Michigan Math. J. 48 (2000)

Determinantal Hypersurfaces

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Introduction

(0.1) In this paper we discuss which homogeneous form on $\mathbf{P}^n$ can be written as the determinant of a matrix with homogeneous entries (possibly symmetric), or the pfaffian of a skew-symmetric matrix. This question has been considered in various particular cases (see the historical comments that follow), and we believe that the general result is well-known to the experts; but we have been unable to find it in the literature. The aim of this paper is to fill this gap.

We will discuss at the outset the general structure theorems; roughly, they show that expressing a homogeneous form F as a determinant (resp. a pfaffian) is equivalent to produce a line bundle (resp. a rank-2 vector bundle) of a certain type on the hypersurface $\mathrm{F} = 0$. The rest of the paper consists of applications. We have restricted our attention to *smooth* hypersurfaces; in fact, we are particularly interested in the case when the *generic* form of degree d in $\mathbf{P}^n$ can be written in one of the above forms. When this is the case, the moduli space of pairs (X, E), where X is a smooth hypersurface of degree d in $\mathbf{P}^n$ and E a rank-1 or rank-2 vector bundle satisfying appropriate conditions, appears as a quotient of an open subset of a certain vector space of matrices; in particular, this moduli space is *unirational*. This is true, for instance, of the universal family of Jacobians of plane curves (Corollary 3.6), and of intermediate Jacobians of cubic threefolds (Corollary 8.8).

Unfortunately, this situation does not occur very frequently: we will show that only curves and cubic surfaces generically admit a determinantal equation. The situation is slightly better for pfaffians: plane curves of any degree, surfaces of degree ≤ 15, and threefolds of degree ≤ 5 can be generically defined by a linear pfaffian.

(0.2) Historical Comments. The representation of curves and surfaces of small degree as linear determinants is a classical subject. The case of cubic surfaces was already known by the middle of the last century [G]; other examples of curves and surfaces are treated in [S]. The general homogeneous forms that can be expressed as linear determinants are determined in [D]. A modern treatment for plane curves appears in [CT]; the result has been rediscovered a number of times since then.

Received January 5, 2000. Revision received April 28, 2000.

The representation of the plane quartic as a symmetric determinant goes back again to 1855 [H]; plane curves of any degree are treated in [Di]. Cubic and quartic surfaces defined by linear symmetric determinants ("symmetroids") have been also studied early; see [Ca]. Surfaces of arbitrary degree are thoroughly treated in [C1], and an overview of the use of symmetric resolutions can be found in [C2].

Finally, the only reference we know about pfaffians is Adler's proof [AR, Apx. V] that a generic cubic threefold can be written as a linear pfaffian.

(0.3) CONVENTIONS. We work over an arbitrary field k, not necessarily algebraically closed. Unless explicitly stated, all geometric objects are defined over k.

ACKNOWLEDGMENTS. I thank F. Catanese for his useful comments and F.-O. Schreyer for providing the computer-aided proof of Propositions 7.6(b) and 8.9 (see Appendix).

1. General Results: Determinants

(1.1) Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n$. We will say that $\mathcal{F}$ is *arithmetically Cohen–Macaulay* (ACM for short) if:

(a) $\mathcal{F}$ is Cohen–Macaulay—that is, the $\mathcal{O}_x$-module $\mathcal{F}_x$ is Cohen–Macaulay for every x in $\mathbf{P}^n$; and

(b) $\mathrm{H}^i(\mathbf{P}^n, \mathcal{F}(j)) = 0$ for $1 \le i \le \dim(\operatorname{Supp}\mathcal{F}) - 1$ and $j \in \mathbf{Z}$.

Put $\mathsf{S}^n = k[\mathrm{X}_0, \dots, \mathrm{X}_n] = \bigoplus_{j\in\mathbf{Z}} \mathrm{H}^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(j))$ (we will often drop the superscript n if there is no danger of confusion). Following EGA, we denote by $\mathbf{\Gamma}_*(\mathcal{F})$ the S-module $\bigoplus_{j\in\mathbf{Z}} \mathrm{H}^0(\mathbf{P}^n, \mathcal{F}(j))$. The following well-known remark explains the terminology.

PROPOSITION 1.2. *The sheaf $\mathcal{F}$ is ACM if and only if the S-module $\mathbf{\Gamma}_*(\mathcal{F})$ is Cohen–Macaulay.*

Proof. Let $U := \mathbf{A}^{n+1} - \{0\}$. The projection $p\colon U \to \mathbf{P}^n$ is affine and satisfies $p_*\mathcal{O}_U = \bigoplus_{j\in\mathbf{Z}} \mathcal{O}_{\mathbf{P}^n}(j)$. The S-module $\mathbf{\Gamma}_*(\mathcal{F})$ defines a coherent sheaf $\tilde{\mathcal{F}}$ on $\mathbf{A}^{n+1}$, whose restriction to U is isomorphic to $p^*\mathcal{F}$. Therefore, $\mathrm{H}^i(U, \tilde{\mathcal{F}})$ is isomorphic to $\bigoplus_{j\in\mathbf{Z}} \mathrm{H}^i(\mathbf{P}^n, \mathcal{F}(j))$. The long exact sequence of local cohomology,

$$\cdots \longrightarrow \mathrm{H}^i_{\{0\}}(\mathbf{A}^{n+1}, \tilde{\mathcal{F}}) \longrightarrow \mathrm{H}^i(\mathbf{A}^{n+1}, \tilde{\mathcal{F}}) \longrightarrow \mathrm{H}^i(U, \tilde{\mathcal{F}}) \longrightarrow \cdots,$$

implies that $\mathrm{H}^0_{\{0\}}(\mathbf{A}^{n+1}, \tilde{\mathcal{F}}) = \mathrm{H}^1_{\{0\}}(\mathbf{A}^{n+1}, \tilde{\mathcal{F}}) = 0$ and gives isomorphisms

$$\bigoplus_{j\in\mathbf{Z}} \mathrm{H}^i(\mathbf{P}^n, \mathcal{F}(j)) \xrightarrow{\sim} H^{i+1}_{\{0\}}(\mathbf{A}^{n+1}, \tilde{\mathcal{F}}) \quad \text{for } i \ge 1.$$

Thus, condition (b) of (1.1) is equivalent to $\mathrm{H}^i_{\{0\}}(\tilde{\mathcal{F}}) = 0$ for $i < \dim(\tilde{\mathcal{F}})$, that is, to $\tilde{\mathcal{F}}_0$ being Cohen–Macaulay. On the other hand, since p is smooth, condition (a) is equivalent to $\tilde{\mathcal{F}}_v$ being Cohen–Macaulay for all $v \in U$; hence the proposition. □

Let us mention incidentally the following corollary, due to Horrocks.

COROLLARY 1.3. *A locally free sheaf $\mathcal{F}$ on $\mathbf{P}^n$ with $\mathrm{H}^i(\mathbf{P}^n, \mathcal{F}(j)) = 0$ for $1 \le i \le n-1$ and $j \in \mathbf{Z}$ splits as a direct sum of line bundles.*

Proof. The S-module $\mathbf{\Gamma}_*(\mathcal{F})$ is Cohen–Macaulay of maximal dimension and hence projective. It is therefore free as an S-graded module; that is, isomorphic to a direct sum $\mathsf{S}(d_1) \oplus \cdots \oplus \mathsf{S}(d_r)$ [Bo, Sec. 8, Prop. 8]. Since $\mathcal{F}$ is the sheaf on $\mathrm{Proj}(\mathsf{S})$ associated to $\mathbf{\Gamma}_*(\mathcal{F})$, it is isomorphic to $\mathcal{O}_{\mathbf{P}^n}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^n}(d_r)$. □

THEOREM A. *Let $\mathcal{F}$ be an ACM sheaf on $\mathbf{P}^n$ of dimension $n-1$. Then there exists an exact sequence*

$$0 \longrightarrow \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbf{P}^n}(e_i) \xrightarrow{\mathrm{M}} \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbf{P}^n}(d_i) \longrightarrow \mathcal{F} \longrightarrow 0. \tag{A1}$$

Conversely, let $\mathrm{M}\colon \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbf{P}^n}(e_i) \to \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbf{P}^n}(d_i)$ *be an injective homomorphism. Then the cokernel of* M *is ACM and its support is the hypersurface* $\det \mathrm{M} = 0$.

Proof. Suppose that $\mathcal{F}$ is ACM of dimension $n-1$. The Cohen–Macaulay S-module $\mathbf{\Gamma}_*(\mathcal{F})$ has projective dimension 1; by Hilbert's theorem [Bo, Sec. 8, Cor. 3] it admits a free graded resolution of the form

$$0 \longrightarrow \bigoplus_{i=1}^{\ell} \mathsf{S}(e_i) \longrightarrow \bigoplus_{i=1}^{\ell} \mathsf{S}(d_i) \longrightarrow \mathbf{\Gamma}_*(\mathcal{F}) \longrightarrow 0, \tag{A2}$$

which gives (A1) by taking the associated sheaves on $\mathbf{P}^n$.

Conversely, suppose we are given the exact sequence (A1). The support of $\mathcal{F}$ consists of the points x of $\mathbf{P}^n$ where $\mathrm{M}(x)$ is not injective, that is, where $\det \mathrm{M}(x) = 0$. Since M is generically injective, this is a hypersurface in $\mathbf{P}^n$.

For every $x \in \mathbf{P}^n$, the $\mathcal{O}_{\mathbf{P}^n,x}$-module $\mathcal{F}_x$ has projective dimension ≤ 1; hence it has depth $\ge \dim \mathcal{O}_{\mathbf{P}^n,x} - 1 = \dim \mathcal{F}_x$ and thus it is Cohen–Macaulay. From (A1) we deduce that $\mathrm{H}^i(\mathbf{P}^n, \mathcal{F}(j)) = 0$ for $1 \le i \le n-2$; hence $\mathcal{F}$ is ACM. □

(1.4) The homomorphism M is given by a matrix $(m_{ij}) \in \mathbf{M}_\ell(\mathsf{S})$, with m_{ij} homogeneous of degree $(d_i - e_j)$; we will use the same letter M to denote this matrix.

(1.5) Let $\mathcal{F}$ be an ACM sheaf on $\mathbf{P}^n$ of dimension $n-1$. We will always take for (A2) a *minimal* graded free resolution of $\mathbf{\Gamma}_*(\mathcal{F})$; this means that the images in $\mathbf{\Gamma}_*(\mathcal{F})$ of the generators of $\mathsf{S}(d_i)$ $(1 \le i \le \ell)$ form a minimal system of generators of the S-module $\mathbf{\Gamma}_*(\mathcal{F})$. Such a resolution is unique up to isomorphism. The resolution (A2) is minimal if and only if the matrix (m_{ij}) is zero modulo $(\mathrm{X}_0, \dots, \mathrm{X}_n)$, that is, if and only if $m_{ij} = 0$ whenever $d_i = e_j$.

With a slight abuse of terminology, we will refer to the corresponding exact sequence (A1) as the *minimal resolution* of the sheaf $\mathcal{F}$.

(1.6) The minimal resolution $0 \to \mathrm{L}_1 \to \mathrm{L}_0 \to \mathcal{F} \to 0$, with $\mathrm{L}_1 = \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbf{P}^n}(e_i)$ and $\mathrm{L}_0 = \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbf{P}^n}(d_i)$, is unique up to isomorphism, but this isomorphism is not unique in general; *it is unique if* $\max(e_j) < \min(d_i)$ (in particular, in the linear case). Indeed, this condition implies that $\operatorname{Hom}(L_0, L_1) = 0$ and thus the map $\operatorname{End}(L_0) \to \operatorname{Hom}(L_0, \mathcal{F})$ is injective. Hence, the only automorphism of L_0 that induces the identity on $\mathcal{F}$ is the identity. If, moreover, every automorphism of $\mathcal{F}$ is scalar, then we see that the only pairs of automorphisms $(P, Q) \in \operatorname{Aut} L_0 \times \operatorname{Aut} L_1$ such that $PM = MQ$ are the pairs (λ, λ) for $\lambda \in k^*$.

(1.7) In this paper we will mainly use Theorem A in the following way. We will start from an integral (usually smooth) hypersurface X and a vector bundle E of rank r on X; then we will still say that E is ACM if it is so as an $\mathcal{O}_{\mathbf{P}^n}$-module, that is, if $H^i(X, \mathcal{F}(j)) = 0$ for $1 \le i \le n-2$ and $j \in \mathbf{Z}$. For such a sheaf, Theorem A provides a minimal resolution (A1); localizing at the generic point of X and using the structure theorem for matrices over a principal ring yields $\det M = F^r$, where $F = 0$ is an equation of X. This gives the following corollary.

COROLLARY 1.8. *Let* X *be a smooth hypersurface in* $\mathbf{P}^n$ *given by an equation* $F = 0$.

(a) *Let* L *be a line bundle on* X, *with* $H^i(X, L(j)) = 0$ *for* $1 \le i \le n-2$ *and all* $j \in \mathbf{Z}$. *Then* L *admits a minimal resolution*

$$0 \longrightarrow \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbf{P}^n}(e_i) \xrightarrow{M} \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbf{P}^n}(d_i) \longrightarrow L \longrightarrow 0$$

with $F = \det M$.

(b) *Conversely, let* $M = (m_{ij}) \in \mathbf{M}_\ell(\mathsf{S})$, *with* m_{ij} *homogeneous of degree* $(d_i - e_j)$ *and* $F = \det M$. *Then the cokernel of* $M\colon \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbf{P}^n}(e_i) \to \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbf{P}^n}(d_i)$ *is a line bundle* L *on* X *with the foregoing properties.* □

(1.9) The apparent generality of this corollary is somewhat misleading: taking for L the line bundle $\mathcal{O}_X(j)$ gives rise to the trivial case $\ell = 1$, $M = (F)$. Thus, if $\operatorname{Pic}(X)$ is generated by $\mathcal{O}_X(1)$ then the hypersurface can *not* be defined by a $\ell \times \ell$ determinant with $\ell > 1$. So interesting situations occur only for curves and surfaces. In particular, we infer from the Noether–Lefschetz theorem that the generic hypersurface of degree d in $\mathbf{P}^n$ can be expressed in a nontrivial way as a determinant only if $n = 2$ or $n = 3$ and $d \le 3$. On the other hand, we will see in (3.1) and (6.4) that any smooth plane curve or cubic surface can be defined by a linear determinant.

(1.10) Conversely, given integers e_i and d_j, one may ask whether a general matrix $(m_{ij}) \in \mathbf{M}_\ell(\mathsf{S})$ with $\deg m_{ij} = d_i - e_j$ defines a smooth curve or surface. If we order the numbers e_i, d_j so that $e_1 \le \cdots \le e_\ell$ and $d_1 \le \cdots \le d_\ell$, *a sufficient condition is the inequality* $d_i > e_{i+1}$ *for* $1 \le i < \ell$. Indeed, we can consider the matrix

$$M = \begin{pmatrix} F_1 & G_1 & 0 & \cdots & 0 \\ 0 & F_2 & G_2 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & F_{\ell-1} & G_{\ell-1} \\ G_\ell & 0 & \cdots & 0 & F_\ell \end{pmatrix},$$

where the entries are product of linear forms. Then $\det M$ can be written in the form $\prod L_i + \prod P_i$, where L_i and P_j are arbitrary linear forms. In this way we obtain, for instance, the Fermat hypersurface $\sum X_i^d = 0$ in $\mathbf{P}^2$ or $\mathbf{P}^3$. (If $\mathrm{char}(k) \mid d$, consider the surface $X_0(X_0^{d-1} + X_1^{d-1}) + (X_1 + X_2)(X_2^{d-1} + X_3^{d-1}) = 0$.)

The integers e_i, d_j that occur in the minimal resolution are determined by the S-module $\mathbf{\Gamma}_*(\mathcal{F})$; we will see some examples in the next sections. We will be particularly interested in the case where the entries (m_{ij}) are linear forms; in this case we will say for short that the matrix M is *linear*. There is a handy characterization of the sheaves which give rise to linear matrices, as follows.

PROPOSITION 1.11. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n$. Then the following conditions are equivalent*:

(i) *there exists an exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n}(-1)^{\ell} \longrightarrow \mathcal{O}_{\mathbf{P}^n}^{\ell} \longrightarrow \mathcal{F} \longrightarrow 0;$$

(ii) *$\mathcal{F}$ is ACM of dimension $n-1$, and*

$$H^0(\mathbf{P}^n, \mathcal{F}(-1)) = H^{n-1}(\mathbf{P}^n, \mathcal{F}(1-n)) = 0.$$

Proof. In view of Theorem A the implication (i) $\Rightarrow$ (ii) is clear, so assume that (ii) holds. Then $H^i(\mathbf{P}^n, \mathcal{F}(-i)) = 0$ for $i \geq 1$; that is, $\mathcal{F}$ is 0-*regular* in the sense of Mumford [Mu, lec. 14]. Again by [Mu], this implies that $\mathcal{F}$ is spanned by its global sections and that the natural map

$$H^0(\mathbf{P}^n, \mathcal{F}(j)) \otimes H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)) \longrightarrow H^0(\mathbf{P}^n, \mathcal{F}(j+1))$$

is surjective for $j \geq 0$. Since $H^0(\mathbf{P}^n, \mathcal{F}(-1)) = 0$, this means that the multiplication map $\mathsf{S} \otimes_k H^0(\mathbf{P}^n, \mathcal{F}) \to \mathbf{\Gamma}_*(\mathcal{F})$ is surjective, and therefore the minimal resolution of $\mathcal{F}$ takes the form

$$0 \longrightarrow \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbf{P}^n}(e_i) \xrightarrow{M} \mathcal{O}_{\mathbf{P}^n}^{\ell} \xrightarrow{p} \mathcal{F} \longrightarrow 0$$

with $\ell = \dim H^0(\mathbf{P}^n, \mathcal{F})$. Since $H^0(p)$ is bijective and $H^{n-1}(\mathbf{P}^n, \mathcal{F}(1-n)) = 0$, we must have $e_i = -1$ for all i. □

This result likewise can be reformulated, as follows.

COROLLARY 1.12. *Let X be a smooth hypersurface of degree d in $\mathbf{P}^n$ given by an equation* $F = 0$.

(a) *Let L be a line bundle on X with $H^i(X, L(j)) = 0$ for $1 \leq i \leq n-2$ and all $j \in \mathbf{Z}$, and let $H^0(X, L(-1)) = H^{n-1}(X, L(1-n)) = 0$. Then there exists a $d \times d$ linear matrix M such that $F = \det M$ and also an exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n}(-1)^d \xrightarrow{M} \mathcal{O}_{\mathbf{P}^n}^d \longrightarrow L \longrightarrow 0.$$

(b) *Conversely, let M be a $d \times d$ linear matrix such that $F = \det M$. Then the cokernel of $M\colon \mathcal{O}_{\mathbf{P}^n}(-1)^d \to \mathcal{O}_{\mathbf{P}^n}^d$ is a line bundle L on X with the foregoing properties.*

2. General Results: Symmetric Determinants and Pfaffians

(2.1) We will now put extra data on our ACM sheaf. Let $\mathcal{F}$ be a torsion-free sheaf on an integral variety X, and let L be a line bundle on X; a bilinear form $\varphi\colon \mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{F}\to \mathrm{L}$ is said to be *invertible* if the associated homomorphism $\kappa : \mathcal{F}\to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathrm{L})$ is an isomorphism. We will consider forms that are ε-symmetric ($\varepsilon=\pm 1$), that is, such that ${}^t\kappa=\varepsilon\kappa$.

THEOREM B. *Assume that* char$(k)\neq 2$. *Let* X *be an integral hypersurface of degree d in* $\mathbf{P}^n$, *and let $\mathcal{F}$ be a torsion-free ACM sheaf on* X *that is equipped with an ε-symmetric invertible form $\mathcal{F}\otimes\mathcal{F}\to\mathcal{O}_X(d+t)$ $(t\in\mathbf{Z})$. Then $\mathcal{F}$ admits a resolution*

$$0\longrightarrow \mathrm{L}_0^*(t)\xrightarrow{\ \mathrm{M}\ }\mathrm{L}_0\longrightarrow\mathcal{F}\longrightarrow 0, \tag{B1}$$

where $\mathrm{L}_0=\bigoplus\mathcal{O}_{\mathbf{P}^n}(d_i)$ *and* M *is ε-symmetric (i.e.,* ${}^t\mathrm{M}=\varepsilon\mathrm{M}$).

Conversely, if a sheaf $\mathcal{F}$ on X *fits into the exact sequence* (B1)*, then it is ACM, torsion-free, and admits an ε-symmetric invertible form $\mathcal{F}\otimes\mathcal{F}\to\mathcal{O}_X(d+t)$.*

Proof. Consider a minimal resolution

$$0\longrightarrow \mathrm{L}_1\xrightarrow{\ \mathrm{M}\ }\mathrm{L}_0\xrightarrow{\ p\ }\mathcal{F}\longrightarrow 0$$

of $\mathcal{F}$. Applying the functor $\mathcal{H}om_{\mathcal{O}_{\mathbf{P}^n}}(*,\mathcal{O}_{\mathbf{P}^n}(t))$ gives an exact sequence

$$0\longrightarrow \mathrm{L}_0^*(t)\xrightarrow{\ {}^t\mathrm{M}\ }\mathrm{L}_1^*(t)\longrightarrow \mathcal{E}xt^1_{\mathcal{O}_{\mathbf{P}^n}}(\mathcal{F},\mathcal{O}_X(t))\longrightarrow 0$$

and the vanishing of $\mathcal{E}xt^i_{\mathcal{O}_{\mathbf{P}^n}}(\mathcal{F},\mathcal{O}_X(t))$ for $i\neq 1$.

Let i be the embedding of X into $\mathbf{P}^n$, and put $\mathcal{F}'=\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{O}_X(d+t))$. Grothendieck duality provides a canonical isomorphism $\mathcal{E}xt^1_{\mathcal{O}_{\mathbf{P}^n}}(\mathcal{F},\mathcal{O}_X(t))\xrightarrow{\sim} i_*\mathcal{F}'$. Thus the above exact sequence gives a minimal resolution of the $\mathcal{O}_{\mathbf{P}^n}$-module $\mathcal{F}'$, and the isomorphism $\kappa:\mathcal{F}\to\mathcal{F}'$ extends to an isomorphism of resolutions:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{L}_1 & \xrightarrow{\ \mathrm{M}\ } & \mathrm{L}_0 & \xrightarrow{\ p\ } & \mathcal{F} & \longrightarrow & 0\\
 & & {\scriptstyle \mathrm{B}}\downarrow & & {\scriptstyle \mathrm{A}}\downarrow & & {\scriptstyle \kappa}\downarrow & & \\
0 & \longrightarrow & \mathrm{L}_0^*(t) & \xrightarrow{\ {}^t\mathrm{M}\ } & \mathrm{L}_1^*(t) & \xrightarrow{\ q\ } & \mathcal{F}' & \longrightarrow & 0.
\end{array}$$

Applying the functor $\mathcal{H}om_{\mathcal{O}_{\mathbf{P}^n}}(*,\mathcal{O}_{\mathbf{P}^n}(t))$ leads to another commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{L}_1 & \xrightarrow{\ \mathrm{M}\ } & \mathrm{L}_0 & \xrightarrow{\ p\ } & \mathcal{F} & \longrightarrow & 0\\
 & & {\scriptstyle {}^t\mathrm{A}}\downarrow & & {\scriptstyle {}^t\mathrm{B}}\downarrow & & {\scriptstyle {}^t\kappa}\downarrow & & \\
0 & \longrightarrow & \mathrm{L}_0^*(t) & \xrightarrow{\ {}^t\mathrm{M}\ } & \mathrm{L}_1^*(t) & \xrightarrow{\ q\ } & \mathcal{F}' & \longrightarrow & 0.
\end{array}$$

Since ${}^t\kappa=\varepsilon\kappa$, we have $q\circ{}^t\mathrm{B}={}^t\kappa\circ p=\varepsilon q\circ\mathrm{A}$, which means that there exists a map $\mathrm{C}\colon \mathrm{L}_0\to\mathrm{L}_0^*(t)$ such that ${}^t\mathrm{B}-\varepsilon\mathrm{A}={}^t\mathrm{MC}$.

Since ${}^t\mathrm{BM} = {}^t\mathrm{M}\,{}^t\mathrm{A}$, we have

$$ {}^t\mathrm{MCM} = ({}^t\mathrm{B} - \varepsilon\mathrm{A})\mathrm{M} = {}^t(\mathrm{AM}) - \varepsilon(\mathrm{AM}) = -\varepsilon\,{}^t\mathrm{M}\,{}^t\mathrm{CM} $$

and thus the map $\mathrm{A}' := \mathrm{A} + (\varepsilon/2)\,{}^t\mathrm{MC}$ satisfies ${}^t(\mathrm{A}'\mathrm{M}) = \varepsilon\mathrm{A}'\mathrm{M}$. Moreover, we still have $q \circ \mathrm{A}' = \kappa \circ p$, so A' is an isomorphism. We have an exact sequence

$$ 0 \longrightarrow \mathrm{L}_0^*(t) \xrightarrow{\mathrm{M}'} \mathrm{L}_0 \xrightarrow{p} \mathcal{F} \longrightarrow 0, $$

where $\mathrm{M}' := \mathrm{A}'^{-1}\,{}^t\mathrm{M}$ satisfies ${}^t\mathrm{M}' = \varepsilon\mathrm{M}'$.

Conversely, starting from the exact sequence (B1), Grothendieck duality implies as above an isomorphism $\kappa : \mathcal{F} \to \mathcal{H}om(\mathcal{F}, \mathcal{O}_\mathrm{X}(d+t))$; applying again the functor $\mathcal{H}om_{\mathcal{O}_{\mathbf{P}^n}}(*, \mathcal{O}_{\mathbf{P}^n}(t))$, we obtain ${}^t\kappa = \varepsilon\kappa$. □

Remark 2.2. The result remains valid in characteristic 2 under the extra hypothesis $\max(e_j) < \min(d_i)$. Indeed, using our notation, the relation $q \circ \mathrm{A} = q \circ {}^t\mathrm{B}$ implies then directly $\mathrm{A} = {}^t\mathrm{B}$ (1.6), and we can take $\mathrm{M}' = \mathrm{A}^{-1}\,{}^t\mathrm{M}$.

Catanese pointed out that his proof [C1] for symmetric surfaces extends readily to the case considered here; it has the advantage of working equally well in characteristic 2, without the restriction on the degrees.

(2.3) Assume again that $\max(e_j) < \min(d_i)$, and let

$$ 0 \longrightarrow \mathrm{P}_0^*(t') \xrightarrow{\mathrm{M}'} P_0 \xrightarrow{p'} \mathcal{F} \longrightarrow 0 $$

be another resolution (B1) of $\mathcal{F}$. Then we have $t = t'$ and a commutative diagram

$$ \begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{L}_0^*(t) & \xrightarrow{\mathrm{M}} & \mathrm{L}_0 & \xrightarrow{p} & \mathcal{F} & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{B}}\downarrow & & {\scriptstyle \mathrm{A}}\downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{P}_0^*(t) & \xrightarrow{\mathrm{M}'} & \mathrm{P}_0 & \xrightarrow{q} & \mathcal{F} & \longrightarrow & 0, \end{array} $$

where the vertical arrows are isomorphisms.

We have $\mathrm{AM} = \mathrm{M}'\mathrm{B}$ and hence, since M and M' are ε-symmetric, $\mathrm{M}\,{}^t\mathrm{A} = {}^t\mathrm{BM}'$ and so ${}^t\mathrm{BAM} = \mathrm{M}\,{}^t\mathrm{AB}$. By (1.6) this implies that ${}^t\mathrm{AB} = \lambda\mathrm{I}$ for some $\lambda \in k^*$. Multiplying A by a scalar yields $\mathrm{M}' = \mathrm{AM}\,{}^t\mathrm{A}$. Thus, all ε-symmetric matrices providing a minimal resolution of $\mathcal{F}$ are conjugate under the action of $\mathrm{Aut}(\mathrm{L}_0)$. In the same way, we see that every automorphism of $\mathcal{F}$ is induced by a matrix $\mathrm{A} \in \mathrm{Aut}(\mathrm{L}_0)$ such that $\mathrm{AM}\,{}^t\mathrm{A} = \lambda\mathrm{M}$ for some $\lambda \in k^*$.

As before, let us rephrase Theorem B in the way we will mostly use it.

Corollary 2.4. *Assume that* $\mathrm{char}(k) \neq 2$. *Let* X *be an integral hypersurface of degree* d *in* $\mathbf{P}^n$, *and let* E *be an ACM line bundle on* X *with* $\mathrm{E}^2 \cong \mathcal{O}_\mathrm{X}(d+t)$ (*resp., an ACM rank*-2 *vector bundle on* X *with determinant* $\mathcal{O}_\mathrm{X}(d+t)$). *There exists a symmetric* (*resp. skew-symmetric*) *matrix* $\mathrm{M} = (m_{ij}) \in \mathbf{M}_\ell(\mathsf{S})$, *with* m_{ij} *homogeneous of degree* $d_i + d_j - t$, *and an exact sequence*

$$ 0 \longrightarrow \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbf{P}^n}(t - d_i) \xrightarrow{\mathrm{M}} \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbf{P}^n}(d_i) \longrightarrow \mathrm{E} \longrightarrow 0; $$

X *is defined by the equation* $\det M = 0$ (*resp.* $\operatorname{pf} M = 0$). *If* $H^0(X, E(-1)) = 0$ *and* $t = -1$, *then the matrix* M *is linear and the exact sequence takes the form*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n}(-1)^{rd} \xrightarrow{M} \mathcal{O}_{\mathbf{P}^n}^{rd} \longrightarrow E \longrightarrow 0$$

with $r = \operatorname{rk} E$.

Proof. By assumption, E carries an ε-symmetric form $E \otimes E \to \mathcal{O}_X(d+t)$ with $\varepsilon = (-1)^{r-1}$. Then Theorem B provides the desired minimal resolution; by (1.7), we have $F = \det M$ if $r = 1$ and $F^2 = \det M = (\operatorname{pf} M)^2$ if $r = 2$. Moreover, if $t = -1$ then we have $H^{n-1}(X, E(1-n)) \cong H^0(X, E(-1))^*$ by Serre duality, so the last assertion follows from Proposition 1.11. □

3. Plane Curves as Determinants

Let C be a smooth plane curve of degree d defined by an equation $F = 0$. We denote by $g = \frac{1}{2}(d-1)(d-2)$ the genus of C. Any line bundle L on C is ACM and hence admits a minimal resolution (A1) with $\det M = F$.

The case of line bundles of degree $g-1$ follows directly from Corollary 1.12 (applied to L(1)) to yield the following.

PROPOSITION 3.1. (a) *Let* L *be a line bundle of degree* $g-1$ *on* C *with* $H^0(X, L) = 0$. *Then there exists a* $d \times d$ *linear matrix* M *such that* $F = \det M$ *and an exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-2)^d \xrightarrow{M} \mathcal{O}_{\mathbf{P}^2}(-1)^d \longrightarrow L \longrightarrow 0.$$

(b) *Conversely, let* M *be a* $d \times d$ *linear matrix such that* $F = \det M$. *Then the cokernel of* $M\colon \mathcal{O}_{\mathbf{P}^2}(-2)^d \to \mathcal{O}_{\mathbf{P}^2}(-1)^d$ *is a line bundle* L *on* C *of degree* $g-1$ *with* $H^0(X, L) = 0$.

(3.2) Let $|\mathcal{O}_{\mathbf{P}^2}(d)|_{sm}$ be the open subset of the projective space $|\mathcal{O}_{\mathbf{P}^2}(d)|$ parameterizing smooth plane curves of degree d. For $\delta \in \mathbf{Z}$, let $\mathcal{J}_d^\delta \to |\mathcal{O}_{\mathbf{P}^2}(d)|_{sm}$ be the family of degree-δ Jacobians: $\mathcal{J}_d^\delta$ parameterizes pairs (C, L) of a smooth plane curve of degree d and a line bundle of degree δ on C. Finally, we denote by Θ_d the divisor in $\mathcal{J}_d^{g-1}$ consisting of pairs (C, L) with $H^0(C, L) \neq 0$. It is an ample divisor, so its complement in $\mathcal{J}_d^{g-1}$ is affine.

Let $\mathcal{L}_d$ be the open subset of the vector space of linear matrices $M \in \mathbf{M}_d(\mathsf{S}^2)$ such that the equation $\det M = 0$ defines a smooth plane curve C_M in $\mathbf{P}^2$. By associating to M the curve C_M and the line bundle $L_M := \operatorname{Coker}\bigl[\mathcal{O}_{\mathbf{P}^n}(-2)^d \xrightarrow{M} \mathcal{O}_{\mathbf{P}^n}(-1)^d\bigr]$ on C_M, we define a morphism $\pi : \mathcal{L}_d \to \mathcal{J}_d^{g-1} - \Theta_d$. The group $GL(d) \times GL(d)$ acts on $\mathcal{L}_d$ by $(P, Q) \cdot M = PMQ^{-1}$; this action factors through the quotient G_d of $GL(d) \times GL(d)$ by $\mathbf{G}_m$ embedded diagonally.

PROPOSITION 3.3. *The group* G_d *acts freely and properly on* $\mathcal{L}_d$, *and the morphism* π *induces an isomorphism* $\mathcal{L}_d/G_d \to \mathcal{J}_d^{g-1} - \Theta_d$.

Proof. This is proved, for instance, in [B3, Sec. 3]; let us give a proof based on our present methods. Let $M \in \mathcal{L}_d$, $(P, Q) \in GL(d) \times GL(d)$, and $M' = PMQ^{-1}$. Then $\det M' = \det M$ up to a scalar, and we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^2}(-1)^d & \xrightarrow{M} & \mathcal{O}_{\mathbf{P}^2}^d & \xrightarrow{p} & L_M & \longrightarrow & 0 \\ & & {\scriptstyle Q}\downarrow & & {\scriptstyle P}\downarrow & & \downarrow\wr & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^2}^d & \xrightarrow{M'} & \mathcal{O}_{\mathbf{P}^2}(-1)^d & \longrightarrow & L_{M'} & \longrightarrow & 0; \end{array} \tag{3.3.a}$$

thus, π factors through a morphism $\mathcal{L}_d/G_d \to \mathcal{J}_d^{g-1} - \Theta_d$. Conversely, if two matrices M and M′ give rise to isomorphic pairs, then the minimal resolution of L_M and $L_{M'}$ are isomorphic and so we again have diagram (3.3.a), which shows that M and M′ are conjugate under G_d. Thus the orbits of G_d in $\mathcal{L}_d$ are isomorphic to the fibres of π and hence are closed. Moreover, by (1.6) the stabilizer of M in $GL(d) \times GL(d)$ reduces to $\mathbf{G}_m$ embedded diagonally, so G_d acts freely on $\mathcal{L}_d$. This proves our assertions. □

REMARK 3.4. A simpler birational presentation of the quotient $GL(d)\backslash\mathcal{L}_d/GL(d)$ (and therefore of $\mathcal{J}_d^{g-1}$) is obtained as follows. Let $\mathcal{D}_d$ be the closed subset of $\mathcal{L}_d$ consisting of matrices of the form $X_0 I_d + X_1 M_1 + X_2 M_2$; it is isomorphic to an affine open subset of $\mathbf{M}_d \times \mathbf{M}_d$, where $\mathbf{M}_d$ denotes the k-variety of $(d \times d)$-matrices. Then $G_d\mathcal{D}_d$ is an open affine subset of $\mathcal{L}_d$, and the stabilizer of $\mathcal{D}_d$ in G_d is $PGL(d)$ acting on $\mathbf{M}_d \times \mathbf{M}_d$ by conjugation. We thus have an open embedding $\mathcal{D}_d/PGL(d) \hookrightarrow GL(d)\backslash\mathcal{L}_d/GL(d)$.

These quotients are of course unirational. It is a classical question to decide whether they are rational: this would have interesting applications in algebra (where the function field of $\mathcal{D}_d/PGL(d)$ is known as the “center of the generic division algebra”) and in geometry ($\mathcal{D}_d/PGL(d)$ is birationally equivalent to the moduli space of stable rank-d vector bundles on $\mathbf{P}^2$ with $c_1 = 0$ and $c_2 = d$). The rationality is known only for $d \leq 4$. We refer to [L] for an excellent survey of these questions.

It is amusing to observe that the universal Jacobian $\mathcal{J}_d^g$ *is rational* [B3, 3.4]: using the rational map $\mathcal{J}_d^g \dashrightarrow \mathrm{Sym}^g(\mathbf{P}^2)$ which maps a general pair (C, L) to the unique element of |L|, we see that $\mathcal{J}_d^g$ is birational to a projective fibre bundle over the rational variety $\mathrm{Sym}^g(\mathbf{P}^2)$. Unfortunately, this does not seem to have any implication on the more interesting question of the rationality of $\mathcal{J}_d^{g-1}$.

We will now determine the minimal resolution of a generic line bundle L of arbitrary degree on a generic plane curve. Replacing L by $L(t)$ for some $t \in \mathbf{Z}$, we can assume that $g - 1 \leq \deg L \leq g - 1 + d$.

PROPOSITION 3.5. *Let* L *be a line bundle of degree* $g-1+p$ *on* C, *with* $0 \leq p \leq d$. *Then the following conditions are equivalent*.

(i) $H^0(C, L(-1)) = H^1(C, L) = 0$, *and the natural map*

$$\mu_0 \colon H^0(C, L) \otimes H^0(C, \mathcal{O}_C(1)) \to H^0(C, L(1))$$

is of maximal rank (*that is, injective for* $p \leq \frac{d}{2}$ *and surjective for* $p \geq \frac{d}{2}$).

(ii) *There is an exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-2)^{d-p} \xrightarrow{\mathrm{M}} \mathcal{O}_{\mathbf{P}^2}(-1)^{d-2p} \oplus \mathcal{O}_{\mathbf{P}^2}^{p} \longrightarrow \mathrm{L} \longrightarrow 0 \quad \textit{if } p \le \tfrac{d}{2},$$

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-2)^{d-p} \oplus \mathcal{O}_{\mathbf{P}^2}(-1)^{2p-d} \xrightarrow{\mathrm{M}} \mathcal{O}_{\mathbf{P}^2}^{p} \longrightarrow \mathrm{L} \longrightarrow 0 \quad \textit{if } p \ge \tfrac{d}{2},$$

with $\det \mathrm{M} = \mathrm{F}$.

The set of pairs (C, L) *satisfying these conditions is Zariski dense in* $\mathcal{J}_d^{g-1+p}$ (*and open if* $k = \bar{k}$).

Proof. Assume that (i) holds. The natural maps

$$\mu_j : \mathrm{H}^0(\mathrm{C}, \mathrm{L}(j)) \otimes \mathrm{H}^0(\mathrm{C}, \mathcal{O}_{\mathrm{C}}(1)) \to \mathrm{H}^0(\mathrm{C}, \mathrm{L}(j+1))$$

are surjective for $j \ge 1$ because $\mathrm{H}^1(\mathrm{C}, \mathrm{L}) = 0$ [Mu]; since $\mathrm{H}^0(\mathrm{C}, \mathrm{L}(-1)) = 0$, this means that the $\mathbf{S}^2$-module $\mathbf{\Gamma}_*(\mathrm{L})$ is generated by homogeneous elements of degree 0 and 1. In other words, the minimal resolution of L takes the form

$$0 \longrightarrow \bigoplus_{i=1}^{p+q} \mathcal{O}_{\mathbf{P}^2}(e_i) \xrightarrow{\mathrm{M}} \mathcal{O}_{\mathbf{P}^2}(-1)^q \oplus \mathcal{O}_{\mathbf{P}^2}^{p} \longrightarrow \mathrm{L} \longrightarrow 0$$

for some integer $q \ge 0$ (observe that $\dim \mathrm{H}^0(\mathrm{C}, \mathrm{L}) = p$ by Riemann–Roch). The vanishing of $\mathrm{H}^1(\mathrm{C}, \mathrm{L})$ and the minimality of the resolution imply $e_i \in \{-2, -1\}$, so we have

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-2)^{d-p} \oplus \mathcal{O}_{\mathbf{P}^2}(-1)^r \xrightarrow{\mathrm{M}} \mathcal{O}_{\mathbf{P}^2}(-1)^q \oplus \mathcal{O}_{\mathbf{P}^2}^{p} \longrightarrow \mathrm{L} \longrightarrow 0, \quad (3.5.a)$$

with $r = 2p - d + q$. After tensor product with $\mathcal{O}_{\mathbf{P}^2}(1)$, the cohomology exact sequence gives

$$q = \dim \operatorname{Coker} \mu_0, \qquad r = \dim \operatorname{Ker} \mu_0, \quad (3.5.b)$$

from which (ii) follows.

If (ii) holds, we have the exact sequence (3.5.a) with $r = 0$ $\big($if $p \le \frac{d}{2}\big)$ or $q = 0$ $\big($if $p \ge \frac{d}{2}\big)$. By (3.5.b), μ_0 is of maximal rank; the vanishing of $\mathrm{H}^0(\mathrm{C}, \mathrm{L}(-1))$ and $\mathrm{H}^1(\mathrm{C}, \mathrm{L})$ is clear.

Let V be the vector space of matrices M appearing in (ii), and let V_0 be the open subset of matrices whose determinant defines a smooth curve; observe that V_0 is non-empty by (1.10). As in (3.3), we have a morphism $\pi : \mathrm{V}_0 \to \mathcal{J}_d^{g-1+p}$; since property (i) is open in $\mathcal{J}_d^{g-1+p}$, it follows that π is dominant. The last assertion of the proposition follows. □

We just also proved the following corollary.

COROLLARY 3.6. *The variety* $\mathcal{J}_d^{\delta}$ *is unirational for all* $\delta \in \mathbf{Z}$.

EXAMPLE 3.7. Consider the relative Jacobian $\mathcal{J}_d^0$. We have $g - 1 = \frac{1}{2}d(d-3)$, so if d is odd then $\mathcal{J}_d^0$ is canonically isomorphic to $\mathcal{J}_d^{g-1}$. Assume $d = 2e$, so that $\mathcal{J}_d^0$ is canonically isomorphic to $\mathcal{J}_d^{g-1+e}$. For (C, L) generic in $\mathcal{J}_d^{g-1+e}$, the minimal resolution of L takes the form

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-2)^e \xrightarrow{\mathrm{M}} \mathcal{O}_{\mathbf{P}^2}^e \longrightarrow \mathrm{L} \longrightarrow 0;$$

hence, the equation of C can be written as the determinant of a matrix $\mathrm{M} \in \mathbf{M}_e(\mathsf{S}^2)$ with quadratic entries. Writing such a matrix as $\mathrm{M} = \sum \mathrm{X}_i \mathrm{X}_j \mathrm{M}_{ij}$, we see as in (3.4) that $\mathcal{J}_d^0$ is birationally equivalent to the quotient of $\mathbf{M}_e^5$ by $\mathrm{GL}(e)$ acting by conjugation. This quotient is birationally equivalent to a vector bundle over $\mathbf{M}_e^2/\mathrm{GL}(e)$ [L]; in particular, we see that the variety $\mathcal{J}_d^0$ is rational for $d = 4$, 6, or 8.

4. Plane Curves as Symmetric Determinants

By Corollary 2.4, any line bundle L on C with $\mathrm{L}^{\otimes 2} \cong \mathcal{O}_{\mathrm{C}}(s)$ admits a symmetric minimal resolution. There are (at least) two interesting applications.

(4.1) Theta Characteristics. Recall that a *theta characteristic* on a smooth curve C is a line bundle κ such that $\kappa^{\otimes 2} \cong \mathrm{K}_{\mathrm{C}}$. We write $h^0(\kappa) := \dim \mathrm{H}^0(\mathrm{C}, \kappa)$.

Proposition 4.2. *Let* C *be a smooth plane curve defined by an equation* $\mathrm{F} = 0$, *and let* κ *be a theta characteristic on* C.

(a) *If* $h^0(\kappa) = 0$ *then* κ *admits a minimal resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-2)^d \xrightarrow{\mathrm{M}} \mathcal{O}_{\mathbf{P}^2}(-1)^d \longrightarrow \kappa \longrightarrow 0,$$

where the matrix $\mathrm{M} \in \mathbf{M}_d(\mathsf{S}^2)$ *is symmetric* (*linear*) *and* $\det \mathrm{M} = \mathrm{F}$.

(b) *If* $h^0(\kappa) = 1$ *then* κ *admits a minimal resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-2)^{d-3} \oplus \mathcal{O}_{\mathbf{P}^2}(-3) \xrightarrow{\mathrm{M}} \mathcal{O}_{\mathbf{P}^2}(-1)^{d-3} \oplus \mathcal{O}_{\mathbf{P}^2} \longrightarrow \kappa \longrightarrow 0,$$

with a symmetric matrix $\mathrm{M} \in \mathbf{M}_{d-2}(\mathsf{S}^2)$ *satisfying* $\det \mathrm{M} = \mathrm{F}$ *and of the form*

$$\mathrm{M} = \begin{pmatrix} \mathrm{L}_{1,1} & \cdots & \mathrm{L}_{1,d-3} & \mathrm{Q}_1 \\ \vdots & & \vdots & \vdots \\ \mathrm{L}_{1,d-3} & \cdots & \mathrm{L}_{d-3,d-3} & \mathrm{Q}_{d-3} \\ \mathrm{Q}_1 & \cdots & \mathrm{Q}_{d-3} & \mathrm{H} \end{pmatrix},$$

where the forms L_{ij}, Q_i, *and* H *are* (*respectively*) *linear, quadratic, and cubic.*

Conversely, the cokernel of a symmetric matrix M *as in* (a) (*resp.* (b)) *is a theta characteristic* κ *on* C *with* $h^0(\kappa) = 0$ (*resp.* $h^0(\kappa) = 1$).

Part (a) is well known and goes back essentially to Dixon [Di]. Part (b) is stated (without proof), for instance, in [B1, 6.27]. Geometrically, when $\mathrm{char}(k) \neq 2$, (a) means that C is the discriminant curve of a net of quadrics in $\mathbf{P}^{d-1}$, and (b) means that C is the discriminant curve of the quadric bundle obtained by projecting the cubic hypersurface $\sum \mathrm{U}_i \mathrm{U}_j \mathrm{L}_{ij} + \sum \mathrm{U}_i \mathrm{Q}_i + \mathrm{H} = 0$ in the projective space $\mathbf{P}^{d-1}$ with coordinates $\mathrm{U}_1, \dots, \mathrm{U}_{d-3}, \mathrm{X}_0, \mathrm{X}_1, \mathrm{X}_2$ from the subspace $\mathrm{X}_0 = \mathrm{X}_1 = \mathrm{X}_2 = 0$.

Proof. Part (a) follows directly from Corollary 2.4 (applied to $\mathrm{E} = \kappa(1)$).

Let κ be a theta characteristic on C, with $h^0(\kappa) = 1$. Then $\mathrm{H}^1(\mathrm{C}, \kappa(1)) = \mathrm{H}^0(\mathrm{C}, \kappa(-1))^* = 0$, so $\boldsymbol{\Gamma}_*(\kappa)$ is generated by its elements of degree 0, 1, and 2. In view of (2.4), the minimal resolution of κ is of the form

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-1)^q \oplus \mathcal{O}_{\mathbf{P}^2}(-2)^p \oplus \mathcal{O}_{\mathbf{P}^2}(-3) \xrightarrow{\mathrm{M}} \mathcal{O}_{\mathbf{P}^2}(-2)^q \oplus \mathcal{O}_{\mathbf{P}^2}(-1)^p \oplus \mathcal{O}_{\mathbf{P}^2} \longrightarrow \kappa \longrightarrow 0$$

for some nonnegative integers p, q. Since the resolution is minimal, the summand $\mathcal{O}_{\mathbf{P}^2}(-1)^q$ in the first term is mapped into $\mathcal{O}_{\mathbf{P}^2}$; this implies $q \le 1$, and in fact $q = 0$ because otherwise the nonzero section of κ would be annihilated by some linear form. This gives the form of the resolution (and of the matrix M) in part (b). □

Assume now that $\mathrm{char}(k) = 0$. (This works equally well in all characteristics $\neq 2$, but references are lacking.) The moduli space of pairs (C, κ), where C is a smooth plane curve of degree d and κ a theta characteristic on C, has two components corresponding to the parity of $h^0(\kappa)$ plus one special component when d is odd, consisting of the pairs $(\mathrm{C}, \mathcal{O}_C((d-3)/2))$ [B2, Prop. 3]; a general element (C, κ) in a nonspecial component satisfies $h^0(\kappa) \le 1$.

COROLLARY 4.3. *Each component of the moduli space of smooth plane curves with a theta characteristic is unirational.*

REMARK 4.4. If k is algebraically closed, then any smooth curve admits a theta characteristic L with $\mathrm{H}^0(\mathrm{L}) = 0$; this follows (via the Riemann singularity theorem) from the classical fact that the theta divisor of a principally polarized Abelian variety cannot contain all points of order 2 (see e.g. [I, Ch. IV, Lemma 11]). Thus *every* smooth plane curve can be defined by a symmetric linear determinant. Actually, every plane curve C admits such a representation: one reduces readily to the case when C is integral; then Theorem B is applied to the sheaf $\pi_*\mathrm{L}$, where $\pi : \mathrm{C}' \to \mathrm{C}$ is the normalization of C and L is a theta characteristic on C′ with $\mathrm{H}^0(\mathrm{C}', \mathrm{L}) = 0$. (This remark answers a question of F. Catanese.)

(4.5) HALF-PERIODS. We assume again that $\mathrm{char}(k) = 0$. Let us consider now the moduli space $\mathcal{R}_d$ of pairs (C, α), where C is a smooth plane curve of degree d and α is a *half-period*, that is, a nontrivial line bundle on C with $\alpha^{\otimes 2} \cong \mathcal{O}_{\mathrm{C}}$. If d is odd then the map $(\mathrm{C}, \alpha) \mapsto (\mathrm{C}, \alpha((d-3)/2))$ gives an isomorphism of $\mathcal{R}_d$ onto the above moduli space; we thus restrict to the case of d even—say, $d = 2e$. It follows then from [B2, Prop. 2] that $\mathcal{R}_d$ is *irreducible.*

PROPOSITION 4.6. *For* (C, α) *general in* $\mathcal{R}_d$, *the line bundle* α *admits a minimal resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-e-1)^e \xrightarrow{\mathrm{M}} \mathcal{O}_{\mathbf{P}^2}(-e+1)^e \longrightarrow \alpha \longrightarrow 0,$$

where the matrix $\mathrm{M} \in \mathbf{M}_e(\mathsf{S}^2)$ *is symmetric* (*with quadratic entries*) *and* $\det \mathrm{M} = \mathrm{F}$.

Proof. In view of Corollary 2.4, this amounts to saying that the line bundle $\alpha(e-1)$ satisfies the equivalent conditions of Proposition 3.5. As in that proposition, it suffices to exhibit a symmetric matrix $\mathrm{M} \in \mathbf{M}_e(\mathsf{S}^2)$ with quadratic entries such that the equation $\det \mathrm{M} = 0$ defines a smooth plane curve.

Start with a symmetric linear matrix $(\mathrm{L}_{ij}) \in \mathbf{M}_e(\mathrm{S})$ such that the curve Γ defined by $\det(\mathrm{L}_{ij}) = 0$ is smooth (such a matrix exists by Proposition 4.2). Changing coordinates if necessary, we can assume that Γ is transverse to the coordinate axes and does not pass through the intersection point of any two axes. Consider the covering $\pi : \mathbf{P}^2 \to \mathbf{P}^2$ given by $\pi(X_0, X_1, X_2) = (X_0^2, X_1^2, X_2^2)$. The pull-back of Γ by π is smooth by our hypotheses; it is defined by the determinant of the symmetric matrix $M = (L_{ij}(X_0^2, X_1^2, X_2^2))$ with quadratic entries. □

COROLLARY 4.7. *The moduli space $\mathcal{R}_d$ is unirational.*

5. Plane Curves as Pfaffians

Again, any rank-2 vector bundle E on the plane curve C with determinant $\mathcal{O}_C(s)$ for some integer s admits a skew-symmetric resolution. Let us restrict our attention to the linear case. Corollary 2.4 applied to E(1) gives the following.

PROPOSITION 5.1. *Let* C *be a smooth plane curve of degree d, and let* E *be a rank-*2 *vector bundle on* C *with* $\det E \cong K_C$ *and* $H^0(C, E) = 0$. *Then* E *admits a minimal resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-2)^{2d} \xrightarrow{M} \mathcal{O}_{\mathbf{P}^2}(-1)^{2d} \longrightarrow E \longrightarrow 0,$$

where the matrix $M \in \mathbf{M}_{2d}(\mathsf{S}^2)$ *is linear skew-symmetric and* $\operatorname{pf} M = F$.

Note that the condition $H^0(C, E) = 0$ implies that E is semi-stable.

COROLLARY 5.2. *The moduli space of pairs* (C, E), *where* C *is a smooth plane curve of degree d and* E *is a semi-stable rank-*2 *vector bundle on* C *with determinant* K_C, *is unirational.*

This is not surprising in this case, since the fibres of the projection to $|\mathcal{O}_{\mathbf{P}^2}(d)|$ are already unirational.

(5.3) Another consequence of Proposition 5.1 is that if $d \geq 4$ and M is general enough, then the corresponding vector bundle $E_M = \operatorname{Coker} M$ is stable and therefore simple; that is, $\operatorname{End}(M) = k$. This means in view of (2.3) that, given three generic skew-symmetric matrices $M_0, M_1, M_2 \in \mathbf{M}_{2d}(k)$, the equations ${}^t A M_i A = M_i$ for $i = 0, 1, 2$ imply $A = \pm I$.

6. Surfaces as Determinants

(6.1) Let S be a smooth surface of degree d in $\mathbf{P}^3$ defined by an equation $F = 0$. Let C be a curve in S and $L = \mathcal{O}_S(C)$. Using the exact sequence $0 \to L^{-1} \to \mathcal{O}_S \to \mathcal{O}_C \to 0$ and Serre duality, we see that L *is* ACM *if and only if* C *is projectively normal in* $\mathbf{P}^3$; that is, the restriction map $H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(j)) \to H^0(C, \mathcal{O}_C(j))$ is surjective for every $j \in \mathbf{Z}$. Since any line bundle is of the form $\mathcal{O}_S(C)$ after some twist, this characterizes the ACM line bundles on S. Thus, any projectively normal curve contained in S gives rise to an expression of F as the determinant of a matrix $M \in \mathbf{M}_k(\mathsf{S}^3)$. Recall, however, that a hypersurface section of S gives the trivial case $\mathrm{M} = (\mathrm{F})$; a curve C defined in $\mathbf{P}^3$ by two equations $\mathrm{A} = \mathrm{B} = 0$ produces a 2×2-matrix $\mathrm{M} = \begin{pmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{pmatrix}$.

We will now restrict our study to *linear* determinants.

PROPOSITION 6.2. *Let* C *be a projectively normal curve on* S *of degree* $\frac{1}{2}d(d-1)$ *and genus* $\frac{1}{6}(d-2)(d-3)(2d+1)$. *Then the line bundle* $\mathcal{O}_{\mathrm{S}}(\mathrm{C})$ *admits a minimal resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3}(-1)^d \xrightarrow{\mathrm{M}} \mathcal{O}_{\mathbf{P}^3}^d \longrightarrow \mathcal{O}_{\mathrm{S}}(\mathrm{C}) \longrightarrow 0$$

with $\det \mathrm{M} = \mathrm{F}$.

Conversely, let $\mathrm{M} \in \mathbf{M}_d(\mathbf{S}^3)$ *be a linear matrix such that* $\det \mathrm{M} = \mathrm{F}$. *Then the cokernel of* $\mathrm{M}\colon \mathcal{O}_{\mathbf{P}^3}(-1)^d \to \mathcal{O}_{\mathbf{P}^3}^d$ *is isomorphic to* $\mathcal{O}_{\mathrm{S}}(\mathrm{C})$, *where* C *is a smooth projectively normal curve on* S *with the above degree and genus.*

Proof. Let C be a curve on S, and put $\mathrm{L} = \mathcal{O}_{\mathrm{S}}(\mathrm{C})$. A straightforward Riemann–Roch computation shows that the given condition on the degree and genus of C is equivalent to $\chi(\mathrm{L}(-1)) = \chi(\mathrm{L}(-2)) = 0$. If C is projectively normal then the spaces $\mathrm{H}^1(\mathrm{S}, \mathrm{L}(j))$ vanish, by (6.1); therefore, the preceding condition is also equivalent to $\mathrm{H}^0(\mathrm{S}, \mathrm{L}(-1)) = \mathrm{H}^2(\mathrm{S}, \mathrm{L}(-2)) = 0$. This is exactly what we need to apply Corollary 1.12.

Conversely, given a matrix M, let $\mathrm{L} = \operatorname{Coker} \mathrm{M}$; in view of the foregoing we need only prove that the linear system $|\mathrm{L}|$ contains a smooth curve. This is obvious in characteristic 0, since L is spanned by its global sections. In the general case, we first observe that the restriction of L to any smooth hyperplane section H of S is very ample: indeed, from the resolution $0 \to \mathcal{O}_{\mathbf{P}^2}(-1)^d \to \mathcal{O}_{\mathbf{P}^2}^d \to \mathrm{L}\big|_{\mathrm{H}} \to 0$ we obtain $\mathrm{H}^1(\mathrm{H}, \mathrm{L}\big|_{\mathrm{H}}(-1)) = 0$ and hence $\mathrm{H}^1(\mathrm{H}, \mathrm{L}\big|_{\mathrm{H}}(-x-y)) = 0$ for all $x, y \in \mathrm{H}$. It follows that the linear system $|\mathrm{L}|$ on S separates two points $x, y \in \mathrm{S}$ (possibly infinitely close) unless the line $\langle x, y\rangle$ lies in S; in other words, the morphism $\varphi_{\mathrm{L}}\colon \mathrm{S} \to \mathbf{P}^{d-1}$ defined by $|\mathrm{L}|$ contracts finitely many lines and embeds the complement of these lines. Then a general hyperplane in $\mathbf{P}^{d-1}$ cuts down a smooth curve $\mathrm{C} \in |\mathrm{L}|$. □

(6.3) Under the hypotheses of Proposition 6.2, Grothendieck duality provides a dual exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3}(-1)^d \xrightarrow{{}^t\mathrm{M}} \mathcal{O}_{\mathbf{P}^3}^d \longrightarrow \mathrm{L}^{-1}(d-1) \longrightarrow 0$$

(see the proof of Theorem B); in other words, *the involution* $\mathrm{M} \mapsto {}^t\mathrm{M}$ *on the space of linear matrices corresponds to the involution* $\mathrm{L} \mapsto \mathrm{L}^{-1}(d-1)$ *on* $\operatorname{Pic}(\mathrm{S})$.

As we have already pointed out, a general form of degree d on $\mathbf{P}^3$ can be represented as a linear determinant only for $d \leq 3$, the only nontrivial case being $d = 3$. There we find the following classical result [G].

COROLLARY 6.4. *Assume that* k *is algebraically closed. A smooth cubic surface can be defined by an equation* $\det \mathrm{M} = 0$, *where* M *is a* 3×3 *linear matrix. There are* 72 *such representations* (*up to the action of* $\mathrm{GL}(3) \times \mathrm{GL}(3)$ *by left and right* *multiplication*), *corresponding in a one-to-one way to the linear systems of twisted cubics on* S.

There are various ways of describing the set of linear systems of twisted cubics on S: they also correspond to the birational morphisms $S \to \mathbf{P}^2$, or to the sets of six lines on S that do not intersect each other. In terms of these, the involution $M \mapsto {}^tM$ corresponds to the Schäfli involution, which associates to such a set $\{\ell_1, \ldots, \ell_6\}$ the unique set $\{\ell'_1, \ldots, \ell'_6\}$ such that the twelve lines ℓ_i, ℓ'_j form a *double-six*; that is, they satisfy $\ell_i \cap \ell'_i = \varnothing$ and $\ell_i \cdot \ell'_j = 1$ for $i \neq j$.

As a consequence, the space of pairs (S, λ), where S is a smooth cubic surface and λ a set of six non-intersecting lines, is *rational*; as in (3.4), it is birational to the quotient of $(\mathbf{M}_3)^3$ by the group GL(3) acting by conjugation, and we know that this quotient is rational.

In the case of a not necessarily algebraically closed field, we find the following result of Segre [Se].

COROLLARY 6.5. *Let* S *be a smooth cubic surface. The following conditions are equivalent*:

(i) S *can be defined by an equation* $\det M = 0$, *where* M *is a* 3×3 *linear matrix*;
(ii) S *contains a twisted cubic*;
(iii) S *admits a birational morphism to* $\mathbf{P}^2$;
(iv) S *contains a rational point and a set* (*defined over* k) *of six non-intersecting lines.*

Proof. The equivalence of (i), (ii), and (iii) follows from Proposition 6.2. The implication (iii) ⇒ (iv) is clear. If (iv) holds, then the surface obtained from S by blowing down the set of six non-intersecting lines is isomorphic to $\mathbf{P}^2$ over $\bar{k}$, contains a rational point, and hence is k-isomorphic to $\mathbf{P}^2$. □

COROLLARY 6.6. *A smooth quartic surface is determinantal if and only if it contains a nonhyperelliptic curve of genus* 3 *embedded in* $\mathbf{P}^3$ *by a linear system of degree* 6.

Proof. The only point to check is that a curve C of genus 3 embedded in $\mathbf{P}^3$ by a linear system $|L|$ of degree 6 is projectively normal if and only if it is not hyperelliptic. Since $H^1(C, L) = 0$, the projective normality reduces (using the basepoint-free pencil trick) to the surjectivity of the restriction map $H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2)) \to H^0(C, L^{\otimes 2})$ or, equivalently (since both spaces have the same dimension), to its injectivity. One checks that C is contained in a quadric if and only if it is hyperelliptic. □

(6.7) There is another approach to Proposition 6.2, which we will now sketch. Given the linear matrix M, let C be the divisor of the section of $L = \operatorname{Coker} M$ corresponding to the first basis vector of $\mathcal{O}^d_{\mathbf{P}^3}$. Using (6.3), we see easily that the curve C is defined in $\mathbf{P}^3$ by the maximal minors of the matrix N obtained from M by deleting the first row. Conversely, since C is projectively normal, it admits a resolution

$$0 \longrightarrow \bigoplus_{j=1}^{\ell-1} \mathcal{O}_{\mathbf{P}^3}(e_j) \xrightarrow{\mathrm{N}} \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbf{P}^3}(d_i) \xrightarrow{\Delta} \mathcal{O}_{\mathbf{P}^3} \longrightarrow \mathcal{O}_{\mathrm{C}} \longrightarrow 0,$$

where Δ is given by the maximal minors of N; with some work one finds $\ell = d$, $e_1 = \cdots = e_{d-1} = -d$, and $d_1 = \cdots = d_d = -(d-1)$. It follows easily that any surface of degree d containing C is defined by the determinant of a linear matrix obtained by adding one row to N.

(6.8) We will not consider surfaces defined by symmetric determinants, though this is again a classical and rich story. See [C1] or [C2] for a modern treatment.

7. Surfaces as Pfaffians

From now on we assume $\mathrm{char}(k) = 0$ (see Remark 7.3(a)).

(7.1) Again we will restrict ourselves to the linear case—that is, to surfaces $\mathrm{S} \subset \mathbf{P}^3$ defined by an equation $\mathrm{pf\,M} = 0$, where M is a $(2d) \times (2d)$ skew-symmetric linear matrix.

Let Z be a finite reduced subscheme of $\mathbf{P}^n$ of degree c (the degree of Z is by definition $\dim_k \mathrm{H}^0(\mathrm{Z}, \mathcal{O}_{\mathrm{Z}})$), and let I_{Z} be its homogeneous ideal in S^n. Then Z is said to be *arithmetically Gorenstein* if the algebra $\mathsf{R} := \mathsf{S}/\mathsf{I}_{\mathrm{Z}}$ is Gorenstein. This implies that there exists an integer N such that

(a) $\dim \mathsf{R}_p + \dim \mathsf{R}_{\mathrm{N}-p} = c$ for all $p \in \mathbf{Z}$.

The integer N is uniquely determined: it is the largest integer such that $\dim \mathsf{R}_{\mathrm{N}} < c$. For lack of a better name, we will call it the *index* of Z.

Assume that $k = \bar{k}$. By [DGO, Thm. 5], the subscheme Z is arithmetically Gorenstein if and only if it satisfies both (a) and

(b) Z has the Cayley–Bacharach property w.r.t. the linear system $|\mathcal{O}_{\mathbf{P}^n}(\mathrm{N})|$; that is, for each point $z \in \mathrm{Z}$, every element of $|\mathcal{O}_{\mathbf{P}^n}(\mathrm{N})|$ containing $\mathrm{Z} - z$ contains Z.

In general, Z is arithmetically Gorenstein if and only if $\mathrm{Z} \otimes_k \bar{k}$ has the same property.

Let $\mathrm{Z} \subset \mathbf{P}^3$ be a finite arithmetically Gorenstein subscheme contained in a surface S of degree d. Let $\mathcal{I}_{\mathrm{Z}}$ be the sheaf of ideals of Z in $\mathcal{O}_{\mathrm{S}}$. Using the exact sequence $0 \to \mathcal{I}_{\mathrm{Z}} \to \mathcal{O}_{\mathrm{S}} \to \mathcal{O}_{\mathrm{Z}} \to 0$, property (a) for $p = \mathrm{N}$ gives $\dim \mathrm{H}^1(\mathrm{S}, \mathcal{I}_{\mathrm{Z}}(\mathrm{N})) = 1$. Thus there exists a unique nontrivial extension (up to automorphism)

$$0 \longrightarrow \mathcal{O}_{\mathrm{S}} \longrightarrow \mathrm{E} \longrightarrow \mathcal{I}_{\mathrm{Z}}(\mathrm{N} - d + 4) \longrightarrow 0.$$

We claim that E is locally free. To check this we can assume that k is algebraically closed; then (b) is equivalent to $\mathrm{H}^1(\mathrm{S}, \mathcal{I}_{\mathrm{Z}'}(\mathrm{N})) = 0$ for each proper subset $\mathrm{Z}' \subset \mathrm{Z}$, which implies our assertion by [GH]. We will say that E is the vector bundle associated to Z.

PROPOSITION 7.2. *Let* S *be a smooth surface of degree* d *in* $\mathbf{P}^3$. *Then the following conditions are equivalent*:

(i) S *can be defined by an equation* pf M $= 0$, *where* M *is a skew-symmetric linear* $(2d) \times (2d)$ *matrix*;

(ii) S *contains a finite arithmetically Gorenstein reduced subscheme* Z *of index* $2d - 5$, *not contained in any surface of degree* $d - 2$.

More precisely, under hypothesis (ii), *the rank*-2 *vector bundle* E *associated to* Z *admits a minimal resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n}(-1)^{2d} \xrightarrow{\ \mathrm{M}\ } \mathcal{O}_{\mathbf{P}^n}^{2d} \longrightarrow \mathrm{E} \longrightarrow 0;$$

the degree of Z *is* $\frac{1}{6}d(d-1)(2d-1)$.

Proof. If (i) holds then the vector bundle E := Coker M is spanned by its global sections. Let Z be the zero locus of a general section of E. Under (i) or (ii) we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathrm{S}} \longrightarrow \mathrm{E} \longrightarrow \mathcal{I}_{\mathrm{Z}}(d-1) \longrightarrow 0. \tag{7.2.a}$$

In view of Proposition 2.4, we have to prove the equivalence of:

(1) E is ACM and $\mathrm{H}^0(\mathrm{S}, \mathrm{E}(-1)) = 0$;

(2) Z is arithmetically Gorenstein and $\mathrm{H}^0(\mathrm{S}, \mathcal{I}_{\mathrm{Z}}(d-2)) = 0$.

Toward that end, we may assume that $k = \bar{k}$. That E is locally free implies that Z has the Cayley–Bacharach property w.r.t. $|\mathcal{O}_{\mathbf{P}^3}(2d-5)|$ [GH]. The sequence (7.2.a) provides an isomorphism

$$\mathrm{H}^0(\mathrm{S}, \mathrm{E}(-1)) \xrightarrow{\ \sim\ } \mathrm{H}^0(\mathrm{S}, \mathcal{I}_{\mathrm{Z}}(d-2))$$

and gives rise, for each $j \in \mathbf{Z}$, to an exact sequence

$$0 \longrightarrow \mathrm{H}^1(\mathrm{S}, \mathrm{E}(j)) \longrightarrow \mathrm{H}^1(\mathrm{S}, \mathcal{I}_{\mathrm{Z}}(d-1+j)) \xrightarrow{\ \partial\ } \mathrm{H}^2(\mathrm{S}, \mathcal{O}_{\mathrm{S}}(j)).$$

Using the exact sequence $0 \to \mathcal{I}_{\mathrm{Z}} \to \mathcal{O}_{\mathrm{S}} \to \mathcal{O}_{\mathrm{Z}} \to 0$, we can identify $\mathrm{H}^1(\mathrm{S}, \mathcal{I}_{\mathrm{Z}}(k))$ with the cokernel of the restriction map $r_k\colon \mathrm{H}^0(\mathrm{S}, \mathcal{O}_{\mathrm{S}}(k)) \to \mathrm{H}^0(\mathrm{Z}, \mathcal{O}_{\mathrm{Z}}(k))$; the map $\mathrm{H}^0(\mathrm{Z}, \mathcal{O}_{\mathrm{Z}}(d-1+j) \to \mathrm{H}^2(\mathrm{S}, \mathcal{O}_{\mathrm{S}}(j))$ deduced from ∂ is identified by Serre duality to the transpose of r_{d-4-j}. Therefore, the vanishing of $\mathrm{H}^1(\mathrm{S}, \mathrm{E}(j))$ is equivalent to $\operatorname{Im} r_{d-1+j} = \operatorname{Ker} {}^t r_{d-4-j} = (\operatorname{Im} r_{d-4-j})^{\perp}$, that is, to $\dim \mathsf{R}_{d-1+j} = c - \dim \mathsf{R}_{d-4-j}$. This proves the equivalence of (i) and (ii).

Under these equivalent conditions, we have Card Z $= c_2(\mathrm{E})$; this number can be computed, for instance, using Riemann–Roch and $\chi(\mathrm{E}) = 2d$. □

REMARKS 7.3. (a) We must restrict to the characteristic 0 case because we do not know how to prove that the zero locus of a general section of E is smooth in characteristic p. The same problem occurs in higher dimension as well.

(b) As in (6.7), we could follow another approach: Using the Buchsbaum–Eisenbud theorem [BE], one shows that I_{Z} is generated by the $(2d-2) \times (2d-2)$ pfaffians extracted from a skew-symmetric linear $(2d-1) \times (2d-1)$ matrix N; then X is defined by the pfaffian of the matrix $\begin{pmatrix} \mathrm{N} & \mathrm{C} \\ -{}^t\mathrm{C} & 0 \end{pmatrix}$, where C is a column of linear forms.

EXAMPLES 7.4. For a cubic surface we have $\deg Z = 5$ and $N = 1$. If $k = \bar{k}$ then a subset Z is arithmetically Gorenstein if and only if any four points in Z are linearly independent, that is, Z is in general position.

For a quartic, the subset Z should have 14 points, not be contained in a quadric, and satisfy the Cayley–Bacharach property w.r.t. $|\mathcal{O}_S(3)|$.

(7.5) Observe that, for each d, there exist smooth surfaces that are defined by the pfaffian of a $(2d) \times (2d)$ skew-symmetric linear matrix and therefore contain a subset Z with the properties of the proposition. For instance, we could take $M = \begin{pmatrix} 0 & N \\ -{}^tN & 0 \end{pmatrix}$, where N is a linear $d \times d$ matrix; we have $\mathrm{pf}\, M = \det N$, and we can choose N so that the surface $\det N = 0$ is smooth (see (1.10)). The corresponding vector bundle E is $L \oplus L^{-1}(d-1)$, where L is the line bundle Coker N; the zero set Z of a general section of E is the intersection of two curves on S of the type described in Property 6.2 (see also (8.3)).

We will now investigate when a generic surface of degree d can be written as a linear pfaffian.

PROPOSITION 7.6. *Assume that k is algebraically closed.*

(a) *Every cubic surface can be defined by a linear pfaffian.*

(b) *A general surface of degree d in $\mathbf{P}^3$ can be defined by a linear pfaffian if and only if $d \leq 15$.*

Proof. (a) follows from Proposition 7.2 and Example 7.4. Let $\mathcal{S}_d$ be the variety of linear skew-symmetric matrices $M \in \mathbf{M}_{2d}(\mathsf{S}^3)$ such that the equation $\mathrm{pf}\, M = 0$ defines a smooth surface in $\mathbf{P}^3$. Consider the map $\mathrm{pf} : \mathcal{S}_d \to |\mathcal{O}_{\mathbf{P}^3}(d)|$. We have $\dim \mathcal{S}_d/\mathrm{GL}(2d) = 4d(2d-1) - 4d^2 = 4d(d-1)$; an easy computation gives $4d(d-1) < \dim|\mathcal{O}_{\mathbf{P}^3}(d)|$ for $d \geq 16$, which gives the "only if" part of (b).

To prove the remaining part we use Adler's method [AR, Apx. V]—namely, we prove that the differential of pf is surjective at a general matrix $M \in \mathcal{S}_d$. As in [AR], a standard computation shows that this is equivalent to the fact that the vector space $H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(d))$ is spanned by the forms $X_k M_{ij}$, where M_{ij} is the pfaffian of the skew-symmetric matrix obtained from M by deleting the rows and columns of index i and j. This has been checked by F. Schreyer using the computer algebra system Macaulay 2: a script is provided in the Appendix. □

We do not consider the proof of (b) as completely satisfactory, since it relies on a computer checking which does not provide any clue regarding why the result holds. The following lemma explains better the meaning of the result. Recall that we associate to a matrix $M \in \mathcal{S}_d$ the smooth surface S_M defined by $\mathrm{pf}\, M = 0$ and the vector bundle $E_M := \mathrm{Coker}\big[\mathcal{O}_{\mathbf{P}^n}(-1)^d \xrightarrow{M} \mathcal{O}_{\mathbf{P}^n}^d\big]$ on S_M.

LEMMA 7.7. *The pfaffian map* $\mathrm{pf} : \mathcal{S}_d \to |\mathcal{O}_{\mathbf{P}^3}(d)|$ *is dominant if and only if* $H^2(S_M, \mathcal{E}nd_0(E_M))$ *vanishes for a general* M *in* $\mathcal{S}_d$.

(As usual, $\mathcal{E}nd_0(E)$ denotes the bundle of traceless endomorphisms of E.)

Proof. We will restrict our attention to matrices M such that E_M is *simple* (i.e., has only scalar endomorphisms). According to (2.3), this means that the only matrices $A \in \mathbf{M}_d(k)$ such that $AM\,{}^tA = M$ are $\pm I$. The matrices M with this property form an open subset $\mathcal{S}_d^s$ of $\mathcal{S}_d$, which is non-empty by (5.3).

We consider the map $\mathrm{pf} : \mathcal{S}_d^s \to |\mathcal{O}_{\mathbf{P}^3}(d)|$; its fibre at a point $S \in |\mathcal{O}_{\mathbf{P}^3}(d)|$ is the moduli space of simple ACM rank-2 vector bundles on S with $\det E = \mathcal{O}_S(d-1)$ and $H^0(S, E(-1)) = 0$. A straightforward computation gives

$$\begin{aligned} \dim \mathcal{S}_d/\mathrm{GL}(2d) &= \dim |\mathcal{O}_{\mathbf{P}^3}(d)| - \chi(\mathcal{E}nd_0(E_M)) \\ &= \dim |\mathcal{O}_{\mathbf{P}^3}(d)| + \dim H^1(S_M, \mathcal{E}nd_0(E_M)) \\ &\quad - \dim H^2(S_M, \mathcal{E}nd_0(E_M)) \end{aligned} \tag{7.7.a}$$

for any matrix $M \in \mathcal{S}_d^s$.

If $H^2(S_M, \mathcal{E}nd_0(E_M)) = 0$, then the moduli space of simple vector bundles on S_M is smooth of dimension $\dim H^1(S_M, \mathcal{E}nd_0(E_M))$ at $[E_M]$. It then follows from (7.7.a) that pf is dominant.

Conversely, assume that pf is dominant. Let S be a generic surface of degree d; the fibre $\mathrm{pf}^{-1}(S)$ can be identified with an open subset of the moduli space of simple rank-2 bundles E on S with $\det E = \mathcal{O}_S(d-1)$ and $c_2(E) = \frac{1}{6}d(d-1)(2d-1)$. Because it is smooth, this open subset is of dimension $\dim H^1(S, \mathcal{E}nd_0(E))$. Comparing with (7.7.a) gives $H^2(S, \mathcal{E}nd_0(E)) = 0$. □

(7.8) Assertion (b) of Proposition 7.6 is therefore equivalent to the fact that, on a general surface S of degree d, the moduli space of ACM rank-2 vector bundles with $\det E = \mathcal{O}_S(d-1)$ and $H^0(S, E(-1)) = 0$ is smooth and of the *expected dimension* $-\chi(\mathcal{E}nd_0(E))$ for $d \le 15$. We were not able to prove this directly except in the obvious case of $d = 4$, where the vanishing of $H^2(S, \mathcal{E}nd_0(E))$ follows from Serre duality.

8. Threefolds as Linear Pfaffians

(8.1) Let us first briefly recall Serre's construction. Let X be a projective manifold of dimension ≥ 3, and let E be a rank-2 vector bundle on X that is spanned by its global sections; put $L = \det E$. Then the zero locus of a general section s of E is a submanifold V of codimension 2 in X, and there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} E \longrightarrow \mathcal{I}_V L \longrightarrow 0;$$

it follows that K_V is isomorphic to $(K_X \otimes L)\big|_V$. Conversely, given a codimension-2 submanifold $V \subset X$ and a line bundle L on X such that $K_V \cong (K_X \otimes L)\big|_V$, there exists a rank-2 vector bundle E and a section $s \in H^0(X, E)$ such that $Z(s) = V$. Moreover, if V is connected then the pair (E, s) is uniquely determined up to isomorphism. We will refer to E as the vector bundle associated to V.

Recall that a submanifold V of $\mathbf{P}^n$ is said to be *arithmetically Cohen–Macaulay* if the sheaf $\mathcal{O}_V$ is ACM *and* V is projectively normal. This implies in particular that $H^0(V, \mathcal{O}_V) = k$, so V is connected.

PROPOSITION 8.2. *Let* X *be a smooth hypersurface of degree* d *in* $\mathbf{P}^n$ $(n = 4$ *or* $5)$. *Then the following conditions are equivalent*:

(i) X *can be defined by an equation* pf M $= 0$, *where* M *is a skew-symmetric linear* $(2d) \times (2d)$ *matrix*;

(ii) X *contains a codimension*-2 *submanifold* V *that is arithmetically Cohen–Macaulay, not contained in any hypersurface of degree* $d - 2$, *and such that* $K_V \cong \mathcal{O}_V(2d - 2 - n)$.

More precisely: *under hypothesis* (ii), *the rank*-2 *vector bundle* E *associated to* V *admits a minimal resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n}(-1)^{2d} \xrightarrow{\ M\ } \mathcal{O}_{\mathbf{P}^n}^{2d} \longrightarrow E \longrightarrow 0;$$

the variety V has degree $\frac{1}{6}d(d-1)(2d-1)$.

Proof. If (i) holds then the vector bundle E := Coker M is spanned by its global sections. Let V be the zero locus of a general section of E. Under (i) or (ii) we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow \mathcal{I}_V(d-1) \longrightarrow 0.$$

By Serre duality, E is ACM if and only if $H^i(X, E(j)) = 0$ for $1 \le i \le n - 3$; in view of the foregoing exact sequence, this is equivalent to V being arithmetically Cohen–Macaulay. Similarly, the condition $H^0(X, E(-1)) = 0$ translates as $H^0(X, \mathcal{I}(d-2)) = 0$; we conclude by Corollary 2.4.

The degree of V can be deduced for instance from (7.2) by restriction to a general 3-dimensional linear subspace. □

(8.3) Note that there do indeed exist smooth threefolds and fourfolds satisfying the equivalent conditions of Proposition 8.2. One way to see this is to consider the vector space $\mathbf{M}_{2d}^{ss}$ of skew-symmetric $(2d) \times (2d)$ matrices and the universal pfaffian hypersurface $\mathcal{X}_d \subset \mathbf{P}(\mathbf{M}_{2d}^{ss})$ consisting of singular matrices. The singular locus of $\mathcal{X}_d$ has codimension 6 and consists of those matrices that are of rank $\le 2d - 4$. Hence, for $n \le 5$, a generic $\mathbf{P}^n \subset \mathbf{P}(\mathbf{M}_{2d}^{ss})$ intersects $\mathcal{X}_d$ along a smooth hypersurface in $\mathbf{P}^n$ defined by the vanishing of a linear pfaffian.

(8.4) THE CUBIC THREEFOLD

PROPOSITION 8.5. *If* $k = \bar{k}$, *then every smooth cubic threefold can be defined by an equation* pf M $= 0$, *where* M *is a skew-symmetric linear* 6×6 *matrix*.

As mentioned in the introduction, this result is due to Adler [AR, Apx. V] in the case of a *generic* cubic threefold.

Proof. Let X be a smooth cubic threefold. In view of Proposition 8.2, we have to prove that X contains a normal elliptic quintic curve. This is essentially in [MT, Remark 4.9]; we sketch the argument since the result we need is not explicitly stated there.

We first observe that X contains a nonnormal elliptic quintic curve C (i.e., contained in a hyperplane); in fact, any smooth hyperplane section S of X contains finitely many 5-dimensional families of such curves (represent S as $\mathbf{P}^2$ blown up at six points and consider the linear system of plane cubics passing through four of the six points). Varying the hyperplane section gives an 8-dimensional family of nonnormal elliptic quintic curves in S.

Let C be one of these curves. Then the normal bundle $N_{C/V}$ fits into an exact sequence

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow N_{C/V} \longrightarrow N_{C/S} \longrightarrow 0,$$

from which one deduces $H^1(C, N_{C/V}) = 0$ and $\dim H^0(C, N_{C/V}) = 10$. Therefore, the Hilbert scheme of curves of degree 5 and arithmetic genus 0 in V is smooth of dimension 10 at C. The general member of the component containing C is a smooth elliptic quintic curve not contained in any hyperplane and thus projectively normal. □

(8.6) By Proposition 2.4, a rank-2 vector bundle E on X is associated to a normal elliptic quintic if and only if $F = E(-1)$ satisfies $\det F = \mathcal{O}_X$ and $H^0(X, F) = 0$; since $\mathrm{Pic}(X) = \mathbf{Z}$, this last condition means that F is *stable* (with respect to $\mathcal{O}_X(1)$). Let $\mathcal{M}_X$ be the moduli space of stable ACM rank-2 vector bundles on X with trivial determinant; it is smooth of dimension 5 [MT]. By a theorem of Druel [Dr], this is also the moduli space of stable rank-2 vector bundles on X with $c_1 = 0$ and $c_2 = 2\ell$, where ℓ denotes the class of a line in $H^4(X, \mathbf{Z})$; we will not need this result here.

Let us now vary X and consider the space $\mathcal{M}$ of pairs (X, F), where X is a smooth element of $|\mathcal{O}_{\mathbf{P}^4}(3)|$ and $F \in \mathcal{M}_X$. By Proposition 8.5 we have a dominant rational map from the space of linear skew-symmetric matrices $M \in \mathbf{M}_6(\mathsf{S}^4)$ onto the space $\mathcal{M}$, which is therefore *unirational*.

(8.7) Thanks to [MT], this has the following nice consequence. We now assume $k = \mathbf{C}$. Let $|\mathcal{O}_{\mathbf{P}^4}(3)|_{sm}$ be the open subset of $|\mathcal{O}_{\mathbf{P}^4}(3)|$ parameterizing smooth cubic threefolds. The intermediate Jacobians of cubic threefolds fit into a universal family $\mathcal{J} \to |\mathcal{O}_{\mathbf{P}^4}(3)|_{sm}$. More generally, for each integer k we can define a twisted intermediate Jacobian $J^k(X)$, which parameterizes 1-dimensional cycles on X with cohomology class $k\ell$; this is a principal homogeneous space under the usual intermediate Jacobian $J^0(X)$. These spaces fit into a family $\mathcal{J}^k$ over $|\mathcal{O}_{\mathbf{P}^4}(3)|_{sm}$; while each $J^k(X)$ is isomorphic to $J^0(X)$, it is not clear that $\mathcal{J}^k$ is isomorphic to $\mathcal{J}$. However, the class of a plane section is a canonical element in each $J^3(X)$, giving a section of the fibration $\mathcal{J}^3 \to |\mathcal{O}_{\mathbf{P}^4}(3)|_{sm}$; this provides canonical isomorphisms $\mathcal{J}^k \xrightarrow{\sim} \mathcal{J}^{k+3}$ above $|\mathcal{O}_{\mathbf{P}^4}(3)|_{sm}$. Note also that, for $p \in \mathbf{Z}$, the multiplication map $\mathcal{J}^k \xrightarrow{\times p} \mathcal{J}^{pk}$ is a finite étale covering, since it is so on each fibre.

COROLLARY 8.8. *The intermediate Jacobian $\mathcal{J}$ of the universal family of cubic threefolds is unirational.*

Proof. Associating to a pair (X, F) in $\mathcal{M}$, the class of $c_2(F)$ defines a morphism $\mathcal{M} \to \mathcal{J}^2$ above $|\mathcal{O}_{\mathbf{P}^4}(3)|_{sm}$. By [MT], this morphism is étale and hence dominant; thus $\mathcal{J}^2$ is unirational. Using the maps $\mathcal{J}^2 \xrightarrow{\times 3} \mathcal{J}^6 \xrightarrow{\sim} \mathcal{J}$, we conclude that $\mathcal{J}$ is unirational. □

Let us now discuss the case of higher-degree threefolds.

PROPOSITION 8.9. *Assume that k is algebraically closed. A general threefold of degree d in $\mathbf{P}^4$ can be defined by a linear pfaffian if and only if $d \le 5$.*

Proof. Let us denote again by $\mathcal{S}_d$ the space of linear skew-symmetric matrices $\mathrm{M} \in \mathbf{M}_{2d}(\mathbf{S}^4)$ such that the equation $\mathrm{pf\,M} = 0$ defines a smooth hypersurface $\mathrm{X_M} \subset \mathbf{P}^4$. As before, the group $\mathrm{GL}(2d)$ acts freely and properly on $\mathcal{S}_d$, and the map $\mathrm{pf}: \mathcal{S}_d \to |\mathcal{O}_{\mathbf{P}^4}(d)|$ factors through $\mathcal{S}_d/\mathrm{GL}(2d)$.

An easy computation gives $\dim \mathcal{S}_d/\mathrm{GL}(2d) < \dim|\mathcal{O}_{\mathbf{P}^4}(d)|$ for $d \ge 6$, so a general threefold of degree ≥ 6 is not pfaffian. For $d = 4$ and 5, one checks as in Proposition 7.6 that the differential of pf at a generic matrix is surjective (see the Appendix; for $d = 4$ this was also observed in [IM]). □

(8.10) Exactly as in Lemma 7.7, we find that the map $\mathrm{pf}: \mathcal{S}_d \to |\mathcal{O}_{\mathbf{P}^4}(d)|$ is dominant if and only if $\mathrm{H}^2(\mathrm{X_M}, \mathcal{E}nd_0(\mathrm{E_M})) = 0$ for M general in $\mathcal{S}_d$—that is, if the moduli space of the vector bundles we are considering on a general quartic or quintic threefold has the expected dimension. We see in particular that there is a finite number of ways of representing a general quintic as a pfaffian; this number is an instance of the *generalized Casson invariant* considered by Thomas [T]. It would be of course quite interesting to determine it.

9. Fourfolds as Linear Pfaffians

(9.1) Let us keep the notation of Proposition 8.9 for *fourfolds* in $\mathbf{P}^5$. We find in this case that $\dim \mathcal{S}_d/\mathrm{GL}(2d) < \dim|\mathcal{O}_{\mathbf{P}^5}(d)|$ for $d \ge 3$, so a general hypersurface of degree ≥ 3 in $\mathbf{P}^5$ cannot be defined by the vanishing of a linear pfaffian (a smooth hyperquadric can of course, since it is isomorphic to the Grassmannian of lines in $\mathbf{P}^3$ in the Plücker embedding). For $d = 3$, one finds $\dim \mathcal{S}_3/\mathrm{GL}(6) = \dim|\mathcal{O}_{\mathbf{P}^5}(3)| - 1$.

PROPOSITION 9.2. (a) *A (smooth) cubic fourfold* $\mathrm{X} \subset \mathbf{P}^5$ *is pfaffian if and only if it contains a Del Pezzo surface of degree* 5.

(b) *Assume* $k = \mathbf{C}$. *The map* $\mathrm{pf}: \mathcal{S}_3/\mathrm{GL}(6) \to |\mathcal{O}_{\mathbf{P}^5}(3)|$ *is generically injective. In particular, pfaffian cubic fourfolds form a hypersurface in the space of all smooth cubic fourfolds.*

The pfaffian cubics play a key role in the proof that the variety of lines contained in a cubic fourfold is irreducible symplectic [BD]. Cubic fourfolds containing a Del Pezzo surface of degree 5 have been considered by Fano [F].

Proof. Part (a) follows at once from Proposition 8.2, so let us prove part (b).

We introduce a 6-dimensional vector space V and the space $\mathrm{Alt(V)}$ of bilinear alternate forms on V; we will view $\mathcal{S}_3$ as an open subset of $\mathrm{Alt(V)}^6 = \mathrm{Alt(V)} \otimes_k k^6$. The map $\mathrm{pf}: \mathcal{S}_3 \to |\mathcal{O}_{\mathbf{P}^5}(3)|$ associates to a sextuple $(\varphi_0, \dots, \varphi_5)$ the hypersurface $\mathrm{pf}\left(\sum_i \mathrm{X}_i \varphi_i\right) = 0$. The group $\mathrm{GL}(6)$ acts on $\mathcal{S}_3$ through its action on k^6; this action commutes with the action of $\mathrm{GL(V)}$, and the map $\mathrm{pf}: \mathcal{S}_3/\mathrm{GL(V)} \to |\mathcal{O}_{\mathbf{P}^5}(3)|$ is

$GL(6)$-equivariant. The orbits of $GL(6)$ in $\mathcal{S}_3$ correspond to 6-dimensional subspaces $L \subset Alt(V)$; to such a subspace is associated the isomorphism class of the cubic hypersurface X_L of degenerate forms in $\mathbf{P}(L)$. Since the action of $GL(6)$ is generically free on $|\mathcal{O}_{\mathbf{P}^5}(3)|$, it is sufficient to prove that the isomorphism class of X_L determines L (up to the action of $GL(V)$).

The orthogonal $L^\perp$ of L in $\bigwedge^2 V$ is 9-dimensional; the locus of rank-2 bivectors in $\mathbf{P}(L^\perp)$ is a K3 surface S of genus 8 [BD]. By [M], a general K3 surface of genus 8 is obtained in this way, and this representation is unique: the surface S determines the space $L^\perp \subset \bigwedge^2 V$ (and therefore also the space $L \subset Alt(V)$) up to the action of $GL(V)$. So what we need to prove is that *the cubic* X_L *determines the* K3 *surface* S up to projective isomorphism.

We proved in [BD] that the variety F of lines contained in X_L is a (complex) symplectic manifold, isomorphic to the Hilbert scheme $S^{[2]}$. In particular, the group $H^2(F, \mathbf{Z})$ carries a canonical quadratic form, and there is a Hodge isometry

$$H^2(F, \mathbf{Z}) \xrightarrow{\sim} H^2(S, \mathbf{Z}) \overset{\perp}{\oplus} \mathbf{Z}\delta,$$

where $H^2(S, \mathbf{Z})$ is endowed with the intersection form and δ is a class of type $(1, 1)$ and square -2. The polarization of F given by the embedding in the Grassmannian $\mathbf{G}(2, 6)$ corresponds to the class $2l - 5\delta$, where l is the polarization on S deduced from the embedding $S \subset \mathbf{P}(L^\perp)$.

Let L and L' be two subspaces of $Alt(V)$ that produce isomorphic cubics; let (S, l) and (S', l') be the corresponding polarized K3 surfaces. We then have a Hodge isometry

$$\varphi\colon H^2(S, \mathbf{Z}) \overset{\perp}{\oplus} \mathbf{Z}\delta \xrightarrow{\sim} H^2(S', \mathbf{Z}) \overset{\perp}{\oplus} \mathbf{Z}\delta',$$

which maps the class $2l - 5\delta$ to the corresponding class $2l' - 5\delta'$. Assume that $\mathrm{Pic}(S) = \mathbf{Z}l$. Then we have $\mathrm{Pic}(S') = \mathbf{Z}l'$, and φ induces an isometry $\mathbf{Z}l \oplus \mathbf{Z}\delta \xrightarrow{\sim} \mathbf{Z}l' \oplus \mathbf{Z}\delta'$, which maps $2l - 5\delta$ onto $2l' - 5\delta'$; an easy computation shows that this implies $\varphi(\delta) = \varphi(\delta')$. Thus φ induces a Hodge isometry of $H^2(S, \mathbf{Z})$ onto $H^2(S', \mathbf{Z})$ mapping l to l'. By the Torelli theorem for K3 surfaces, this implies that (S, l) and (S', l') are isomorphic. □

Appendix:
Hypersurfaces Are Generically Pfaffian in the Expected Range

Frank-Olaf Schreyer

We prove by a Macaulay 2 computation that a generic surface of degree $d \le 15$ in $\mathbf{P}^3$, as well as a general threefold of degree $d \le 5$ in $\mathbf{P}^4$, can be defined by the pfaffian of a skew-symmetric $2d \times 2d$ matrix with linear entries (Propositions 7.6 and 8.9 in the text). As explained in the text, it is sufficient to prove that, for some matrix M of this type, the space of homogeneous forms of degree d is equal to $\mathfrak{m} \cdot \texttt{pfaffians}(2d-2, M)$, where $\mathfrak{m}$ is the ideal spanned by the coordinates and $\texttt{pfaffians}(2d-2, M)$ the ideal of submaximal pfaffians of M. We compute the dimension of the latter space at randomly chosen skew symmetric matrices over a finite field using Macaulay 2 [GS]. The computation is within the range of contemporary computers. On the computer "alice" of the Mathematical Science Research Institute at Berkeley, the following code was executed in about two hours of cpu time. The output verifies the result.

```
isPrime(31991)
  kk=ZZ/31991 – this is a field

randomSkewMatrix = (e,S) -> (
  -- returns a random e x e skew symmetric matrix
  -- with linear entries in the ring S
  N:=binomial(e,2);
  R:=kk[t_0..t_(N-1)];
  G:=genericSkewMatrix(R,t_0,e);
  substitute(G,random(S^{0},S^{N:-1}))
  ) -- end randomSkewMatrix

subPfaffiansViaSyzygies = (M) -> (
  -- This is an alternative to the command pfaffians(2d-2,M).
  -- It returns the generators of the ideal of the 2d-2 pfaffians
  -- of the linear 2d x 2d skew symmetric matrix M computed
  -- using the structure theorem of [B-E]:
  -- Under a mild genericity condition on the submatrix M1
  -- the syzygies of the 2d-1 x 2d-1 skew matrix M1 are its 2d-1
  -- principal pfaffians.
  -- If the computation fails, then the standard way is used.
  d:=lift((rank source M)/2,ZZ);
  syzygiesGivePfaffians=true; i:=0; S:=ring M;
  J:=generators ideal0_S;
  while syzygiesGivePfaffians==true and (i<(2*d)) do (
    -- take i-th 2d-1 x 2d-1 skew submatrix
    M1:=transpose((transpose(M_{0..(i-1),(i+1)..(2*d-1)}))
    _{0..(i-1),(i+1)..(2*d-1)});
    N1:=syz(M1,DegreeLimit=>d);
    syzygiesGivePfaffians=((degrees source N1) == {{d}});
    if syzygiesGivePfaffians==true then
    J=(J|flatten(N1));
    i=i+1;
  );
  if syzygiesGivePfaffians then (mingens image J)
  else (mingens image pfaffians(2*d-1,M))
  ) -- end subPfaffiansViaSyzygies

isDominant=(r,d) -> (
  S:=kk[x_0..x_r]; M:=randomSkewMatrix(2*d,S);
  J:=subPfaffiansViaSyzygies(M);
```

```
  N=syz(J,DegreeLimit=>d);
  -- DegreeLimit=> d is carefully chosen to compute only
  -- linear sysygies.  From this the number of kk-linear
  -- independent elements of degree d in the ideal
  -- with generated by J can be computed:
  cd=binomial(d+r,r)-(r+1)*rank(target N)+(rank source N);
  cd==0) -- end isDominant

lowerBoundForDominantDegree = (r) -> (
  dominant:=true; d:=2;
  while dominant do
  (d=d+1;dominant=isDominant(r,d););
  d-1)

isDominant(5,3)
  cd
  time d4=lowerBoundForDominantDegree(4)
  time d3=lowerBoundForDominantDegree(3)
```

Note that we used the method to compute pfaffians via syzygies, since this is faster than the command `pfaffians(2*d-2,M)`. The reason is that syzygy computations are fast whereas the `pfaffian` command does not utilize much special structure. For comments on the commands and the Macaulay 2 language, refer to the on-line help.

Notice that the computation also shows that the closure of the scheme of pfaffian cubic fourfolds form a hypersurface in $|\mathcal{O}_{\mathbf{P}^5}(3)|$.

References

[AR] A. Adler and S. Ramanan, *Moduli of Abelian varieties*, Lecture Notes in Math., 1644, Springer-Verlag, Berlin, 1996.

[B1] A. Beauville, *Variétés de Prym et jacobiennes intermédiaires*, Ann. Sci. École Norm. Sup. (4) 10 (1977), 309–391.

[B2] ———, *Le groupe de monodromie des familles universelles d'hypersurfaces et d'intersections complètes*, Lecture Notes in Math., 1194, pp. 8–18, Springer-Verlag, Berlin, 1986.

[B3] ———, *Jacobiennes des courbes spectrales et systèmes hamiltoniens complètement intégrables*, Acta Math. 164 (1990), 211–235.

[BD] A. Beauville and R. Donagi, *La variété des droites d'une hypersurface cubique de dimension 4*, C. R. Acad. Sci. Paris Sér. I Math. 301 (1985), 703–706.

[BE] D. Buchsbaum and D. Eisenbud, *Algebra structure on finite free resolutions and some structure theorem for ideals of codimension 3*, Amer. J. Math. 99 (1977), 447–485.

[Bo] N. Bourbaki, *Algèbre*, ch. 10 (algèbre homologique), Masson, Paris, 1980.

[C1] F. Catanese, *Babbage's conjecture, contact of surfaces, symmetric determinantal varieties and applications*, Invent. Math. 63 (1981), 433–465.

[C2] ———, *Homological algebra and algebraic surfaces*, Proc. Sympos. Pure Math., 62, pp. 3–56, Amer. Math. Soc., Providence, RI, 1997.

[Ca] A. Cayley, *A memoir on quartic surfaces,* Proc. London Math. Soc. 3 (1869–71), 19–69.

[CT] R. J. Cook and A. D. Thomas, *Line bundles and homogeneous matrices,* Quart. J. Math. Oxford Ser. (2) 30 (1979), 423–429.

[DGO] E. D. Davis, A. V. Geramita, and F. Orecchia, *Gorenstein algebras and the Cayley–Bacharach theorem,* Proc. Amer. Math. Soc. 93 (1985), 593–597.

[D] L. E. Dickson, *Determination of all general homogeneous polynomials expressible as determinants with linear elements,* Trans. Amer. Math. Soc. 22 (1921), 167–179.

[Di] A. C. Dixon, *Note on the reduction of a ternary quantic to a symmetric determinant,* Proc. Cambridge Phil. Soc. 11 (1902), 350–351.

[Dr] S. Druel, *Espace des modules des faisceaux semi-stables de rang 2 et de classes de Chern $c_1 = 0$, $c_2 = 2$ et $c_3 = 0$ sur une hypersurface cubique lisse de $\mathbf{P}^4$.* Internat. Math. Res. Notices (to appear).

[F] G. Fano, *Sulle forme cubiche dello spazio a cinque dimensioni contenenti rigate razionale del 4° ordine,* Comment. Math. Helv. 15 (1943), 71–80.

[G] H. Grassman, *Die stereometrischen Gleichungen dritten grades, und die dadurch erzeugten Oberflächen,* J. Reine Angew. Math. 49 (1855), 47–65.

[GS] D. Grayson and M. Stillman, *Macaulay 2,* http://www.math.uiuc.edu/Macaulay2/.

[GH] P. Griffiths and J. Harris, *Residues and zero-cycles on algebraic varieties,* Ann. of Math. (2) 108 (1978), 461–505.

[H] O. Hesse, *Ueber determinanten und ihre anwendung in der geometrie, insbesondere auf Curven vierter ordnung,* J. Reine Angew. Math. 49 (1855), 243–264.

[IM] A. Iliev and D. Markushevich, *Quartic 3-fold: Pfaffians, instantons and half-canonical curves,* Doc. Math. 5 (2000), 23–47.

[L] L. Le Bruyn, *Centers of generic division algebras, the rationality problem 1965–1990,* Israel J. Math. 76 (1991), 97–111.

[MT] D. Markushevich and A. Tikhomirov, *The Abel–Jacobi map of a moduli component of vector bundles on the cubic threefold,* J. Algebraic Geom. (to appear).

[M] S. Mukai, *Curves, K3 surfaces and Fano 3-folds of genus* ≤ 10, Algebraic geometry and commutative algebra I, pp. 357–377, Kinokuniya, Tokyo, 1988.

[Mu] D. Mumford, *Lectures on curves on an algebraic surface,* Ann. of Math. Stud., 59, Princeton Univ. Press, Princeton, NJ, 1966.

[S] F. Schur, *Ueber die durch collineare Grundgebilde erzeugten Curven und Flächen,* Math. Ann. 18 (1881), 1–32.

[Se] B. Segre, *On the rational solutions of homogeneous cubic equations in four variables,* Math. Notae 11 (1951), 1–68.

[T] R. P. Thomas, *A holomorphic Casson invariant for Calabi–Yau 3-folds, and bundles on K3 fibrations,* preprint, www.arXiv.org/math/9806111.

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