2,0}(X) \to H^{2,0}(\widehat{X})$ is bijective, and $\widehat{X}$ is ρ-maximal if and only if X is ρ-maximal; so replacing π by $\widehat{\pi}$ we may assume that π is a morphism.

(a) Let $V := (\pi^*)^{-1}(\mathrm{NS}(X) \otimes \mathbb{Q})$. We have

$$V \otimes_{\mathbb{Q}} \mathbb{C} = (\pi^*)^{-1}(\mathrm{NS}(X) \otimes \mathbb{C}) = (\pi^*)^{-1}(H^{1,1}(X)) = H^{1,1}(Y)$$

(the last equality holds because π^* is injective on $H^{2,0}(Y)$ and $H^{0,2}(Y)$), hence Y is ρ-maximal.

(b) Let W be the $\mathbb{Q}$-vector subspace of $H^2(Y,\mathbb{Q})$ such that

$$W \otimes_{\mathbb{Q}} \mathbb{C} = H^{2,0}(Y) \oplus H^{0,2}(Y).$$

Then π^*W is a $\mathbb{Q}$-vector subspace of $H^2(X,\mathbb{Q})$, and

$$(\pi^*W) \otimes \mathbb{C} = \pi^*(W \otimes \mathbb{C}) = \pi^*H^{2,0}(Y) \oplus \pi^*H^{0,2}(Y) = H^{2,0}(X) \oplus H^{0,2}(X),$$

so X is ρ-maximal. □

3. Abelian varieties

There is a nice characterization of ρ-maximal abelian varieties ([Kat75], [Lan75]):

Proposition 3. — *Let A be an abelian variety of dimension g. We have*

$$\operatorname{rk}_{\mathbb{Z}} \operatorname{End}(A) \leqslant 2g^2.$$

The following conditions are equivalent:

(i) *A is ρ-maximal;*

(ii) $\operatorname{rk}_{\mathbb{Z}} \operatorname{End}(A) = 2g^2$*;*

(iii) *A is isogenous to E^g, where E is an elliptic curve with complex multiplication.*

(iv) *A is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication.*

(The equivalence of (i), (ii) and (iii) follows easily from Lemma 1 below; the only delicate point is (iii) $\Rightarrow$ (iv), which we will not use.)

Coming back to the surface case, suppose that our abelian variety A contains a surface S such that the restriction map $H^{2,0}(A) \to H^{2,0}(S)$ is surjective. Then S is ρ-maximal if A is ρ-maximal (Proposition 2(b)). Unfortunately this situation seems to be rather rare. We will discuss below (Proposition 6) the case of $\mathrm{Sym}^2 C$ for a curve C. Another interesting example is the *Fano surface* F_X parametrizing the lines contained in a smooth cubic threefold X, embedded in the intermediate Jacobian JX [CG72]. There are some cases in which JX is known to be ρ-maximal:

Proposition 4

(a) *For $\lambda \in \mathbb{C}$, $\lambda^3 \neq 1$, let X_λ (resp. E_λ) be the cubic in $\mathbb{P}^4$ (resp. $\mathbb{P}^2$) defined by*

$$X_\lambda \colon X^3 + Y^3 + Z^3 - 3\lambda XYZ + T^3 + U^3 = 0, \quad E_\lambda \colon X^3 + Y^3 + Z^3 - 3\lambda XYZ = 0.$$

If E_λ is isogenous to E_0, JX_λ and F_{X_λ} are ρ-maximal. The set of $\lambda \in \mathbb{C}$ for which this happens is countably infinite.

(b) *Let $X \subset \mathbb{P}^4$ be the Klein cubic threefold $\sum_{i\in\mathbb{Z}/5} X_i^2 X_{i+1} = 0$. Then JX and F_X are ρ-maximal.*

Proof. — Part (a) is due to Roulleau [Rou11], who proves that JX_λ (for any λ) is isogenous to $E_0^3 \times E_\lambda^2$. Since the family $(E_\lambda)_{\lambda\in\mathbb{C}}$ is not constant, there is a countably infinite set of $\lambda \in \mathbb{C}$ for which E_λ is isogenous to E_0, hence JX_λ and therefore F_{X_λ} are ρ-maximal.

Part (b) follows from a result of Adler [Adl81], who proves that JX is isogenous (actually isomorphic) to E^5, where E is the elliptic curve whose endomorphism ring is the ring of integers of $\mathbb{Q}(\sqrt{-11})$ (see also [Rou09] for a precise description of the group $\mathrm{NS}(X)$). □

4. Products of curves

Proposition 5. — *Let C, C' be two smooth projective curves, of genus g and g' respectively. The following conditions are equivalent:*

(i) *The surface $C \times C'$ is ρ-maximal;*

(ii) *There exists an elliptic curve E with complex multiplication such that JC is isogenous to E^g and JC' to $E^{g'}$.*

Proof. — Let p, p' be the projections from $C \times C'$ to C and C'. We have

$$H^{1,1}(C\times C') = p^*H^2(C,\mathbb{C}) \oplus p'^*H^2(C',\mathbb{C}) \oplus \big(p^*H^{1,0}(C)\otimes p'^*H^{0,1}(C')\big) \oplus \big(p^*H^{0,1}(C)\otimes p'^*H^{1,0}(C')\big),$$

hence $h^{1,1}(C \times C') = 2gg' + 2$. On the other hand we have

$$\mathrm{NS}(C\times C') = p^*\,\mathrm{NS}(C) \oplus p'^*\,\mathrm{NS}(C') \oplus \mathrm{Hom}(JC, JC')$$

([LB92], Th. 11.5.1), hence $C \times C'$ is ρ-maximal if and only if $\mathrm{rk}\,\mathrm{Hom}(JC, JC') = 2gg'$. Thus the Proposition follows from the following (well-known) lemma:

Lemma 1. — *Let A and B be two abelian varieties, of dimension a and b respectively. The $\mathbb{Z}$-module $\mathrm{Hom}(A,B)$ has rank $\leqslant 2ab$; equality holds if and only if there exists an elliptic curve E with complex multiplication such that A is isogenous to E^a and B to E^b.*

Proof. — There exist simple abelian varieties $A_1, \ldots, A_s$, with distinct isogeny classes, and nonnegative integers $p_1, \ldots, p_s, q_1, \ldots, q_s$ such that A is isogenous to $A_1^{p_1} \times \cdots \times A_s^{p_s}$ and B to $A_1^{q_1} \times \cdots \times A_s^{q_s}$. Then

$$\operatorname{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q} \cong M_{p_1,q_1}(K_1) \times \cdots \times M_{p_s,q_s}(K_s),$$

where K_i is the (possibly skew) field $\operatorname{End}(A_i) \otimes_{\mathbb{Z}} \mathbb{Q}$. Put $a_i := \dim A_i$. Since K_i acts on $H^1(A_i, \mathbb{Q})$ we have $\dim_{\mathbb{Q}} K_i \leqslant b_1(A_i) = 2a_i$, hence

$$\operatorname{rk}\operatorname{Hom}(A, B) \leqslant \sum_i 2p_i q_i a_i \leqslant 2\Big(\sum p_i a_i\Big)\Big(\sum q_i a_i\Big) = 2ab.$$

The last inequality is strict unless $s = a_1 = 1$, in which case the first one is strict unless $\dim_{\mathbb{Q}} K_1 = 2$. The lemma, and therefore the Proposition, follow. □

The most interesting case occurs when $C = C'$. Then:

Proposition 6. — *Let C be a smooth projective curve. The following conditions are equivalent:*

(i) *The Jacobian JC is ρ-maximal;*
(ii) *The surface $C \times C$ is ρ-maximal;*
(iii) *The symmetric square* $\operatorname{Sym}^2 C$ *is ρ-maximal.*

Proof. — The equivalence of (i) and (ii) follows from Proposition 5. The Abel-Jacobi map $\operatorname{Sym}^2 C \to JC$ induces an isomorphism

$$H^{2,0}(JC) \cong \wedge^2 H^0(C, K_C) \xrightarrow{\sim} H^{2,0}(\operatorname{Sym}^2 C),$$

thus (i) and (iii) are equivalent by Proposition 2. □

When the equivalent conditions of Proposition 6 hold, we will say that C has *maximal correspondences* (the group $\operatorname{End}(JC)$ is often called the group of divisorial correspondences of C).

By Proposition 3 the Jacobian JC is then isomorphic to a product of isogenous elliptic curves with complex multiplication. Though we know very few examples of such curves, we will give below some examples with $g = 4$ or 10.

For $g = 2$ or 3, there is a countably infinite set of curves with maximal correspondences ([HN65], [Hof91]). The point is that any indecomposable principally polarized abelian variety of dimension 2 or 3 is a Jacobian; thus it suffices to construct an indecomposable principal polarization on E^g, where E is an elliptic curve with complex multiplication, and this is easily translated into a problem about hermitian forms of rank g on certain rings of quadratic integers.

This approach works only for $g = 2$ or 3; moreover it does not give an explicit description of the curves. Another method is by using automorphism groups, with the help of the following easy lemma:

Lemma 2. — *Let G be a finite group of automorphisms of C, and let $H^0(C,K_C)=\oplus_{i\in I}V_i$ be a decomposition of the G-module $H^0(C,K_C)$ into irreducible representations. Assume that there exists an elliptic curve E and for each $i\in I$, a nontrivial map $\pi_i : C\to E$ such that $\pi_i^*H^0(E,K_E)\subset V_i$. Then JC is isogenous to E^g.*

In particular if $H^0(C,K_C)$ is an irreducible G-module and C admits a map onto an elliptic curve E, then JC is isogenous to E^g.

Proof. — Let η be a generator of $H^0(E,K_E)$. Let $i\in I$; the forms $g^*\pi_i^*\eta$ for $g\in G$ generate V_i, hence there exists a subset A_i of G such that the forms $g^*\pi_i^*\eta$ for $g\in A_i$ form a basis of V_i.

Put $\Pi_i=(g\circ\pi_i)_{g\in A_i} : C\to E^{A_i}$, and $\Pi=(\Pi_i)_{i\in I} : C\to E^g$. By construction $\Pi^* : H^0(E^g,\Omega^1_{E^g})\to H^0(C,K_C)$ is an isomorphism. Therefore the map $JC\to E^g$ deduced from Π is an isogeny. □

In the examples which follow, and in the rest of the paper, we put $\omega:=e^{2\pi i/3}$.

Example 1. — We consider the family (C_t) of genus 2 curves given by $y^2=x^6+tx^3+1$, for $t\in\mathbb{C}\smallsetminus\{\pm 2\}$. It admits the automorphisms

$$\tau : (x,y)\longmapsto\Big(\frac{1}{x},\frac{y}{x^3}\Big)\quad\text{and}\quad\psi : (x,y)\longmapsto(\omega x,y).$$

The forms dx/y and xdx/y are eigenvectors for ψ and are exchanged (up to sign) by τ; it follows that the action of the group generated by ψ and τ on $H^0(C_t,K_{C_t})$ is irreducible.

Let E_t be the elliptic curve defined by $v^2=(u+2)(u^3-3u+t)$; the curve C_t maps onto E_t by

$$(x,y)\longmapsto\Big(x+\frac{1}{x},\frac{y(x+1)}{x^2}\Big).$$

By Lemma 2 JC_t is isogenous to E_t^2. Since the j-invariant of E_t is a non-constant function of t, there is a countably infinite set of $t\in\mathbb{C}$ for which E_t has complex multiplication, hence C_t has maximal correspondences.

Example 2. — Let C be the genus 2 curve $y^2=x(x^4-1)$; its automorphism group is a central extension of $\mathfrak{S}_4$ by the hyperelliptic involution σ ([LB92], 11.7); its action on $H^0(C,K_C)$ is irreducible.

Let E be the elliptic curve $E\colon v^2=u(u+1)(u-2\alpha)$, with $\alpha=1-\sqrt{2}$. The curve C maps to E by

$$(x,y)\longmapsto\Big(\frac{x^2+1}{x-1},\frac{y(x-\alpha)}{(x-1)^2}\Big).$$

The j-invariant of E is 8000, so E is the elliptic curve $\mathbb{C}/\mathbb{Z}[\sqrt{-2}]$ ([Sil94], Prop. 2.3.1).

Example 3 (The $\mathfrak{S}_4$-invariant quartic curves). — Consider the standard representation of $\mathfrak{S}_4$ on $\mathbb{C}^3$. It is convenient to view $\mathfrak{S}_4$ as the semi-direct product $(\mathbb{Z}/2)^2\rtimes\mathfrak{S}_3$, with $\mathfrak{S}_3$ (resp. $(\mathbb{Z}/2)^2$) acting on $\mathbb{C}^3$ by permutation (resp. change of sign) of the basis vectors. The quartic forms invariant under this representation form the pencil

$$(C_t)_{t\in\mathbb{P}^1}\colon x^4+y^4+z^4+t(x^2y^2+y^2z^2+z^2x^2)=0.$$

According to [DK93], this pencil was known to Ciani. It contains the Fermat quartic ($t=0$) and the Klein quartic ($t=\frac{3}{2}(1\pm i\sqrt{7})$).

Let us take $t\notin\{2,-1,-2,\infty\}$; then C_t is smooth. The action of $\mathfrak{S}_4$ on $H^0(C_t,K)$, given by the standard representation, is irreducible. Moreover the involution $x\mapsto -x$ has 4 fixed points, hence the quotient curve E_t has genus 1. It is given by the degree 4 equation

$$u^2+tu(y^2+z^2)+y^4+z^4+ty^2z^2=0$$

in the weighted projective space $\mathbb{P}(2,1,1)$. Thus E_t is a double covering of $\mathbb{P}^1$ branched along the zeroes of the polynomial $(t+2)(y^4+z^4)+2ty^2z^2$. The cross-ratio of these zeroes is $-(t+1)$, so E_t is the elliptic curve $y^2=x(x-1)(x+t+1)$. By Lemma 2 *JC_t is isogenous to E_t^3*. For a countably infinite set of t the curve E_t has complex multiplication, thus C_t has maximal correspondences. For $t=0$ we recover the well known fact that the Jacobian of the Fermat quartic curve is isogenous to $(\mathbb{C}/\mathbb{Z}[i])^3$.

Example 4. — Consider the genus 3 hyperelliptic curve $H\colon y^2=x(x^6+1)$. The space $H^0(H,K_H)$ is spanned by dx/y, xdx/y, x^2dx/y. This is a basis of eigenvectors for the automorphism $\tau:(x,y)\mapsto(\omega x,\omega^2y)$. On the other hand the involution $\sigma:(x,y)\mapsto(1/x,-y/x^4)$ exchanges dx/y and x^2dx/y, hence the summands of the decomposition

$$H^0(H,K_H)=\Big\langle\frac{dx}{y},x^2\frac{dx}{y}\Big\rangle\oplus\Big\langle x\frac{dx}{y}\Big\rangle$$

are irreducible under the group $\mathfrak{S}_3$ generated by σ and τ.

Let E_i be the elliptic curve $v^2=u^3+u$, with endomorphism ring $\mathbb{Z}[i]$. Consider the maps f and g from H to E_i given by

$$f(x,y)=(x^2,xy)\qquad g(x,y)=\Big(\lambda^2\big(x+\frac{1}{x}\big),\frac{\lambda^3y}{x^2}\Big)\quad\text{with }\lambda^{-4}=-3.$$

We have

$$f^*\frac{du}{v}=\frac{2xdx}{y}\quad\text{and}\quad g^*\frac{du}{v}=\lambda^{-1}(x^2-1)\frac{dx}{y}.$$

Thus we can apply Lemma 2, and we find that *JH is isogenous to E_i^3*.

Thus JH is isogenous to the Jacobian of the Fermat quartic F_4 (Example 3). In particular we see that the surface $H\times F_4$ is ρ-maximal.

We now arrive to our main example in higher genus. Recall that we put $\omega=e^{2\pi i/3}$.

Proposition 7. — *The Fermat sextic curve $C_6\colon X^6+Y^6+Z^6=0$ has maximal correspondences. Its Jacobian JC_6 is isogenous to E_ω^{10}, where E_ω is the elliptic curve $\mathbb{C}/\mathbb{Z}[\omega]$.*

The first part can be deduced from the general recipe given by Shioda to compute the Picard number of $C_d \times C_d$ for any d [Shi81]. Let us give an elementary proof. Let $G := T \rtimes \mathfrak{S}_3$, where $\mathfrak{S}_3$ acts on $\mathbb{C}^3$ by permutation of the coordinates and T is the group of diagonal matrices t with $t^6 = 1$.

Let $$\Omega = \frac{XdY - YdX}{Z^5} = \frac{YdZ - ZdY}{X^5} = \frac{ZdX - XdZ}{Y^5} \in H^0(C, K_C(-3)).$$

A basis of eigenvectors for the action of T on $H^0(C_6, K)$ is given by the forms $X^aY^bZ^c\,\Omega$, with $a+b+c=3$; using the action of $\mathfrak{S}_3$ we get a decomposition into irreducible components:

$$H^0(C_6, K) = V_{3,0,0} \oplus V_{2,1,0} \oplus V_{1,1,1},$$

where $V_{\alpha,\beta,\gamma}$ is spanned by the forms $X^aY^bZ^c\,\Omega$ with $\{a,b,c\} = \{\alpha,\beta,\gamma\}$.

Let us use affine coordinates $x = X/Z$, $y = Y/Z$ on C_6. We consider the following maps from C_6 onto $E_\omega\colon v^2 = u^3 - 1$:

$$f(x,y) = (-x^2, y^3), \qquad g(x,y) = \Big(2^{-2/3}x^{-2}y^4,\ \frac{1}{2}(x^3 - x^{-3})\Big)\,;$$

and, using for E_ω the equation $\xi^3 + \eta^3 + 1 = 0$, $h(x,y) = (x^2, y^2)$.

We have

$$\begin{aligned} f^*\frac{du}{v} &= -\frac{2xdx}{y^3} = -2XY^2\,\Omega \in V_{2,1,0},\\ g^*\frac{du}{v} &= -2^{4/3}Y^3\,\Omega \in V_{3,0,0},\\ h^*\frac{d\xi}{\eta^2} &= 2XYZ\,\Omega \in V_{1,1,1}, \end{aligned}$$

so the Proposition follows from Lemma 2. □

By Proposition 2 every quotient of C_6 has again maximal correspondences. There are four such quotient which have genus 4:

• The quotient by an involution $\alpha \in T$, which we may take to be $\alpha : (X,Y,Z) \mapsto (X,Y,-Z)$. The canonical model of C_6/α is the image of C_6 by the map

$$(X,Y,Z) \longmapsto (X^2, XY, Y^2, Z^2)\,;$$

its equations in $\mathbb{P}^3$ are $xz - y^2 = x^3 + z^3 + t^3 = 0$. Projecting onto the conic $xz - y^2 = 0$ realizes C_6/α as the cyclic triple covering $v^3 = u^6 + 1$ of $\mathbb{P}^1$.

• The quotient by an involution $\beta \in \mathfrak{S}_3$, say $\beta : (X,Y,Z) \mapsto (Y,X,Z)$. The canonical model of C_6/β is the image of C_6 by the map

$$(X,Y,Z) \longmapsto ((X+Y)^2, Z(X+Y), Z^2, XY)\,;$$

its equations are $xz - y^2 = x(x-3t)^2 + z^3 - 2t^3 = 0$.

Since the quadric containing their canonical model is singular, the two genus 4 curves C_6/α and C_6/β have a unique g^1_3. The associated triple covering $C_6/\alpha \to \mathbb{P}^1$ is cyclic, while the corresponding covering $C_6/\beta \to \mathbb{P}^1$ is not. Therefore the two curves are not isomorphic.

• The quotient by an element of order 3 of T acting freely, say $\gamma : (X, Y, Z) \mapsto (X, \omega Y, \omega^2 Z)$. The canonical model of C_6/γ is the image of C_6 by the map

$$(X, Y, Z) \mapsto (X^3, Y^3, Z^3, XYZ)\,;$$

its equations are $x^2+y^2+z^2 = t^3-xyz = 0$. Projecting onto the conic $x^2+y^2+z^2 = 0$ realizes C_6/γ as the cyclic triple covering $v^3 = u(u^4 - 1)$ of $\mathbb{P}^1$; thus C_6/γ is not isomorphic to C_6/α or C_6/β.

• The quotient by an element of order 3 of $\mathfrak{S}_3$ acting freely, say $\delta : (X, Y, Z) \mapsto (Y, Z, X)$. The canonical model of C_6/δ is the image of C_6 by the map

$$(X, Y, Z) \mapsto (X^3 + Y^3 + Z^3, XYZ, X^2Y + Y^2Z + Z^2X, XY^2 + YZ^2 + ZX^2).$$

It is contained in the smooth quadric $(x+y)^2+5y^2-2zt = 0$, so C_6/δ is not isomorphic to any of the 3 previous curves.

Thus we have found four non-isomorphic curves of genus 4 with Jacobian isogenous to E_ω^4. The product of any two of these curves is a ρ-maximal surface.

Corollary 1. — *The Fermat sextic surface S_6: $X^6+Y^6+Z^6+T^6 = 0$ is ρ-maximal.*

Proof. — This follows from Propositions 7, 2 and Shioda's trick: there exists a rational dominant map $\pi : C_6 \times C_6 \dashrightarrow S_6$, given by

$$\pi\big((X, Y, Z), (X', Y', Z')\big) = (XZ', YZ', iX'Z, iY'Z). \qquad \square$$

Remark 4. — Since the Fermat plane quartic has maximal correspondences (Example 2), the same argument gives the classical fact that the Fermat quartic surface is ρ-maximal. It follows from the explicit formula for $\rho(S_d)$ given in [Aok83] that S_d is ρ-maximal (for $d \geqslant 4$) only for $d = 4$ and 6.

Again every quotient of the Fermat sextic is ρ-maximal. For instance, the quotient of S_6 by the automorphism $(X, Y, Z, T) \mapsto (X, Y, Z, \omega T)$ is the double covering of $\mathbb{P}^2$ branched along C_6: it is a ρ-maximal K3 surface. The quotient of S_6 by the involution $(X, Y, Z, T) \mapsto (X, Y, -Z, -T)$ is given in $\mathbb{P}^5$ by the equations

$$y^2 - xz = v^2 - uw = x^3 + z^3 + u^3 + w^3 = 0\,;$$

it is a complete intersection of degrees $(2, 2, 3)$, with 12 ordinary nodes. Other quotients have p_g equal to 2, 3, 4 or 6.

5. Quotients of self-products of curves

The method of the previous section may sometimes allow to prove that certain quotients of a product $C \times C$ have maximal Picard number. Since we have very few examples we will refrain from giving a general statement and contend ourselves with one significant example.

Let C be the curve in $\mathbb{P}^4$ defined by

$$u^2 = xy, \quad v^2 = x^2 - y^2, \quad w^2 = x^2 + y^2.$$

It is isomorphic to the modular curve $X(8)$ [FSM13]. Let $\Gamma\subset \mathrm{PGL}(5,\mathbb{C})$ be the subgroup of diagonal elements changing an even number of signs of u,v,w; Γ is isomorphic to $(\mathbb{Z}/2)^2$ and acts freely on C.

Proposition 8

(a) *JC is isogenous to $E_i^3\times E_{\sqrt{-2}}^2$, where $E_\alpha=\mathbb{C}/\mathbb{Z}[\alpha]$ for $\alpha=i$ or $\sqrt{-2}$.*
(b) *The surface $(C\times C)/\Gamma$ is ρ-maximal.*

Proof

(a) The form $\Omega:=(xdy-ydx)/uvw$ generates $H^0(C,K_C(-1))$, and is Γ-invariant; thus multiplication by Ω induces a Γ-equivariant isomorphism

$$H^0(\mathbb{P}^4,\mathcal{O}_{\mathbb{P}^4}(1))\xrightarrow{\sim} H^0(C,K_C).$$

Let V and L be the subspaces of $H^0(C,K_C)$ corresponding to $\langle u,v,w\rangle$ and $\langle x,y\rangle$. The projection $(u,v,w,x,y)\mapsto(u,v,w)$ maps C onto the quartic curve $F\colon 4u^4+v^4-w^4=0$; the induced map $f:C\to F$ identifies F with the quotient of C by the involution $(u,v,w,x,y)\mapsto(u,v,w,-x,-y)$, and we have $f^*H^0(F,K_F)=V$.

The quotient curve $H:=C/\Gamma$ is the genus 2 curve $z^2=t(t^4-1)$ [Bea13]. The pull-back of $H^0(H,K_H)$ is the subspace invariant under Γ, that is L. Thus JC is isogenous to $JF\times JH$. From examples 1 and 2 of §4 we conclude that JC is isogenous to $E_i^3\times E_{\sqrt{-2}}^2$.

(b) We have Γ-equivariant isomorphisms

$$\begin{aligned}H^{1,1}(C\times C)&=H^2(C,\mathbb{C})\oplus H^2(C,\mathbb{C})\oplus(H^{1,0}\boxtimes H^{0,1})\oplus(H^{0,1}\boxtimes H^{1,0})\\&=\mathbb{C}^2\oplus \mathrm{End}(H^0(C,K_C))^{\oplus 2}\end{aligned}$$

(where Γ acts trivially on $\mathbb{C}^2$), hence

$$H^{1,1}((C\times C)/\Gamma)=\mathbb{C}^2\oplus\mathrm{End}_\Gamma(H^0(C,K_C))^{\oplus 2}.$$

As a Γ-module we have $H^0(C,K_C)=L\oplus V$, where Γ acts trivially on L and V is the sum of the 3 nontrivial one-dimensional representations of Γ. Thus

$$\mathrm{End}_\Gamma(H^0(C,K_C))=\mathbb{M}_2(\mathbb{C})\times\mathbb{C}^3.$$

Similarly we have $\mathrm{NS}((C\times C)/\Gamma)\otimes\mathbb{Q}=\mathbb{Q}^2\oplus(\mathrm{End}_\Gamma(JC)\otimes\mathbb{Q})$ and

$$\mathrm{End}_\Gamma(JC)\otimes\mathbb{Q}=(\mathrm{End}(JH)\otimes\mathbb{Q})\times(\mathrm{End}_\Gamma(JF)\otimes\mathbb{Q})^3=\mathbb{M}_2(\mathbb{Q}(\sqrt{-2}))\times\mathbb{Q}(i)^3,$$

hence the result. □

Corollary 2 ([ST10]). — *Let $\Sigma\subset\mathbb{P}^6$ be the* surface of cuboids*, defined by*

$$t^2=x^2+y^2+z^2,\quad u^2=y^2+z^2,\quad v^2=x^2+z^2,\quad w^2=x^2+y^2.$$

Σ has 48 ordinary nodes; its minimal desingularization S is ρ-maximal.

Indeed Σ is a quotient of $(C\times C)/\Gamma$ [Bea13]. □

(The result has been obtained first in [ST10] with a very different method.)

6. Other examples

6.1. Elliptic modular surfaces. — Let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ such that $-I \notin \Gamma$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the Poincaré upper half-plane $\mathbb{H}$; let Δ_Γ be the compactification of the Riemann surface $\mathbb{H}/\Gamma$. The universal elliptic curve over $\mathbb{H}$ descends to $\mathbb{H}/\Gamma$, and extends to a smooth projective surface B_Γ over Δ_Γ, the *elliptic modular surface* attached to Γ. In [Shi69] Shioda proves that B_Γ *is ρ-maximal.*(1)

Now take $\Gamma = \Gamma(5)$, the kernel of the reduction map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/5)$. In [Liv81] Livne constructed a $\mathbb{Z}/5$-covering $X \to B_{\Gamma(5)}$, branched along the sum of the 25 5-torsion sections of $B_{\Gamma(5)}$. The surface X satisfies $c_1^2 = 3c_2\,(= 225)$, hence it is a ball quotient and therefore rigid. By analyzing the action of $\mathbb{Z}/5$ on $H^{1,1}(X)$ Livne shows that $H^{1,1}(X)$ is not defined over $\mathbb{Q}$, hence X *is not ρ-maximal.* This seems to be the only known example of a surface which cannot be deformed to a ρ-maximal surface.

6.2. Surfaces with $p_g = K^2 = 1$. — The minimal surfaces with $p_g = K^2 = 1$ have been studied by Catanese [Cat79] and Todorov [Tod80]. Their canonical model is a complete intersection of type $(6,6)$ in the weighted projective space $\mathbb{P}(1,2,2,3,3)$. The moduli space $\mathcal{M}$ is smooth of dimension 18.

Proposition 9. — *The ρ-maximal surfaces are dense in $\mathcal{M}$.*

Proof. — We can replace $\mathcal{M}$ by the Zariski open subset $\mathcal{M}_a$ parametrizing surfaces with ample canonical bundle. Let $S \in \mathcal{M}_a$, and let $f : \mathcal{S} \to (B, o)$ be a local versal deformation of S, so that $S \cong \mathcal{S}_o$. Let L be the lattice $H^2(S,\mathbb{Z})$, and $k \in L$ the class of K_S. We may assume that B is simply connected and fix an isomorphism of local systems $R^2 f_*(\mathbb{Z}) \xrightarrow{\sim} L_B$, compatible with the cup-product and mapping the canonical class $[K_{\mathcal{S}/B}]$ onto k. This induces for each $b \in B$ an isometry $\varphi_b : H^2(\mathcal{S}_b, \mathbb{C}) \xrightarrow{\sim} L_\mathbb{C}$, which maps $H^{2,0}(\mathcal{S}_b)$ onto a line in $L_\mathbb{C}$; the corresponding point $\wp(b)$ of $\mathbb{P}(L_\mathbb{C})$ is the period of $\mathcal{S}_b$. It belongs to the complex manifold

$$\Omega := \big\{[x] \in \mathbb{P}(L_\mathbb{C}) \mid x^2 = 0,\ x\cdot k = 0,\ x\cdot \overline{x} > 0\big\}.$$

Associating to $x \in \Omega$ the real 2-plane $P_x := \langle \mathrm{Re}(x), \mathrm{Im}(x)\rangle \subset L_\mathbb{R}$ defines an isomorphism of Ω onto the Grassmannian of positive oriented 2-planes in $L_\mathbb{R}$.

The key point is that the image of the *period map* $\wp : B \to \Omega$ is open [Cat79]. Thus we can find b arbitrarily close to o such that the 2-plane P_b is defined over $\mathbb{Q}$, hence $H^{2,0}(\mathcal{S}_b) \oplus H^{0,2}(\mathcal{S}_b) = P_b \otimes_\mathbb{R} \mathbb{C}$ is defined over $\mathbb{Q}$. □

Remark 5. — The proof applies to all surfaces with $p_g = 1$ for which the image of the period map is open (for instance to K3 surfaces); unfortunately this seems to be a rather exceptional situation.

(1) I am indebted to I. Dolgachev and B. Totaro for pointing out this reference.

6.3. Todorov surfaces. — In [Tod81] Todorov constructed a series of regular surfaces with $p_g = 1$, $2 \leqslant K^2 \leqslant 8$, which provide counter-examples to the Torelli theorem. The construction is as follows: let $K \subset \mathbb{P}^3$ be a Kummer surface. We choose k double points of K in general position (this can be done with $0 \leqslant k \leqslant 6$), and a general quadric $Q \subset \mathbb{P}^3$ passing through these k points. The *Todorov surface* S is the double covering of K branched along $K \cap Q$ and the remaining $16 - k$ double points. It is a minimal surface of general type with $p_g = 1$, $K^2 = 8 - k$, $q = 0$. If moreover we choose K ρ-maximal (that is, $K = E^2/\{\pm 1\}$, where E is an elliptic curve with complex multiplication), then S is ρ-maximal by Proposition 2(b).

Note that by varying the quadric Q we get a continuous, non-constant family of ρ-maximal surfaces.

6.4. Double covers. — In [Per82] Persson constructs ρ-maximal double covers of certain rational surfaces by allowing the branch curve to acquire some simple singularities (see also [BE87]). He applies this method to find ρ-maximal surfaces in the following families:

- Horikawa surfaces, that is, surfaces on the "Noether line" $K^2 = 2p_g - 4$, for $p_g \not\equiv -1 \pmod{6}$;
- Regular elliptic surfaces;
- Double coverings of $\mathbb{P}^2$.

In the latter case the double plane admits (many) rational singularities; it is unknown whether there exists a ρ-maximal surface S which is a double covering of $\mathbb{P}^2$ branched along a smooth curve of even degree $\geqslant 8$.

6.5. Hypersurfaces and complete intersections. — Probably the most natural families to look at are smooth surfaces in $\mathbb{P}^3$, or more generally complete intersections. Here we may ask for a smooth surface S, or for the minimal resolution of a surface with rational double points (or even any surface deformation equivalent to a complete intersection of given type). Here are the examples that we know of:

- The quintic surface $x^3yz + y^3zt + z^3tx + t^3xy = 0$ has four A_9 singularities; its minimal resolution is ρ-maximal [Sch11]. It is not yet known whether there exists a smooth ρ-maximal quintic surface.
- The Fermat sextic is ρ-maximal (§4, Corollary 1).
- The complete intersection $y^2 - xz = v^2 - uw = x^3 + z^3 + u^3 + w^3 = 0$ of type $(2, 2, 3)$ in $\mathbb{P}^5$ has 12 nodes; its minimal desingularization is ρ-maximal (end of §4).
- The surface of cuboids is a complete intersection of type $(2, 2, 2, 2)$ in $\mathbb{P}^6$ with 48 nodes; its minimal desingularization is ρ-maximal (§5, Corollary 2).

7. The complex torus associated to a ρ-maximal variety

For a ρ-maximal variety X, let T_X be the $\mathbb{Z}$-module $H^2(X, \mathbb{Z})/\operatorname{NS}(X)$. We have a decomposition

$$T_X \otimes \mathbb{C} = H^{2,0} \oplus H^{0,2}$$

defining a weight 1 Hodge structure on T_X, hence a complex torus $\mathcal{T} := H^{0,2}/p_2(T_X)$, where $p_2 : T_X \otimes \mathbb{C} \to H^{2,0}$ is the second projection. Via the isomorphism $H^{0,2} = H^2(X, \mathcal{O}_X)$, $\mathcal{T}_X$ is identified with the cokernel of the natural map $H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$.

The exponential exact sequence gives rise to an exact sequence

$$0 \longrightarrow \mathrm{NS}(X) \longrightarrow H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}_X) \longrightarrow H^2(X,\mathcal{O}_X^*) \xrightarrow{\partial} H^3(X,\mathbb{Z}),$$

hence to a short exact sequence

$$0 \longrightarrow \mathcal{T}_X \longrightarrow H^2(X,\mathcal{O}_X^*) \xrightarrow{\partial} H^3(X,\mathbb{Z}),$$

so that $\mathcal{T}_X$ appears as the "continuous part" of the group $H^2(X,\mathcal{O}_X^*)$.

Example 5. — Consider the elliptic modular surface B_Γ of Section 6.1. The space $H^0(B_\Gamma, K_{B_\Gamma})$ can be identified with the space of cusp forms of weight 3 for Γ; then the torus $\mathcal{T}_{B_\Gamma}$ is the complex torus associated to this space by Shimura (see [Shi69]).

Example 6. — Let $X = C \times C'$, with JC isogenous to E^g and JC' to $E^{g'}$ (Proposition 5). The torus $\mathcal{T}_X$ is the cokernel of the map

$$i \otimes i' : H^1(C,\mathbb{Z}) \otimes H^1(C',\mathbb{Z}) \longrightarrow H^1(C,\mathcal{O}_C) \otimes H^1(C',\mathcal{O}_{C'}),$$

where i and i' are the embeddings

$$H^1(C,\mathbb{Z}) \hookrightarrow H^1(C,\mathcal{O}_C) \quad \text{and} \quad H^1(C',\mathbb{Z}) \hookrightarrow H^1(C',\mathcal{O}_{C'}).$$

We want to compute $\mathcal{T}_X$ up to isogeny, so we may replace the left hand side by a finite index sublattice. Thus, writing $E = \mathbb{C}/\Gamma$, we may identify i with the diagonal embedding $\Gamma^g \hookrightarrow \mathbb{C}^g$, and similarly for i'; therefore $i \otimes i'$ is the diagonal embedding of $(\Gamma \otimes \Gamma)^{gg'}$ in $\mathbb{C}^{gg'}$. Put $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$; the image Γ' of $\Gamma \otimes \Gamma$ in $\mathbb{C}$ is spanned by $1, \tau, \tau^2$; since E has complex multiplication, τ is a quadratic number, hence Γ has finite index in Γ'. Finally we obtain that *$\mathcal{T}_X$ is isogenous to $E^{gg'}$*.

For the surface $X = (C \times C)/\Gamma$ studied in §5 an analogous argument shows that $\mathcal{T}_X$ is isogenous to $A = E_i^4 \times E_{\sqrt{-2}}^3$. This is still an abelian variety of type CM, in the sense that $\mathrm{End}(A) \otimes \mathbb{Q}$ contains an étale $\mathbb{Q}$-algebra of maximal dimension $2\dim(A)$. There seems to be no reason why this should hold in general. However it is true in the special case $h^{2,0} = 1$ (e.g. for holomorphic symplectic manifolds):

Proposition 10. — *If $h^{2,0}(X) = 1$, the torus $\mathcal{T}_X$ is an elliptic curve with complex multiplication.*

Proof. — Let T'_X be the pull back of $H^{2,0} + H^{0,2}$ in $H^2(X,\mathbb{Z})$; then $p_2(T'_X)$ is a sublattice of finite index in $p_2(T_X)$. Choosing an ample class $h \in H^2(X,\mathbb{Z})$ defines a quadratic form on $H^2(X,\mathbb{Z})$ which is positive definite on T'_X. Replacing again T'_X by a finite index sublattice we may assume that it admits an orthogonal basis (e, f) with $e^2 = a$, $f^2 = b$. Then $H^{2,0}$ and $H^{0,2}$ are the two isotropic lines of $T'_X \otimes \mathbb{C}$; they are spanned by the vectors $\omega = e + \tau f$ and $\overline{\omega} = e - \tau f$, with $\tau^2 = -a/b$. We have $e = \frac{1}{2}(\omega + \overline{\omega})$ and $f = \frac{1}{2\tau}(\omega - \overline{\omega})$; therefore multiplication by $\frac{1}{2\tau}\overline{\omega}$ induces an isomorphism of $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ onto $H^{0,2}/p_2(T'_X)$, hence $\mathcal{T}_X$ is isogenous to $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ and

$$\operatorname{End}(\mathcal{T}_X)\otimes\mathbb{Q}=\mathbb{Q}(\tau)=\mathbb{Q}\Big(\sqrt{-\operatorname{disc}(T'_X)}\Big). \qquad \square$$

8. Higher codimension cycles

A natural generalization of the question considered here is to look for varieties X for which the group $H^{2p}(X,\mathbb{Z})_{\mathrm{alg}}$ of algebraic classes in $H^{2p}(X,\mathbb{Z})$ has maximal rank $h^{p,p}$. Very few nontrivial cases seem to be known. The following is essentially due to Shioda:

Proposition 11. — *Let F^n_d be the Fermat hypersurface of degree d and even dimension $n=2\nu$. For $d=3,4$, the group $H^n(F^n_d,\mathbb{Z})_{\mathrm{alg}}$ has maximal rank $h^{\nu,\nu}$.*

Proof. — According to [Shi79] we have

$$\operatorname{rk} H^n(F^n_3,\mathbb{Z})_{\mathrm{alg}}=1+\frac{n!}{(\nu)!^2}\quad\text{and}\quad \operatorname{rk} H^n(F^n_4,\mathbb{Z})_{\mathrm{alg}}=\sum_{k=0}^{k=\nu+1}\frac{(n+2)!}{(k!)^2(n+2-2k)!}\,\cdot$$

On the other hand, let $R^n_d:=\mathbb{C}[X_0,\dots,X_{n+1}]/(X_0^{d-1},\dots,X_{n+1}^{d-1})$ be the Jacobian ring of F^n_d; Griffiths theory [Gri69] provides an isomorphism of the primitive cohomology $H^{\nu,\nu}(F^n_d)_{\mathrm{o}}$ with the component of degree $(\nu+1)(d-2)$ of R^n_d. Since this ring is the tensor product of $(n+2)$ copies of $\mathbb{C}[T]/(T^{d-1})$, its Poincaré series $\sum_k \dim(R^n_d)_k T^k$ is $(1+T+\cdots+T^{d-2})^{n+2}$. Then an elementary computation gives the result. $\square$

In the particular case of cubic fourfolds we have more examples:

Proposition 12. — *Let F be a cubic form in 3 variables, such that the curve $F(x,y,z)=0$ in $\mathbb{P}^2$ is an elliptic curve with complex multiplication; let X be the cubic fourfold defined by $F(x,y,z)+F(u,v,w)=0$ in $\mathbb{P}^5$. The group $H^4(X,\mathbb{Z})_{\mathrm{alg}}$ has maximal rank $h^{2,2}(X)$.*

Proof. — Let u be the automorphism of X defined by

$$u(x,y,z;u,v,w)=(x,y,z;\omega u,\omega v,\omega w).$$

We observe that u acts trivially on the (one-dimensional) space $H^{3,1}(X)$. Indeed Griffiths theory [Gri69] provides a canonical isomorphism

$$\operatorname{Res}: H^0(\mathbb{P}^5,K_{\mathbb{P}^5}(2X))\xrightarrow{\ \sim\ } H^{3,1}(X)\,;$$

the space $H^0(\mathbb{P}^5,K_{\mathbb{P}^5}(2X))$ is generated by the meromorphic form Ω/G^2, with

$$\Omega = x dy\wedge dz\wedge du\wedge dv\wedge dw - y dx\wedge dz\wedge du\wedge dv\wedge dw+\cdots,$$
$$G=F(x,y,z)+F(u,v,w).$$

The automorphism u acts trivially on this form, and therefore on $H^{3,1}(X)$.

Let F be the variety of lines contained in X. We recall from [BD85] that F is a holomorphic symplectic fourfold, and that there is a natural isomorphism of Hodge structures $\alpha: H^4(X,\mathbb{Z})\xrightarrow{\ \sim\ } H^2(F,\mathbb{Z})$. Therefore the automorphism u_F of F induced by u is symplectic. Let us describe its fixed locus.

The fixed locus of u in X is the union of the plane cubics E given by $x=y=z=0$ and E' given by $u=v=w=0$. A line in X preserved by u must have (at least) two fixed points, hence must meet both E and E'; conversely, any line joining a point of E to a point of E' is contained in X, and preserved by u. This identifies the fixed locus A of u_F to the abelian surface $E\times E'$. Since u_F is symplectic A is a symplectic submanifold, that is, the restriction map $H^{2,0}(F)\to H^{2,0}(A)$ is an isomorphism. By our hypothesis A is ρ-maximal, so F is ρ-maximal by Proposition 2. Since α maps $H^4(X,\mathbb{Z})_{\rm alg}$ onto $\mathrm{NS}(F)$ this implies the Proposition. □

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