On the stability of the direct image of a generic vector bundle

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Introduction

We discuss in this note the following conjecture:

Conjecture .– *Let* $\pi : \mathrm{X}' \rightarrow \mathrm{X}$ *be a finite morphism between smooth projective curves, and* L *a generic vector bundle on* X' *. The vector bundle* $\pi_*\mathrm{L}$ *is stable if* $g(\mathrm{X}) \geq 2$ *, semi-stable if* $g(\mathrm{X}) = 1$ *.*

I do not have a strong motivation towards the conjecture, except that it is a rather natural statement. As we will see below, the crucial case is when L is a line bundle; the (easy) case when π is a double covering was used in [B] to control the theta divisor on the moduli space of rank 2 vector bundles on X . One may hope that a proof of the conjecture would lead to a better understanding of the theta linear system in arbitrary rank.

We have only partial results in the direction of the conjecture: we will show that stability holds with respect to sub-bundles of small degree (§ 1), for small values of $\chi(\mathrm{L})$ (§ 2), or when π is étale (§ 3).

1. General remarks

(1.1) It is of course sufficient to prove the conjecture for *one* vector bundle with the same rank and degree as L . Let $\pi' : \mathrm{X}'' \rightarrow \mathrm{X}'$ be an étale covering of degree $\mathrm{rk}\,\mathrm{L}$, and M a general line bundle on X'' of degree $\deg \mathrm{L}$. Then $\pi'_*\mathrm{M}$ has same rank and degree as L ; so our conjecture holds if it holds for line bundles on X'' w.r.t. the covering $\pi \circ \pi'$. Therefore it is enough to prove the conjecture in the case L is a line bundle.

(1.2) From now on we suppose that L is a line bundle. We denote by r the degree of the covering π , so that $\pi_*\mathrm{L}$ is a rank r vector bundle; we denote by g the genus of X and by g' the genus of X' .

The assertion depends only of course on the degree of L , and actually on the degree of L (mod. r), where r is the degree of the covering π : this is because the (semi-) stability of $\pi_*\mathrm{L}$ is equivalent to that of $\pi_*(\mathrm{L} \otimes \pi^*\mathrm{M})$ for any line bundle M on X . Moreover, the duality isomorphism $\pi_*(\mathrm{K}_{\mathrm{X}'} \otimes \mathrm{L}^{-1}) \cong \mathrm{K}_\mathrm{X} \otimes (\pi_*\mathrm{L})^*$ implies that the conjecture is true for $\chi(\mathrm{L}) = n$ if and only if it is true for $\chi(\mathrm{L}) = -n$.

(1.3) The weaker conclusion of the Conjecture in the case $g = 1$ is due to the fact that there are no stable bundles of rank r and degree d on an elliptic curve if

r and d are not coprime. In the case $\mathrm{X}=\mathbf{P}^1$ the analogous statement would be false for the same reason; the best one can hope for is the following:

For a generic vector bundle L *on* X', $\pi_*\mathrm{L}$ *is "almost stable", that is of the form* $\mathcal{O}_{\mathbf{P}^1}(a-1)^{\oplus p}\oplus\mathcal{O}_{\mathbf{P}^1}(a)^{\oplus q}$ *for some integers* a,p,q.

This is actually quite easy: by (1.1) we may assume that L is a line bundle. Put $\mathcal{O}_{\mathrm{X}'}(1)=\pi^*\mathcal{O}_{\mathbf{P}^1}(1)$. Let a be the integral part of $(\deg\mathrm{L}-g(\mathrm{X}'))/r$. We have $\deg\mathrm{L}(-a-1)<g(\mathrm{X}')\le\deg\mathrm{L}(-a)$, hence for L general enough $\mathrm{H}^0(\mathrm{X}',\mathrm{L}(-a-1))=\mathrm{H}^1(\mathrm{X},\mathrm{L}(-a))=0$. Therefore $\mathrm{H}^0(\mathbf{P}^1,\pi_*\mathrm{L}\,(-a-1))=\mathrm{H}^1(\mathbf{P}^1,\pi_*\mathrm{L}\,(-a))=0$, which is equivalent to our assertion.

2. Sub- and quotient line bundles

Proposition 2.1 .– *If* L *is general, any sub-line bundle* (resp. *quotient bundle*) M *of* $\pi_*\mathrm{L}$ *satisfies*

$$\mu(\mathrm{M})\le\mu(\pi_*\mathrm{L})-(1-\frac{1}{r})(g-1)$$

$$\big(\text{resp.}\quad \mu(\mathrm{M})\ge\mu(\pi_*\mathrm{L})+(1-\frac{1}{r})(g-1)\,\big)\,.$$

An interesting feature of this result is that it is the best possible: by [L] or [H], any vector bundle E of rank r contains a sub-line bundle of degree $\le[\mu(\mathrm{E})-(1-\frac{1}{r})(g-1)]$, where $[x]$ denotes the integral part of x. So $\pi_*\mathrm{L}$ is "as stable as possible" with respect to sub- and quotient line bundles.

Proof: Let M be a sub-line bundle of $\pi_*\mathrm{L}$; put $\deg(\mathrm{M})=m$ and $\deg(\mathrm{L})=d$. The condition $\mathrm{M}\subset\pi_*\mathrm{L}$ means that L can be written as $\pi^*\mathrm{M}(\mathrm{D})$, where D is an effective divisor, of degree $d-rm$. The locus of these line bundles has dimension $\le g+d-rm$; if L is generic we have $g+d-rm\ge g'$, that is $rm\le g-1+\chi(\mathrm{L})$ or $\mu(\mathrm{M})\le\mu(\pi_*\mathrm{L})-(1-\frac{1}{r})(g-1)$. The case of quotient line bundles follows by duality (1.2).

Corollary 2.2 .– *Let* F *be a sub-bundle of* $\pi_*\mathrm{L}$, *of rank* p. *Then:*

a) $$[\mu(\mathrm{F})+\frac{g-1}{p}]\le\mu(\pi_*\mathrm{L})+\frac{g-1}{r}\quad;$$

b) $$\mu(\mathrm{F})<\mu(\mathrm{E})+1\ ;$$

c) $$\textit{If}\quad p\le\frac{gr}{g+r}\quad\textit{or}\quad p\ge\frac{r^2}{g+r}\ ,\quad\textit{then}\quad\mu(\mathrm{F})<\mu(\mathrm{E})\ .$$

Proof : By the above remark, the vector bundle F contains a sub-line bundle M of degree $[\mu(\mathrm{F})-(1-\frac{1}{p})(g-1)]$. Applying 2.1 we get:

$$[\mu(\mathrm{F})-(1-\frac{1}{p})(g-1)]\leq\mu(\pi_*\mathrm{L})-(1-\frac{1}{r})(g-1)\ ,$$

which gives a).

For $x\in\frac{1}{p}\mathbf{Z}$ we have $[x]\geq x-1+\frac{1}{p}$; thus

$$\mu(\mathrm{F})-\mu(\pi_*\mathrm{L})<1-g\frac{r-p}{rp}\ ,$$

from which one deduces b) and the first case of c). The second case of c) follows by duality (1.2). ∎

3. The case $|\chi(\mathrm{L})|$ small

(3.1) Let E be a vector bundle on a curve C, with $\chi(\mathrm{E})\leq 0$; let $\mathrm{W_E}$ be the closed subset of JC consisting of line bundles α such that $\mathrm{H}^0(\mathrm{E}\otimes\alpha)\neq 0$. We claim that if $\mathrm{W_E}$ is not empty, its codimension in JC is $\leq 1-\chi(\mathrm{E})$. Let us recall briefly the proof: we denote by $\mathcal{P}$ be the Poincaré line bundle on $\mathrm{C}\times\mathrm{JC}$, and by p,q the projections of $\mathrm{C}\times\mathrm{JC}$ onto C and JC respectively. The cohomology $\mathrm{R}q_*(p^*\mathrm{E}\otimes\mathcal{P})$ can be represented locally (and even globally) in the derived category $\mathrm{D(JC)}$ by a complex $\mathrm{L}_0\xrightarrow{u}\mathrm{L}_1$; we have $\mathrm{rk}\,(\mathrm{L}_0)-\mathrm{rk}\,(\mathrm{L}_1)=\chi(\mathrm{E})$. Then $\mathrm{W_E}$ is the locus where u is not injective, or equivalently is not of maximal rank. By standard matrix theory this locus is of codimension $\leq 1-\chi(\mathrm{E})$. Moreover it is non-empty if $1-\chi(\mathrm{E})\leq\dim\mathrm{JC}$ ([L]).

Proposition 3.2 .– *For a generic line bundle* L *on* X' *with* $|\chi(\mathrm{L})|\leq g+\dfrac{g^2}{r}$, *the vector bundle* $\pi_*\mathrm{L}$ *is semi-stable, and stable unless* $g=1$.

Proof : We treat the case $\chi(\mathrm{L})\leq 0$; the case $\chi(\mathrm{L})\geq 0$ will follow by duality (1.2). We first assume $-g\leq\chi(\mathrm{L})\leq 0$.

Let $\mathrm{F}\subset\pi_*\mathrm{L}$ be a subbundle of $\pi_*\mathrm{L}$, of rank p . We claim that $\chi(\mathrm{F})\leq\chi(\mathrm{L})$. If $\mathrm{W_F}=\varnothing$ we have $1-\chi(\mathrm{F})>g$, hence $\chi(\mathrm{F})\leq -g\leq\chi(\mathrm{L})$. Assume that $\mathrm{W_F}$ is not empty; it has codimension $\leq 1-\chi(\mathrm{F})$ (3.1). The variety $\mathrm{W}_{\pi_*\mathrm{L}}$ contains $\mathrm{W_F}$, and therefore has also codimension $\leq 1-\chi(\mathrm{F})$. On the other hand we have set-theoretically

$$\mathrm{W}_{\pi_*\mathrm{L}}=\{\alpha\in\mathrm{JX}\,|\,\mathrm{H}^0(\mathrm{X},\mathrm{L}\otimes\pi^*\alpha)\neq 0\}=(\pi^*)^{-1}(\mathrm{W_L})\ .$$

The locus $\mathrm{W_L}$ parameterizes line bundles in JX' of the form $\mathrm{L}^{-1}(\mathrm{E})$, where E is any effective divisor on X' of degree $g'-1+\chi(\mathrm{L})$; it has codimension $1-\chi(\mathrm{L})$.

Thus for generic L the pull-back $(\pi^*)^{-1}(\mathrm{W_L})$ has also codimension $1-\chi(\mathrm{L})$; we conclude that $\chi(\mathrm{F}) \le \chi(\mathrm{L}) = \chi(\pi_*\mathrm{L})$.

Now we have $\chi(\pi_*\mathrm{L}) \le \frac{p}{r}\,\chi(\pi_*\mathrm{L})$ (since $\chi(\mathrm{L}) \le 0$), hence $\mu(\mathrm{F}) \le \mu(\pi_*\mathrm{L})$. The inequality is strict unless $\chi(\mathrm{L}) = 0$, in which case we can only conclude that $\pi_*\mathrm{L}$ is semi-stable. Suppose that we have an extension $0 \to \mathrm{F} \to \pi_*\mathrm{L} \to \mathrm{G} \to 0$, with $\chi(\mathrm{F}) = \chi(\mathrm{G}) = 0$. We then have $\mathrm{W}_{\pi_*\mathrm{L}} = \mathrm{W_F} + \mathrm{W_G}$ in the divisor group of JX . On the other hand the equality $\mathrm{W}_{\pi_*\mathrm{L}} = (\pi^*)^{-1}(\mathrm{W_L})$ holds as an equality of divisors; so when $g \ge 2$ we obtain a contradiction from Lemma 3.3 below.

If $\chi(\mathrm{L}) \le -g$, the above argument still gives the inequality $\chi(\mathrm{F}) \le -g$. By 2.2 c) we may assume $p < \dfrac{r^2}{g+r}$. This implies

$$\frac{\chi(\mathrm{F})}{p} \le -\frac{g}{p} < -\frac{g(g+r)}{r^2} \le \frac{\chi(\mathrm{L})}{r}\ ,$$

hence $\mu(\mathrm{F}) < \mu(\pi_*\mathrm{L})$. ∎

Lemma 3.3 .– *Let* A, B *be abelian varieties of dimension* ≥ 2 *,* $\varphi : \mathrm{B} \to \mathrm{A}$ *be a homomorphism with finite kernel, and* W *an ample, integral divisor on* A *. The pull back by* φ *of a generic translate of* W *is integral.*

Proof: Let $\Phi : \mathrm{B} \times \mathrm{A} \to \mathrm{A}$ be the homomorphism defined by $\Phi(b,a) = \varphi(b) - a$. It is smooth and surjective, so $\Phi^{-1}(\mathrm{W})$ is an integral divisor of $\mathrm{B} \times \mathrm{A}$. Therefore the fibre of the second projection $\Phi^{-1}(\mathrm{W}) \to \mathrm{A}$ at a general point a of A is locally integral. But this fibre can be identified with the divisor $\varphi^*(\mathrm{W}+a)$; since this divisor is ample, it is also connected, hence integral. ∎

Corollary 3.4 .– *The conjecture holds for a covering* $\mathrm{X}' \to \mathrm{X}$ *of degree smaller than* $g(1+\sqrt{3}) - 1$ *.*

Proof: The conjecture will hold if any number is congruent (mod. r) to some number χ satisfying $|\chi| \le g + \frac{g^2}{r}$. This is ensured by the inequality $r < 2(g + \frac{g^2}{r}) - 1$, which holds if $r < g(1+\sqrt{3}) - 1$ (exercise!). ∎

Remarque 3.5 .– Since $[g + \frac{g^2}{r}] \ge g$ the conjecture holds also for $r \le 2g+1$; this is is slightly better than the Corollary when $g = 1$ or 2 .

(3.6) It is tempting to improve the result of 3.2 by applying the same method to the vector bundle $\mathrm{F} \otimes \pi_*\mathrm{L}$, for some appropriate vector bundle F on X: the (semi-) stability of $\mathrm{F} \otimes \pi_*\mathrm{L}$ implies that of F , and we can choose F so that for instance $\chi(\mathrm{F} \otimes \pi_*\mathrm{L}) = 0$. We have $\mathrm{W}_{\mathrm{F}\otimes\pi_*\mathrm{L}} = (\pi^*)^{-1}(\mathrm{W}_{\pi^*\mathrm{F}\otimes\mathrm{L}})$. Inspection of the proof of 3.2 shows that we need the following:

(3.6 a) $\mathrm{W}_{\pi^*\mathrm{F}\otimes\mathrm{L}}$ has codimension *exactly* $1 - \chi(\pi^*\mathrm{F} \otimes \mathrm{L})$.

This gives the stability of $F \otimes \pi_* L$ if $1-g \le \chi(F \otimes \pi_* L) < 0$, and the semi-stability when $\chi(F \otimes \pi_* L) = 0$; to get the stability in the latter case we need moreover:

(3.6 b) The divisor $W_{\pi^* F \otimes L}$ is integral.

Unfortunately (3.6 a) seems rather difficult to check: though we are free to choose F general enough, its pull back $\pi^* F$ will be rather special, and we do not see any way of proving (3.6 a) unless we know that *all* stable bundles with the same degree and rank satisfy it. Here is one case where this works:

Proposition 3.7 .– *If* r *is even and* L *general of degree* $\equiv \frac{r}{2} \pmod{r}$, $\pi_* L$ *is semi-stable.*

Proof : The hypothesis means $\mu(\pi_* L) \in \frac{1}{2} + \mathbf{Z}$, so we can choose a rank 2 bundle F so that $\chi(F \otimes \pi_* L) = 0$. Condition (3.6 a) means that the bundle $\pi^* F \otimes L$ has a theta divisor, that is that $H^0(X', \pi^* F \otimes L \otimes \alpha) = 0$ for α general in JX'; by [R] this is the case if $\pi^* F$ is semi-stable. But the semi-stability of $\pi^* F$ is equivalent to that of F [?]. ■

4. The case of an étale covering

Proposition 4.1 .– *The conjecture holds if* π *is étale.*

Proof : Let $\rho : Y \to X$ be the étale Galois covering associated to π, and Σ the set of X-morphisms $Y \to X'$; we put $Y_\sigma = Y$ for each $\sigma \in \Sigma$. We have a cartesian diagram

$$\begin{array}{ccc} \coprod_{\sigma \in \Sigma} Y_\sigma & \xrightarrow{\pi'} & Y \\ {\scriptstyle \rho'} \downarrow & & \downarrow {\scriptstyle \rho} \\ X' & \xrightarrow{\pi} & X \end{array}$$

where π' is the identity on each Y_σ, while $\rho'_{|Y_\sigma} = \sigma$.

It follows that for any coherent sheaf L on X' we have a canonical isomorphism

$$\rho^* \pi_* L \xrightarrow{\sim} \pi'_* \rho'^* L \cong \bigoplus_{\sigma \in \Sigma} \sigma^* L .$$

Take for L a line bundle. The line bundles $\sigma^* L$, for $\sigma \in \Sigma$, have all the same slope δ. Therefore $\rho^* \pi_* L$ is semi-stable, hence $\pi_* L$ is semi-stable for *every* line bundle L on X'.

Assume now $g > 1$. Suppose that $\pi_*\mathrm{L}$ contains a non-trivial sub-bundle F with $\mu(\mathrm{F}) = \mu(\pi_*\mathrm{L})$. Then $\rho^*\mathrm{F}$ is a sub-bundle of $\bigoplus_{\sigma\in\Sigma} \sigma^*\mathrm{L}$, with slope δ. The category of semi-stable vector bundles on Y with slope δ is an abelian category, whose simple objects are the stable bundles. By general nonsense it follows that any subbundle of $\bigoplus_{\sigma\in\Sigma} \sigma^*\mathrm{L}$ with slope δ is isomorphic to a direct sum $\bigoplus_{\sigma\in\Sigma'} \sigma^*\mathrm{L}$ for some subset Σ' of Σ.

The Galois group G of ρ acts transitively on Σ, by the formula $g\cdot\sigma = \sigma\circ g^{-1}$ for $g \in \mathrm{G}$, $\sigma \in \Sigma$. Our bundle $\rho^*\mathrm{F}$ is G-invariant. We will show that for generic L the line bundles $\sigma^*\mathrm{L}$, for $\sigma \in \Sigma$, are pairwise non-isomorphic; this implies that Σ' must be invariant under G, that is $\Sigma' = \Sigma$ and $\mathrm{F} = \pi_*\mathrm{L}$, which proves the stability of $\pi_*\mathrm{L}$.

To prove the above claim, we choose a particular element σ of Σ, and let H be its stabilizer. We consider the component $\mathrm{J}^{\mathrm{L}}\mathrm{X}'$ of $\mathrm{Pic}(\mathrm{X}')$ containing L. Then $\sigma^*(\mathrm{J}^{\mathrm{L}}\mathrm{X}')$ is a subvariety of $\mathrm{Pic}(\mathrm{Y})$, invariant under H, of dimension g'. Suppose that it is invariant under a sub-group H' of G containing H; since it is connected, it must actually lie in the pull-back of $\mathrm{Pic}(\mathrm{Y}/\mathrm{H}')$. But the Riemann-Hurwitz formula shows that the genus of Y/H' is strictly smaller than that of $\mathrm{X}' = \mathrm{Y}/\mathrm{H}$ unless $\mathrm{H}' = \mathrm{H}$. So for L general enough $\sigma^*\mathrm{L}$ cannot be fixed by any element of $\mathrm{G} - \mathrm{H}$, which proves our claim. ∎

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