



Algebraic cycles on Jacobian varieties

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ABSTRACT

Let J be the Jacobian of a smooth curve C of genus g , and let $A(J)$ be the ring of algebraic cycles modulo algebraic equivalence on J , tensored with \mathbb{Q} . We study in this paper the smallest \mathbb{Q} -vector subspace R of $A(J)$ which contains C and is stable under the natural operations of $A(J)$: intersection and Pontryagin products, pull back and push down under multiplication by integers. We prove that this ‘tautological subring’ is generated (over \mathbb{Q}) by the classes of the subvarieties $W_1 = C$, $W_2 = C + C, \dots, W_{g-1}$. If C admits a morphism of degree d onto \mathbb{P}^1 , we prove that the last $d - 1$ classes suffice.

1. Introduction

Let C be a compact Riemann surface of genus g . Its Jacobian variety J carries a number of natural subvarieties, defined up to translation: first of all the curve C embeds into J , then we can use the group law of J to form $W_2 = C + C$, $W_3 = C + C + C, \dots$ until W_{g-1} which is a theta divisor on J . Then we can intersect these subvarieties, add again, pull back or push down under multiplication by integers, and so on. Thus we get a rather large number of algebraic subvarieties which live naturally in J .

If we look at the classes obtained in this way in rational cohomology, the result is disappointing. We just find the subalgebra of $H^*(J, \mathbb{Q})$ generated by the class θ of the theta divisor. In fact, the polynomials in θ are the only algebraic cohomology classes which live on a generic Jacobian. The situation becomes more interesting if we look at the \mathbb{Q} -algebra $A(J)$ of algebraic cycles modulo algebraic equivalence on J ; here a result of Ceresa [Cer83] implies that, for a generic curve C , the class of W_{g-p} in $A^p(J)$ is *not* proportional to θ^p for $2 \leq p \leq g - 1$. This leads naturally to investigate the ‘tautological subring’ of $A(J)$, that is, the smallest \mathbb{Q} -vector subspace R of $A(J)$ which contains C and is stable under the natural operations of $A(J)$: intersection and Pontryagin products (see start of § 2), pull back and push down under multiplication by integers. Our main result states that this space is not too complicated. Let $w^p \in A^p(J)$ be the class of W_{g-p} . Then we can state the following theorem.

THEOREM.

- a) R is the sub- \mathbb{Q} -algebra of $A(J)$ generated by w^1, \dots, w^{g-1} .
- b) If C admits a morphism of degree d onto \mathbb{P}^1 , R is generated by w^1, \dots, w^{d-1} .

In particular we see that R is finite-dimensional, a fact which does not seem to be *a priori* obvious (the space $A(J)$ is known to be infinite-dimensional for C generic of genus 3, see [Nor89]).

The proof rests in an essential way on the properties of the Fourier transform, a \mathbb{Q} -linear automorphism of $A(J)$ with remarkable properties. We recall these properties in § 1; in § 2 we look at the case of Jacobian varieties, computing in particular the Fourier transform of the class of C

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in $A(J)$. This is the main ingredient in the proof of part a of the Theorem, which we give in § 3. Part b turns out to be an easy consequence of a result of Colombo and van Geemen [CG93]; this is explained in § 4, together with a few examples.

2. Algebraic cycles on abelian varieties

2.1 Let X be an abelian variety over \mathbf{C} . We will denote by p and q the two projections of $X \times X$ onto X , and by $m : X \times X \rightarrow X$ the addition map.

Let $A(X)$ be the group of algebraic cycles on X modulo algebraic equivalence, tensored with \mathbb{Q} . It is a \mathbb{Q} -vector space, graded by the codimension of the cycle classes. It carries two natural multiplication laws $A(J) \otimes_{\mathbb{Q}} A(J) \rightarrow A(J)$, which are associative and commutative: the intersection product, which is homogeneous with respect to the graduation, and the Pontryagin product, defined by

$$x * y := m_*(p^*x \cdot q^*y),$$

which is homogeneous of degree $-g$. If Y and Z are subvarieties of X , the cycle class $[Y] * [Z]$ is equal to $(\deg \mu)[Y + Z]$ if the addition map $\mu : Y \times Z \rightarrow Y + Z$ is generically finite, and is zero otherwise.

2.2 For $k \in \mathbb{Z}$, we will still denote by k the endomorphism $x \mapsto kx$ of X . According to [Bea86], there is a second graduation on $A(X)$, leading to a bigraduation

$$A(X) = \bigoplus_{s,p} A^p(X)_{(s)}$$

such that

$$k^*x = k^{2p-s}x, \quad k_*x = k^{2g-2p+s}x \quad \text{for } x \in A^p(X)_{(s)}.$$

Both products are homogeneous with respect to the second graduation. We have $A^p(X)_{(s)} = 0$ for $s < p - g$ or $s \geq p$ (use [Bea86, Proposition 4]). It is conjectured that negative degrees actually do not occur; this will not concern us here, as we will only consider cycles in $A(X)_{(s)}$ for $s \geq 0$.

2.3 A crucial tool in what follows will be the Fourier transform for algebraic cycles, defined in [Bea83]. Let us recall briefly the results we will need, the proofs can be found in [Bea83] and [Bea86]. We will concentrate on the case of a *principally polarized* abelian variety (X, θ) , and use the polarization to identify X with its dual abelian variety.

Let $\ell := p^*\theta + q^*\theta - m^*\theta \in A^1(X \times X)$; this is the class of the *Poincaré line bundle* \mathcal{L} on $X \times X$. The Fourier transform $\mathcal{F} : A(X) \rightarrow A(X)$ is defined by $\mathcal{F}x = q_*(p^*x \cdot e^\ell)$. It satisfies the following properties:

- i) $\mathcal{F} \circ \mathcal{F} = (-1)^g(-1)^*$;
- ii) $\mathcal{F}(x * y) = \mathcal{F}x \cdot \mathcal{F}y$ and $\mathcal{F}(x \cdot y) = (-1)^g \mathcal{F}x * \mathcal{F}y$;
- iii) $\mathcal{F}A^p(X)_{(s)} = A^{g-p+s}(X)_{(s)}$;
- iv) let $x \in A(X)$; put $\bar{x} = (-1)^*x$. Then $\mathcal{F}x = e^\theta((\bar{x}e^\theta) * e^{-\theta})$.

Let us prove property iv, which is not explicitly stated in [Bea83] or [Bea86]. Replacing ℓ by its definition, we get $\mathcal{F}x = e^\theta q_*(p^*(xe^\theta) \cdot e^{-m^*\theta})$. Let ω be the automorphism of $A \times A$ defined by $\omega(a, b) = (-a, a + b)$. We have $p \circ \omega = -p$, $q \circ \omega = m$, and $m \circ \omega = q$. Hence

$$\mathcal{F}x = e^\theta q_* \omega_* \omega^*(p^*(xe^\theta) \cdot e^{-m^*\theta}) = e^\theta m_*(p^*(\bar{x}e^\theta) \cdot q^*e^{-\theta}) = e^\theta((\bar{x}e^\theta) * e^{-\theta}),$$

and thus property iv is proved.

3. The Fourier transform on a Jacobian

3.1 From now on we take for our abelian variety the Jacobian (J, θ) of a smooth projective curve C of genus g . We choose a base point $o \in C$, which allows us to define an embedding $\varphi : C \hookrightarrow J$ by $\varphi(p) = \mathcal{O}_C(p - o)$. Since we are working modulo algebraic equivalence, all our constructions will be independent of the choice of the base point.

We will denote simply by C the class of $\varphi(C)$ in $A^{g-1}(J)$. For $0 \leq d \leq g$, we put $w^{g-d} := (1/d!)C^{*d} \in A^{g-d}(J)$; it is the class of the subvariety W_d of J parameterizing line bundles of the form $\mathcal{O}_C(E_d - do)$, where E_d is an effective divisor of degree d . We have $w^1 = \theta$ by the Riemann theorem, $w^{g-1} = C$, and w^g is the class of a point. We define the *Newton polynomials* in the classes w^i by

$$N^k(w) = \frac{1}{k!} \sum_{i=1}^g \lambda_i^k$$

in the ring obtained by adjoining to $A(J)$ the roots $\lambda_1, \dots, \lambda_g$ of the equation $\lambda^g - \lambda^{g-1}w^1 + \dots + (-1)^g w^g = 0$. We have $N^k(w) \in A^k(J)$; for instance

$$N^1(w) = \theta, \quad N^2(w) = \frac{1}{2}\theta^2 - w^2, \quad N^3(w) = \frac{1}{6}\theta^3 - \frac{1}{2}\theta \cdot w^2 - \frac{1}{2}w^3, \dots$$

3.2 The class $N^k(w)$ is a polynomial in w^1, \dots, w^k ; conversely, w^k is a polynomial in $N^1(w), \dots, N^k(w)$.

PROPOSITION 3.3. We have $-\mathcal{F}C = N^1(w) + N^2(w) + \dots + N^{g-1}(w)$.

Proof. We use the notation of § 2, and denote moreover by \bar{p}, \bar{q} the projections of $C \times J$ onto C and J . Consider the cartesian diagram

$$\begin{array}{ccc} C \times J & \xrightarrow{\Phi} & J \times J \\ \bar{p} \downarrow & & \downarrow p \\ C & \xrightarrow{\varphi} & J \end{array}$$

with $\Phi = (\varphi, 1_J)$. Put $\bar{\ell} := \Phi^*\ell$. We have $p^*C \cdot e^\ell = \Phi_*1 \cdot e^\ell = \Phi_*e^{\bar{\ell}}$, and therefore

$$\mathcal{F}C = \bar{q}_*e^{\bar{\ell}}.$$

The line bundle $\bar{\mathcal{L}} := \Phi^*\mathcal{L}$ is the Poincaré line bundle on $C \times J$: that is, we have $\bar{\mathcal{L}}_{C \times \{\alpha\}} = \alpha$ for all $\alpha \in J$, and $\bar{\mathcal{L}}_{\{o\} \times J} = \mathcal{O}_J$. We will now work exclusively on $C \times J$, and suppress the bar above the letters p, q, \mathcal{L} and ℓ . We apply the Grothendieck–Riemann–Roch theorem to q and \mathcal{L} . Since we are working modulo algebraic equivalence, the Todd class of C is simply $1 + (1 - g)o$. Let $i_o : J \hookrightarrow C \times J$ be the map $\alpha \mapsto (o, \alpha)$; we have

$$q_*(p^*o \cdot e^\ell) = q_*i_{o*}i_o^*e^\ell = i_o^*e^\ell = 1,$$

since $i_o^*\mathcal{L}$ is trivial. Thus

$$\text{ch } q_!\mathcal{L} = q_*(p^* \text{Todd}(C) \cdot \text{ch } \mathcal{L}) = q_*e^\ell - (g - 1).$$

The Chern classes of $q_!\mathcal{L}$ are computed in [Mat61]: we have

$$c(-q_!\mathcal{L}) = 1 + w^1 + \dots + w^g.$$

Putting things together we obtain

$$\mathcal{F}C = q_*e^\ell = g - 1 - \text{ch}(-q_!\mathcal{L}) = -(N^1(w) + N^2(w) + \dots + N^g(w)). \quad \square$$

Let $C = \sum_{s=0}^{g-1} C_{(s)}$ be the decomposition of C in $\bigoplus_s A^{g-1}(J)_{(s)}$. From Proposition 3.3, and properties iii and i in § 2, we obtain a corollary.

COROLLARY 3.4. We have $N^k(w) = -\mathcal{F}C_{(k-1)} \in A^k(J)_{(k-1)}$ and $\mathcal{F}(N^k(w)) = (-1)^{g+k}C_{(k-1)}$.

COROLLARY 3.5. The \mathbb{Q} -subalgebra R of $A(J)$ generated by w^1, \dots, w^{g-1} is bigraded. In particular, it is stable under the operations k^* and k_* for each $k \in \mathbb{Z}$.

Indeed R is also generated by the elements $N^1(w), \dots, N^{g-1}(w)$ given above, which are homogeneous for both graduations.

4. Proof of the main result

In order to prove part a of the Theorem, it remains to prove that the \mathbb{Q} -subalgebra R of $A(J)$ generated by w^1, \dots, w^{g-1} is stable under the Pontryagin product. In view of property ii in § 2, it suffices to prove the following.

PROPOSITION 4.1. R is stable under \mathcal{F} .

Proof. Let $\mathcal{F}R$ denote the image of R under the Fourier transform; it is a vector space over \mathbb{Q} , stable under the Pontryagin product (property ii). We will prove that $\mathcal{F}R$ is stable under \mathcal{F} , that is, $\mathcal{F}\mathcal{F}R \subset \mathcal{F}R$; since $\mathcal{F}\mathcal{F}R = R$ (property i), this implies $R \subset \mathcal{F}R$, then $\mathcal{F}R \subset R$ by applying \mathcal{F} again.

We observe that it is enough to prove that $\mathcal{F}R$ is stable under multiplication by θ . Indeed, it is then stable under multiplication by e^θ , and finally under \mathcal{F} in view of property iv, $\mathcal{F}x = e^\theta((\bar{x}e^\theta) * e^{-\theta})$.

Since the \mathbb{Q} -algebra R is generated by the classes $N^p(w)$, $\mathcal{F}R$ is spanned as a \mathbb{Q} -vector space by the elements

$$\mathcal{F}(N^{p_1}(w) \cdots N^{p_r}(w)) = \pm C_{(p_1-1)} * \cdots * C_{(p_r-1)}$$

(we are using property ii and Corollary 3.4).

LEMMA 4.2. $\mathcal{F}R$ is spanned by the classes $(k_{1*}C) * \cdots * (k_{r*}C)$, for all sequences (k_1, \dots, k_r) of positive integers.

Proof. For $k \in \mathbb{Z}$ we have from § 2 that

$$k_*C = \sum_{s=0}^{g-1} k^{2+s}C_{(s)}.$$

Therefore

$$(k_{1*}C) * \cdots * (k_{r*}C) = (k_1 \cdots k_r)^2 \sum_{s_1, \dots, s_r} k_1^{s_1} \cdots k_r^{s_r} C_{(s_1)} * \cdots * C_{(s_r)},$$

where $\mathbf{s} = (s_1, \dots, s_r)$ runs in $[0, g-1]^r$; this shows in particular that $(k_{1*}C) * \cdots * (k_{r*}C)$ belongs to $\mathcal{F}R$. We claim that we can choose g^r r -tuples $\mathbf{k} = (k_1, \dots, k_r)$ so that the matrix $(a_{\mathbf{k}, \mathbf{s}})$ with entries $a_{\mathbf{k}, \mathbf{s}} = (k_1^{s_1} \cdots k_r^{s_r})$ is invertible: if we take for instance the sequence of r -tuples $\mathbf{k}_\ell = (\ell, \ell^g, \dots, \ell^{g^{r-1}})$, for $1 \leq \ell \leq g^r$, we get for $\det(a_{\mathbf{k}, \mathbf{s}})$ a non-zero Vandermonde determinant. Thus each element $C_{(s_1)} * \cdots * C_{(s_r)}$ is a \mathbb{Q} -linear combination of classes of the form $(k_{1*}C) * \cdots * (k_{r*}C)$, which proves Lemma 4.2. □

We now return to the proof of Proposition 4.1.

Thus it suffices to prove that each product $\theta \cdot ((k_{1*}C) * \dots * (k_{r*}C))$ belongs to \mathcal{FR} . We observe that $(k_{1*}C) * \dots * (k_{r*}C)$ is a multiple of the image of the composite map

$$u : C^r \xrightarrow{\varphi} J^r \xrightarrow{\mathbf{k}} J^r \xrightarrow{m} J,$$

where $\mathbf{k} = (k_1, \dots, k_r)$, $\varphi = (\varphi, \dots, \varphi)$ and m is the addition morphism. Thus the class $\theta \cdot ((k_{1*}C) * \dots * (k_{r*}C))$ is proportional to $u_*u^*\theta$.

Let $p_i : J^r \rightarrow J$ (respectively $p_{ij} : J^r \rightarrow J^2$) denote the projection onto the i th factor (respectively the i th and j th factors). In $A^1(J^r)$, we have

$$m^*\theta = \sum_i p_i^*\theta - \sum_{i < j} p_{ij}^*\ell;$$

indeed for $r = 2$ this is the definition of ℓ , and the general case follows from the theorem of the cube. We have also $k_i^*\theta = k_i^2\theta$ and $(k_i, k_j)^*\ell = k_i k_j \ell$. Thus

$$\mathbf{k}^*m^*\theta = \sum_i k_i^2 p_i^*\theta - \sum_{i < j} k_i k_j p_{ij}^*\ell;$$

denoting by q_i, q_{ij} the projections of C^r onto C and C^2 , we find

$$u^*\theta = \sum_i k_i^2 q_i^* \varphi^* \theta - \sum_{i < j} k_i k_j q_{ij}^* (\varphi, \varphi)^* \ell.$$

Let Δ be the diagonal in C^2 . The theorem of the square gives

$$(\varphi, \varphi)^* \mathcal{L} = \mathcal{O}_{C^2}(\Delta - C \times o - o \times C).$$

Therefore $u^*\theta$ is algebraically equivalent to a linear combination of divisors of the form q_i^*o and $q_{ij}^*\Delta$. Under u_* each of these divisors is mapped to a multiple of the cycle $(l_{1*}C) * \dots * (l_{r-1*}C)$, where the sequence $(l_1 \dots l_{r-1})$ is $(k_1, \dots, \widehat{k_i}, \dots, k_r)$ in the first case and $(k_1, \dots, \widehat{k_i}, \dots, \widehat{k_j}, \dots, k_r, k_i + k_j)$ in the second one (as usual the symbol $\widehat{k_i}$ means that k_i is omitted). This proves our claim, and therefore Proposition 4.1. □

5. d-gonal curves

PROPOSITION 5.1. Assume that the curve C is d -gonal, that is, admits a degree d morphism onto \mathbb{P}^1 . We have $N^k(w) = 0$ for $k \geq d$, and the \mathbb{Q} -algebra R is generated by w^1, \dots, w^{d-1} .

Proof. By now this is an immediate consequence of a result of Colombo and van Geemen, which says that for a d -gonal curve $C_{(s)} = 0$ for $s \geq d - 1$ [CG93, Proposition 3.6]. (Our class $C_{(s)}$ is denoted $\pi_{2g-2-s}C$ in [CG93].) This implies $N^k(w) = 0$ for $k \geq d$ [CG93, Proposition 3.2], so that R is a polynomial ring in $N^1(w), \dots, N^{d-1}(w)$, hence in w^1, \dots, w^{d-1} (from § 3). □

The case $d = 2$ of Proposition 5.1 had already been observed by Collino [Col75].

COROLLARY 5.2. If C is hyperelliptic, $R = \mathbb{Q}[\theta]/(\theta^{g+1})$.

COROLLARY 5.3. If C is trigonal, R is generated by θ and the class $\eta = N^2(w)$ in $A^2(J)$. There exists an integer $k \leq g/3$ such that

$$R = \mathbb{Q}[\theta, \eta]/(\theta^{g+1}, \theta^{g-2}\eta, \dots, \theta^{g+1-3k}\eta^k, \eta^{k+1}).$$

Proof. By Proposition 5.1 R is generated by θ and η . For $p, s \in \mathbb{N}$, the class $\theta^{p-2s}\eta^s$ is the only monomial in θ, η which belongs to $A^p(J)_{(s)}$; therefore it spans the \mathbb{Q} -vector space $R_{(s)}^p$ (in particular, this space is zero for $p < 2s$). This implies that the relations between θ and η are monomial, that is, of the form $\theta^r \eta^s = 0$ for some pairs $(r, s) \in \mathbb{N}^2$.

Similarly, as a \mathbb{Q} -algebra for the Pontryagin product, R is generated by $C_{(0)}$ and $C_{(1)}$. The \mathbb{Q} -vector space $R_{(s)}^p$ is spanned by $C_{(0)}^{*(g-p-s)} * C_{(1)}^{*s}$, hence is zero for $p + s > g$. In particular we see that $\theta^r \eta^s = 0$ as soon as $r + 3s > g$.

Let k be the smallest integer such that $\eta^k \neq 0$, $\eta^{k+1} = 0$. By what we have just seen the first relation implies $3k \leq g$. Suppose we have $\theta^r \eta^s = 0$ for some integers r, s with $r + 3s \leq g$ and $s \leq k$. Then we have $R_{(s)}^{r+2s} = 0$ and $C_{(0)}^{*(g-r-3s)} * C_{(1)}^{*s} = 0$. Taking $*$ -product with $C_{(0)}^{**r}$ we arrive at $C_{(0)}^{*(g-3s)} * C_{(1)}^{*s} = 0$, which implies $\eta^s = 0$, contradicting the definition of k . \square

In the general case, since any curve of genus g has a g_d^1 with $d \leq (g + 3)/2$ [ACGH85, ch. V, Theorem 1.1] we get a corollary.

COROLLARY 5.4. Put $d := [(g + 1)/2]$. The \mathbb{Q} -algebra R is generated by w^1, \dots, w^d .

5.5 We may now ask how many of the classes w^i are really needed to generate R . Since $N^k(w)$ belongs to $A^k(J)_{k-1}$, it is readily seen that it cannot be a polynomial in $N^1(w), \dots, N^{k-1}(w)$ unless it is zero. Thus the question is to determine when these classes vanish. I know only two results in that direction: Ceresa's result [Cer83] implies that $N^2(w)$ is non-zero for a generic curve of genus ≥ 3 , and Fakhruddin proved that $N^3(w)$ is non-zero for a generic curve of genus ≥ 11 [Fak96, Corollary 4.6]. It would be interesting to extend this to higher-codimensional classes.

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