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Jacobians among abelian threefolds: a geometric approach

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Abstract Let (A, θ) be a principally polarized abelian threefold over a perfect field k, not isomorphic to a product over $\bar{k}$. There exists a canonical extension k'/k, of degree ≤ 2, such that (A, θ) becomes isomorphic to a Jacobian over k'. The aim of this note is to give a geometric construction of this extension.

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1 Introduction

Let (A, θ) be a principally polarized abelian variety of dimension 3 over a field k. If k is algebraically closed, (A, θ) is the Jacobian variety of a curve C (or a product of Jacobians). If k is an arbitrary perfect field the situation is more subtle (see Proposition 3 below): there is still a curve C defined over k, but either (A, θ) is isomorphic to JC, or they become isomorphic only after a quadratic extension k' of k, uniquely determined by (A, θ).

Now given (A, θ), how can we decide if it is a Jacobian, and more precisely determine the extension k'/k? For $k \subset \mathbb{C}$, a solution is given in [9] in terms of modular forms. Here we propose a geometric approach, based on a construction of Recillas. We have to make the following assumptions:

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– A admits a theta divisor Θ defined over k;
– There exists a point $a \in A(k)$ outside Σ_A,

where $\Sigma_A \subset A$ is an explicit divisor containing 0, which we will define in Sect. 3.

We put $\tilde{X}_a := \Theta \cap (\Theta + a)$. This is a curve defined over k, and the second assumption guarantees that it is smooth. There exists $b \in A(k)$ such that the involution $\iota : z \mapsto b - z$ exchanges Θ and $\Theta + a$, hence acts on $\tilde{X}_a$. This action is free; the quotient $X_a := \tilde{X}_a/\iota$ is a non-hyperelliptic genus 4 curve, whose canonical model lies on a unique quadric $Q \subset \mathbb{P}^3$. Then for $\text{char}(k) \neq 2$ the extension k' is $k(\sqrt{\text{disc}(Q)})$ (we will give more detailed statements in Sect. 3).

The proof has two steps. We consider first the case where (A, θ) is a Jacobian, and prove that in that case the quadric Q is k-isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (Sect. 2). Then we treat the case where (A, θ) is *not* a Jacobian, and prove that the nontrivial automorphism of the extension k'/k exchanges the two rulings of Q (Sect. 3); this is enough to prove the theorem.

2 Recillas' construction

Throughout the paper we work over a perfect field k.

In this section we fix a non-hyperelliptic curve C of genus 3 (that is, a smooth plane quartic curve), defined over k. We will denote by K its canonical class. We assume that the principal polarization of JC can be defined by a theta divisor Θ *defined over* k – we do not assume that Θ is symmetric. This is equivalent to the existence of a divisor class $D \in JC^2(k)$ such that Θ is the image of $\text{Sym}^2 C - D$ in JC. Note that since C is not hyperelliptic, Θ is smooth and the map $E \mapsto E - D$ induces an isomorphism of $\text{Sym}^2 C$ onto Θ.

We assume that $JC(k)$ contains a point $a \neq 0$, and consider the curve $\tilde{X}_a := \Theta \cap (\Theta + a)$. Put $b = K + a - 2D \in JC(k)$; we have $-\Theta = \Theta + a - b$. The involution $z \mapsto b - z$ exchanges Θ and $\Theta + a$, hence induces an involution ι of $\tilde{X}_a$. We define a divisor $\Sigma_{JC} \subset JC$ by $\Sigma_{JC} = \Sigma'_{JC} \cup \Sigma''_{JC}$, where

$$\Sigma'_{JC} = \{2E - K \mid E \in \text{Sym}^2 C\} \quad \text{and} \quad \Sigma''_{JC} = C - C.$$

Proposition 1 *The curve $\tilde{X}_a := \Theta \cap (\Theta + a)$ is smooth and connected if and only if $a \in JC \smallsetminus \Sigma_{JC}$. If this is the case, the involution ι of $\tilde{X}_a$ is fixed point free.*

Proof Throughout the paper it will be convenient to use the following notation: given a divisor class d of degree 2 on C with $h^0(\mathcal{O}_C(d)) = 1$, we denote by $\langle d \rangle \in \text{Sym}^2 C$ the unique effective divisor in the class d.

Let $z \in \tilde{X}_a$. By [7, thm. 2], the tangent space $\mathbb{P}T_z(\Theta) \subset \mathbb{P}T_z(JC) = \mathbb{P}^2$ is identified with the line spanned by the divisor $\langle D + z \rangle \in \text{Sym}^2 C$. Similarly $\mathbb{P}T_z(\Theta + a)$ is identified with the line spanned by the divisor $\langle D + z - a \rangle \in \text{Sym}^2 C$; the intersection $\tilde{X}_a$ is singular at z if and only if these two lines coincide. If this happens, then either

- the two divisors $\langle D + z \rangle$ and $\langle D + z - a \rangle$ have a common point, which implies $a \in C - C$; or
- $\langle D + z \rangle + \langle D + z - a \rangle \sim K$, which implies $a \in \Sigma'_{JC}$.

Conversely, if $a \in \Sigma'_{JC}$, we have $K + a \sim 2E$ with $E \in \mathrm{Sym}^2 C$; then $z = E - D$ is a singular point of $\tilde{X}_a$. If $a \sim p - q$, with $p, q \in C$, the intersection $\tilde{X}_a$ is reducible, equal to $(C + p - D) \cup (K - D - q - C)$.

Assume now $a \notin \Sigma_{JC}$, so that $\tilde{X}_a$ is smooth; the exact sequence

$$\begin{aligned} 0 \;\to\; &\mathcal{O}_{JC}(-\Theta - (\Theta + a)) \longrightarrow \mathcal{O}_{JC}(-\Theta) \oplus \mathcal{O}_{JC}(-(\Theta + a)) \longrightarrow \\ &\longrightarrow \mathcal{O}_{JC} \longrightarrow \mathcal{O}_{\tilde{X}_a} \to 0 \end{aligned}$$

gives $h^0(\mathcal{O}_{\tilde{X}_a}) = 1$, hence $\tilde{X}_a$ is connected. If a point $z \in \tilde{X}_a$ is fixed by ι it satisfies $2(D + z) \sim K + a$, which implies $a \in \Sigma'_{JC}$. □

We assume from now on $a \notin \Sigma_{JC}$. We denote by X_a the quotient curve $\tilde{X}_a/\iota$. The adjunction formula gives

$$K_{\tilde{X}_a} \sim (\Theta + (\Theta + a))_{|\tilde{X}_a} = 2\Theta^3 = 12\,, \quad \text{hence} \quad g(\tilde{X}_a) = 7 \;\text{ and }\; g(X_a) = 4.$$

If $\mathrm{char}(k) \neq 2$, the principally polarized abelian variety JC is canonically isomorphic to the Prym variety associated to the étale double covering $\tilde{X}_a \to X_a$ ([2, Sect. 3.b], [13]).

We embed $\tilde{X}_a$ into $\mathrm{Sym}^2 C \times \mathrm{Sym}^2 C$ by

$$z \mapsto (\langle D + z\rangle, \langle K + a - (D + z)\rangle).$$

Then $\tilde{X}_a$ is identified with $s^{-1}(|K + a|)$, where

$$s : \mathrm{Sym}^2 C \times \mathrm{Sym}^2 C \to \mathrm{Sym}^4 C$$

is the sum map. The involution ι is induced by the involution of the product $\mathrm{Sym}^2 C \times \mathrm{Sym}^2 C$ which exchanges the factors. The map

$$s : \tilde{X}_a \to |K + a|$$

factors through ι, hence induces a 3-to-1 map $t : X_a \to |K + a|$ $(\cong \mathbb{P}^1)$. The fibre of t above $E \in |K + a|$ parametrizes the decompositions $E = d + d'$, with d, d' in $\mathrm{Sym}^2 C$.

We now consider the involution $(d, d') \mapsto (\langle K - d\rangle, \langle K - d'\rangle)$ of $\mathrm{Sym}^2 C \times \mathrm{Sym}^2 C$; it maps $\tilde{X}_a$ onto $s^{-1}(|K - a|) = \tilde{X}_{-a}$ and commutes with ι, hence induces an isomorphism $X_a \xrightarrow{\sim} X_{-a}$. By composition with the map $X_{-a} \to |K - a|$ defined above we obtain another degree 3 map $t' : X_a \to |K - a|$.

The maps t and t' are defined over k; they define two g^1_3 on X_a, that is, two linear series of degree 3 and projective dimension 1, defined over k.

Lemma 1 *The two g^1_3 defined by t and t' on X_a are distinct.*

Proof Let us first observe that the degree 4 morphism $f : C \to \mathbb{P}^1$ defined by the linear system $|K+a|$ is separable. If this is not the case, we have $\mathrm{char}(k) = 2$ and f factors as $C \xrightarrow{F} C_1 \xrightarrow{g} \mathbb{P}^1$, where C_1/k is the pull back of C/k by the automorphism $\lambda \mapsto \lambda^2$ of k, F is the Frobenius k-morphism and g is separable of degree 2 (see [6, IV.2]). But then C_1 is hyperelliptic, hence also C.

Assume that the two linear series are the same. By the previous observation there exists a divisor $E = p+q+r+s$ in $|K+a|$ consisting of 4 distinct points. There must exist $E' \in |K-a|$ such that $t^{-1}(E') = t'^{-1}(E)$. This means that for each decomposition $E = d+d'$ with d, d' in $\mathrm{Sym}^2 C$, we have $E' = \langle K-d\rangle + \langle K-d'\rangle$.

Let us write $\langle K-p-q\rangle = p'+q'$ and $\langle K-r-s\rangle = r'+s'$, so that $E' = p'+q'+r'+s'$. We must have $E' = \langle K-p-r\rangle + \langle K-q-s\rangle$, so we can suppose $\langle K-p-r\rangle = p'+r'$. Then $K-p-p' \sim q+q' \sim r+r'$, which implies $r' = q$, $q' = r$. But then we get $K-p-q-r \sim p'$ and $a \sim s-p'$, which contradicts the hypothesis $a \notin \Sigma_{JC}$. □

We can now conclude:

Proposition 2 *For $a \notin \Sigma_{JC}$, the genus 4 curve X_a is not hyperelliptic; the unique quadric $Q \subset \mathbb{P}^3$ containing its canonical model is smooth and split over k (that is, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ over k).*

Proof Since X_a admits a base point free g_3^1 it cannot be hyperelliptic [1, p. 13]. Let us denote the two distinct g_3^1 of X_a by $|E|$ and $|E'|$. We have $E+E' \sim K_{X_a}$; by the base-point free pencil trick [1, p. 126], the multiplication map $H^0(X_a, E) \otimes H^0(X_a, E') \to H^0(X_a, K_{X_a})$ is an isomorphism. Thus the canonical map of X_a is the composition of $(t, t') : X_a \to \mathbb{P}^1 \times \mathbb{P}^1$ and of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. The genus 4 canonical curve $X_a \subset \mathbb{P}^3$ of genus 4 is contained in a unique quadric, therefore this quadric is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. □

Remark 1 If C is hyperelliptic, Proposition 1 still holds, with essentially the same proof. However Lemma 1 fails: in fact, we have $t' = \sigma \circ t$, where $\sigma : |K+a| \xrightarrow{\sim} |K-a|$ is induced by the hyperelliptic involution. Actually in that case X_a has a unique g_3^1, at least if $\mathrm{char}(k) \neq 2$. Indeed Θ has a singular point, given by the g_2^1 of C; on the other hand JC is isomorphic to the Prym variety of $\tilde{X}_a/X_a$. By [11, Sect. 7, Thm. (c)], this happens if and only if X_a admits a unique g_3^1.

Remark 2 The divisor Σ'_{JC} is equal to $\mathbf{2}_*\Xi$, where $\mathbf{2}$ is the endomorphism $z \mapsto 2z$ of JC and Ξ is any symmetric theta divisor; thus it can be defined on any absolutely indecomposable principally polarized abelian threefold (A, θ), with no reference to the isomorphism $A \xrightarrow{\sim} JC$. The same holds for Σ''_{JC} provided $\mathrm{char}(k) \neq 2$. Recall indeed that there is a canonical linear system on A, denoted $|2\theta|$, which contains the double of each symmetric theta divisor. Then:

Lemma 2 *If $\mathrm{char}(k) \neq 2$ and the curve C is not hyperelliptic, the divisor $\Sigma''_{JC} = C - C$ is the unique divisor in $|2\theta|$ with multiplicity ≥ 4 at 0.*

This is quite classical if $k = \mathbb{C}$, see [5]. We do not know whether it still holds when $\mathrm{char}(k) = 2$.

Proof The difference map $C\times C \to C-C$ is an isomorphism outside the diagonal Δ, and contracts Δ to 0; therefore the multiplicity of $C-C$ at 0 is $-\Delta^2=4$.

Let us prove the unicity; we may assume $k=\bar{k}$. We denote by $|2\theta|_0$ the subspace of elements of $|2\theta|$ containing 0. The multiplicity at 0 of an element of $|2\theta|$ is even: this follows from the "inverse formula" of [10, p. 331]. Thus we have a projective linear map $\tau : |2\theta|_0 \dashrightarrow |\mathcal{O}_{\mathbb{P}^2}(2)|$ which associates to a divisor its quadratic tangent cone at 0. Since $\dim |2\theta|_0 = 6$ and $\dim |\mathcal{O}_{\mathbb{P}^2}(2)| = 5$, it suffices to prove that τ is surjective. For each $E \in \mathrm{Sym}^2 C$, the divisor $(\mathrm{Sym}^2 C - E) + (\mathrm{Sym}^2 C - (K-E))$ belongs to $|2\theta|_0$; by [7, thm. 2], its tangent cone at 0 is twice the line in $\mathbb{P}^2$ spanned by E. Since the double lines span the space of conics, τ is surjective. □

3 The main result

In this section we fix a principally polarized abelian threefold (A,θ) over k. We assume that it is absolutely indecomposable, that is, (A,θ) is not isomorphic over $\bar{k}$ to a product of two principally polarized abelian varieties. It is equivalent to say that the theta divisor of A is irreducible (over $\bar{k}$), or that $(A,\theta)_{\bar{k}}$ is isomorphic to the Jacobian of a curve [12]. This does not imply that (A,θ) itself is a Jacobian; indeed we have [14]:

Proposition 3 *There exists a curve C over k and a character $\varepsilon_A : \mathrm{Gal}(\bar{k}/k) \to \{\pm 1\}$, uniquely determined, such that (A,θ) is k-isomorphic to JC twisted by ε_A. If C is hyperelliptic, ε_A is trivial.*

Remark 3 In more down-to-earth terms this means the following. Let k' be the extension of k defined by the character ε_A. Then:

- if $\varepsilon_A = 1$ (that is, $k'=k$), JC is k-isomorphic to (A,θ). This is the case if C is hyperelliptic.
- if $\varepsilon_A \neq 1$ (that is, k' is a quadratic extension of k), JC is isomorphic to (A,θ) over k' *but not over* k. More precisely, let σ be the nontrivial automorphism of k'/k; there exists an isomorphism $\varphi : (A,\theta) \to JC$ defined over k' such that ${}^{\sigma}\varphi = -\varphi$.

Our aim is to describe geometrically the character ε_A. We will compare it to the character associated to a smooth quadric $Q \subset \mathbb{P}^3_k$ in the following way: such a quadric admits two rulings defined over $\bar{k}$, so the action of $\mathrm{Gal}(\bar{k}/k)$ on these rulings provides a character $\varepsilon_Q : \mathrm{Gal}(\bar{k}/k) \to \{\pm 1\}$. We will describe this character in more concrete terms below.

We define the divisor $\Sigma_A = \Sigma'_A \cup \Sigma''_A$ on A as in Remark 2: we put $\Sigma'_A = \mathbf{2}_*\Delta$ for any symmetric theta divisor Δ; if $\mathrm{char}(k)\neq 2$, Σ''_A is the unique divisor in $|2\theta|$ with multiplicity ≥ 4 at 0. An alternative definition, which works in all characteristics, is as follows: we choose an isomorphism $A \xrightarrow{\sim} JC$ over k' and put $\Sigma''_A = \varphi^{-1}(C-C)$. Since $C-C$ is symmetric this definition does not depend on the choice of φ.

We make the following assumptions:

– A admits a theta divisor Θ defined over k;
– There exists a point $a \in A(k)$ outside Σ_A.

The theta divisors of (A, θ) form a torsor under A, and the first assumption means that this torsor is trivial. Let us observe that this is automatic when k is finite, since then any torsor under A is trivial by a theorem of Lang [8].

If Θ is singular, C is hyperelliptic, hence $A \cong JC$ by Proposition 3. Thus we may assume that Θ is smooth.

The divisor $-\Theta$ is in the class of the polarization θ, hence there is a unique $b \in A(k)$ such that $(-\Theta) + b = \Theta + a$; the involution $z \mapsto b - z$ exchanges Θ and $\Theta + a$.

Theorem 1 *Let X_a be the quotient of the curve $\Theta \cap (\Theta + a)$ by the involution $z \mapsto b - z$. Then X_a is a smooth curve of genus 4, non hyperelliptic. Its canonical model lies in a smooth quadric $Q \subset \mathbb{P}^3$, and we have $\varepsilon_A = \varepsilon_Q$.*

Proof Following Remark 3, we choose an isomorphism $\varphi : (A, \theta) \to JC$ defined over k'. It induces an isomorphism of $\tilde{X}_a$ onto the corresponding curve $\tilde{X}_{\varphi(a)} \subset JC$, hence of X_a onto $X_{\varphi(a)}$. By remark 2 φ maps Σ_A onto Σ_{JC}, thus $\varphi(a) \notin \Sigma_{JC}$; then Proposition 2 tells us that X_a is not hyperelliptic and that its canonical model is contained in a unique smooth quadric $Q \subset \mathbb{P}^3$ which is split over k'. This means that the character ε_Q is trivial on the subgroup $\mathrm{Gal}(\bar{k}/k')$ of $\mathrm{Gal}(\bar{k}/k)$; in other words, ε_Q is either trivial or equal to ε_A.

It remains to prove that ε_Q is nontrivial when $k' \neq k$, that is, the nontrivial automorphism σ of k'/k exchanges the two rulings of Q, or equivalently the two g^1_3 of X_a.

We have ${}^\sigma\varphi = -\varphi$ (Remark 3). We write as before $\varphi(\Theta) = \mathrm{Sym}^2 C - D$; we observe that ${}^\sigma(\varphi(\Theta)) = -\varphi(\Theta)$, hence ${}^\sigma D \sim K - D$. Recall that the maps $t : X_a \to |K + \varphi(a)|$ and $t' : X_a \to |K - \varphi(a)|$ defining the two g^1_3 are given by

$$t(\bar{z}) = \langle D + \varphi(z) \rangle + \langle K - D - \varphi(z) + \varphi(a) \rangle$$
$$t'(\bar{z}) = \langle K - D - \varphi(z) \rangle + \langle D + \varphi(z) - \varphi(a) \rangle ,$$

where z is a point of $\tilde{X}_a$ and $\bar{z}$ its image in X_a.

Using ${}^\sigma\varphi(z) = -\varphi({}^\sigma z)$ and ${}^\sigma D \sim K - D$ we get

$${}^\sigma t(\bar{z}) = \langle K - D - \varphi({}^\sigma z) \rangle + \langle D + \varphi({}^\sigma z) - \varphi(a) \rangle = t'({}^\sigma \bar{z});$$

thus σ exchanges t and t', hence the two rulings of Q. □

One can describe the extension k'/k (hence the character ε_Q) using the even Clifford algebra $C^+(Q)$ [4]: its center is isomorphic to k' if $k' \neq k$ and to $k \times k$ otherwise. From the description of this center (see [3, Sect. 9, no. 4, Remarque 2]), we obtain:

Proposition 4 *Assume* $\mathrm{char}(k) \neq 2$, *and let $\delta \in k^*$ be the discriminant of Q (well defined* mod. k^{*2}*). The extension k' is isomorphic to $k(\sqrt{\delta})$.*

Similarly, if $\mathrm{char}(k) = 2$, we have $k' = k(\lambda)$ with $\lambda^2 + \lambda = \Delta$, where Δ is the *pseudo-discriminant* of Q [3, Sect. 9, exerc. 9].

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