

Riemannian Holonomy and Algebraic Geometry

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Introduction

This survey is devoted to a particular instance of the interaction between Riemannian geometry and algebraic geometry, the study of manifolds with special holonomy. The holonomy group is one of the most basic objects associated with a Riemannian metric; roughly, it tells us what are the geometric objects on the manifold (complex structures, differential forms, ...) which are *parallel* with respect to the metric (see 1.3 for a precise statement).

There are two surprising facts about this group. The first one is that, despite its very general definition, there are few possibilities – this is Berger’s theorem (1.2). The second one is that apart from the generic case in which the holonomy group is $SO(n)$, all other cases appear to be related in some way to algebraic geometry. Indeed the study of *compact* manifolds with special holonomy brings into play some special, and quite interesting, classes of algebraic varieties: Calabi-Yau, complex symplectic or complex contact manifolds. I would like to convince algebraic geometers that this interplay is interesting on two accounts: on one hand the general theorems on holonomy give deep results on the geometry of these special varieties; on the other hand Riemannian geometry provides us with good problems in algebraic geometry – see 4.3 for a typical example.

I have tried to make these notes accessible to students with little knowledge of Riemannian geometry, and a basic knowledge of algebraic geometry. Two appendices at the end recall the basic results of Riemannian (resp. algebraic) geometry which are used in the text.

This is an updated version of the notes of the “Emmy Noether lectures” which I gave at Bar Ilan University (Fall 1998). I want to thank the Emmy Noether Institute for the invitation, and Mina Teicher for her warm hospitality.

1. Holonomy

1.1. Definition

Perhaps the most fundamental object associated to a Riemannian metric on a manifold M is a canonical connection on the tangent bundle $T(M)$, the *Levi-Civita connection*. A connection gives an isomorphism between the tangent spaces at infinitesimally near points; more precisely, to each path γ on M with origin p and extremity q , the connection associates an isomorphism $\varphi_\gamma : T_p(M) \rightarrow T_q(M)$ (“parallel transport”), which is actually an isometry with respect to the scalar products on $T_p(M)$ and $T_q(M)$ induced by the metric (see App. A for more details). If δ is another path from q to r , the isomorphism associated to the path composed of γ and δ is $\varphi_\delta \circ \varphi_\gamma$.



Let $p \in M$; the above construction associates in particular to every loop γ at p an isometry of $T_p(M)$. The set of all such isometries is a subgroup H_p of the orthogonal group $O(T_p(M))$, called the *holonomy subgroup* of M at p . If q is another point of M and γ a path from p to q , we have $H_q = \varphi_\gamma H_p \varphi_\gamma^{-1}$, so that the H_p 's define a unique conjugacy class $H \subset O(n)$; the group H is often called simply the *holonomy group* of M . Similarly the representations of the groups H_p on $T_p(M)$ are isomorphic, so we can talk about the *holonomy representation* of H .

There is a variant of this definition, the *restricted* holonomy group, obtained by considering only those loops which are homotopically trivial. This group actually behaves more nicely: it is a connected, closed Lie subgroup of $SO(T_p(M))$. To avoid technicalities, we will always assume that our varieties are *simply-connected*, so that the two notions coincide. We will also usually consider *compact* manifolds: this is somehow the most interesting case, at least for the applications to algebraic geometry.

1.2. The theorems of De Rham and Berger

With such a degree of generality we would expect very few restrictions, if any, on the holonomy group. This is far from being the case: thanks to a remarkable theorem of Berger, we can give a complete (and rather small) list of possible holonomy groups. First of all, let us say that a Riemannian manifold is *irreducible* if its holonomy representation is irreducible.

Theorem (De Rham).— *Let M be a compact simply-connected Riemannian manifold. There exists a canonical decomposition $M \xrightarrow{\sim} \prod M_i$, where each M_i is an irreducible Riemannian manifold. Let $p = (p_i)$ be a point of M , and let $H_i \subset O(T_{p_i}(M_i))$ be the holonomy group of M_i at p_i ; then the holonomy group of M at p is the product $\prod H_i$, acting on $T_p(M) = \prod T_{p_i}(M_i)$ by the product representation.*

The reader fluent in Riemannian geometry may replace compact by complete. On the other hand, both completeness and simple connectedness are essential here. The proof is far from trivial, see for instance [K-N], IV.6.

We are thus reduced to *irreducible* (compact, simply-connected) Riemannian manifolds. Among these are some very classical manifolds, the *symmetric spaces*; they are of the form G/H , where G is a compact Lie group and H is the neutral component of the fixed locus of an involution of G . These spaces are completely classified, and their geometry is well-known; the holonomy group is H itself. Excluding this case, we get:

Theorem (Berger).— *Let M be an irreducible (simply-connected) Riemannian manifold, which is not isomorphic to a symmetric space. Then the holonomy group H of M belongs to the following list:*

H	$\dim(M)$	<i>metric</i>
$SO(n)$	n	generic
$U(m)$	$2m$	Kähler
$SU(m)$ ($m \geq 3$)	$2m$	Calabi-Yau
$Sp(r)$	$4r$	hyperkähler
$Sp(r)Sp(1)$ ($r \geq 2$)	$4r$	quaternion-Kähler
G_2	7	
$Spin(7)$	8	

We have eliminated $SU(2)$ ($= Sp(1)$) and $Sp(1)Sp(1)$ ($= SO(4)$) so that a given group appears only once in the list. We should point out that a third exceptional

case, $\text{Spin}(9) \subset \text{SO}(16)$, appeared in Berger's list, but has been eliminated later (see [B-G]).

Which groups in this list do effectively occur for some compact, simply-connected, non-symmetric manifold? That $\text{O}(n)$ and $\text{U}(m)$ occur is classical and easy: one starts from an arbitrary Riemannian (resp. Kählerian) metric on M and perturbs it in the neighborhood of a point. The other groups required much more efforts. The case of $\text{SU}(m)$ is a direct consequence of the Calabi conjecture, proved by Yau [Y]; examples with $H = \text{Sp}(r)$ were found in 1982 [B1], again using Yau's result. Examples in the last cases, G_2 and $\text{Spin}(7)$, were found only recently [J1, J2]. As for $\text{Sp}(1)\text{Sp}(r)$, no example is known, and in fact it is generally conjectured that they should not exist – we will discuss this in § 4.

1.3. The holonomy principle

Before describing the subgroups which appear in the list, let us discuss the geometric meaning of such a restriction on the holonomy. We say that a tensor field θ on M is *parallel* if for any path γ from p to q , the isomorphism φ_γ transports $\theta(p)$ onto $\theta(q)$ (this is equivalent to $\nabla\theta = 0$, see App. A). This implies in particular that $\theta(p)$ is invariant under the holonomy subgroup H_p . Conversely, given a tensor $\theta(p)$ on $T_p(M)$ invariant under H_p , we can transport it at q by any path from p to q and obtain a tensor $\theta(q)$ independent of the chosen path; the tensor field θ thus constructed is parallel. We have thus established:

Holonomy principle: Evaluation at p establishes a one-to-one correspondence between parallel tensor fields and tensors on $T_p(M)$ invariant under H_p .

In the next sections we will illustrate this principle by going through Berger's list. Let us start with the two simplest cases:

a) $H = \text{SO}(n)$ means that there are no parallel tensor fields (apart from the metric and the orientation). Such a metric is often called *generic*.

b) $\text{U}(m)$ is the subgroup of $\text{SO}(2m)$ preserving a complex structure J on \mathbf{R}^{2m} which is orthogonal (that is, $J \in \text{SO}(2m)$, $J^2 = -1$). Therefore the manifolds with holonomy contained in $\text{U}(m)$ are the Riemannian manifolds with a complex structure J which is orthogonal and parallel. This is one of the classical characterization of *Kähler* manifolds.

We claimed in the introduction that compact manifolds with special holonomy are related to algebraic geometry. Indeed compact Kähler manifolds share many properties with projective manifolds, in particular the Hodge decomposition of the cohomology spaces. Actually for a long time the only known examples of compact Kähler manifolds were deformations of projective ones, like complex tori or K3

surfaces; Kodaira asked whether every compact Kähler manifold was obtained in this way. This has been recently answered negatively by C. Voisin [V2].

We will discuss the groups $SU(m)$, $Sp(r)$ and $Sp(1)Sp(r)$ in the next sections. We will not discuss the exotic holonomies G_2 and $Spin(7)$ here; I refer to [J3] for a readable account.

2. Calabi-Yau manifolds

We now consider manifolds with holonomy contained in $SU(m)$. We view $SU(m)$ as the subgroup of $U(m)$ preserving an alternate complex m -form on \mathbf{C}^m ; therefore a manifold X with holonomy contained in $SU(m)$ is a Kähler manifold (of complex dimension m) with a parallel form of type $(m, 0)$. This means that the canonical line bundle $K_X := \Omega_X^m$ is flat; in other words, the Ricci curvature (which for a Kähler manifold is just the curvature of K_X) is zero. Thus the manifolds with holonomy $SU(m)$ are exactly the Ricci-flat manifolds.

It is easy to see that a parallel form is closed, hence in this case holomorphic: thus the canonical bundle K_X of X is trivial (as a holomorphic bundle). Conversely, the Calabi conjecture, proved by Yau [Y], implies that a *Calabi-Yau* manifold, namely a compact, simply-connected Kähler manifold with trivial canonical bundle, admits a Ricci-flat metric. So *the compact (simply-connected) complex manifolds which admit a metric with holonomy contained in $SU(m)$ are the Calabi-Yau manifolds.*

This fact has strong implications in algebraic geometry, in particular thanks to the following result:

Proposition (Bochner's principle). — *On a compact Kähler Ricci-flat manifold, any holomorphic tensor field (covariant or contravariant) is parallel.*

The proof rests on the following formula, which follows from a tedious but straightforward computation ([B-Y], p. 142): if τ is any tensor field,

$$\Delta(\|\tau\|^2) = \|\nabla\tau\|^2 .$$

Therefore $\Delta(\|\tau\|^2)$ is nonnegative, hence 0 since its mean value over X is 0 by Stokes' formula. It follows that τ is parallel. ■

As a consequence we get

Proposition. — *Let X be a compact Kähler manifold, of dimension $m \geq 3$, with holonomy group $SU(m)$. Then X is projective, and $H^0(X, \Omega_X^p) = 0$ for $0 < p < m$.*

Proof: Let $x \in X$, and $V = T_x(X)$. Using the Bochner and holonomy principles, we see that the space $H^0(X, \Omega_X^p)$ can be identified with the $SU(V)$ -invariant subspace of $\Lambda^p V^*$. Because $SU(V)$ acts irreducibly on $\Lambda^p V^*$, the invariant subspace is zero unless $p = 0$ or $p = m$. Since $H^0(X, \Omega_X^2)$ is zero, X is projective (App. B). ■

Manifolds with holonomy $\mathrm{Sp}(r)$, called *hyperkähler* manifolds, have very special properties; we will study them in detail in the next section. Since the only groups in Berger's list which are contained in $\mathrm{SU}(m)$ are of the form $\mathrm{SU}(p)$ or $\mathrm{Sp}(q)$, we get the following structure theorem:

Theorem. — *Any (simply-connected) Calabi-Yau manifold is a product $\prod_i V_i \times \prod_j X_j$, where:*

a) *Each V_i is a projective Calabi-Yau manifold, with $H^0(V_i, \Omega_{V_i}^p) = 0$ for $0 < p < \dim(V_i)$;*

b) *The manifolds X_j are irreducible hyperkähler.*

(There is a more general statement for non simply-connected manifolds, see for instance [B1]).

Further developments

Calabi-Yau manifolds have been at the center of a flurry of activity in the last 15 years, principally under the influence of mathematical physics. The keyword here is *mirror symmetry*, a (conjectural) duality between families of Calabi-Yau manifolds. I will not try to be more precise, because this goes far beyond the scope of these notes. A good introduction to the beginning of the story is given in [V1]. The current main trends are the Strominger-Yau-Zaslow conjecture [S-Y-Z] and Kontsevich homological mirror symmetry conjecture [Ko]. Both give an important role to the (real) symplectic aspect.

3. Symplectic manifolds

3.1. Hyperkähler versus symplectic

The group $\mathrm{Sp}(r)$ is the quaternionic unitary group, that is, the group of \mathbf{H} -linear automorphisms of \mathbf{H}^r which preserve the standard hermitian form $\psi(z, z') = \sum_r z_i \bar{z}'_i$. Viewing \mathbf{H}^r as \mathbf{R}^{4r} realizes $\mathrm{Sp}(r)$ as a subgroup of the orthogonal group $\mathrm{SO}(4r)$. The manifolds of dimension $4r$ with holonomy $\mathrm{Sp}(r)$ are called *hyperkähler* manifolds.

There are two ways of making this definition explicit. We can characterize $\mathrm{Sp}(r)$ as the subgroup of orthogonal transformations of \mathbf{R}^{4r} which are linear with respect to the complex structures I, J, K (here $(1, I, J, K)$ is the standard basis of \mathbf{H} over \mathbf{R} , with $IJ = -JI = K$). By the holonomy principle, hyperkähler manifolds are therefore characterized by the existence of 3 complex structures I, J, K , with $IJ = -JI = K$, such that the metric is Kähler with respect to each of these. Actually any pure quaternion $aI + bJ + cK$ with $a^2 + b^2 + c^2 = 1$ defines such a structure,

so hyperkähler manifolds admit a family of complex Kähler structures parametrized by the sphere \mathbf{S}^2 (hence their name).

A second way to look at $\mathrm{Sp}(r)$ is to give a special role to one of these complex structures, say \mathbf{I} , and to view \mathbf{H} as $\mathbf{C}(\mathbf{J})$ (and \mathbf{C} as $\mathbf{R}(\mathbf{I})$). We identify \mathbf{H}^r with $\mathbf{C}^r \oplus \mathbf{C}^r \mathbf{J} = \mathbf{C}^{2r}$. The hermitian form ψ can be written as $h + \varphi \mathbf{J}$, where h is the standard (complex) hermitian form and φ the standard \mathbf{C} -bilinear symplectic form on \mathbf{C}^{2r} . Therefore $\mathrm{Sp}(r)$ is the intersection in $\mathrm{SO}(4r)$ of the unitary group $\mathrm{U}(2r)$ and the complex symplectic group $\mathrm{Sp}(2r, \mathbf{C})$ (incidentally, this implies that $\mathrm{Sp}(r)$ is a maximal compact subgroup of $\mathrm{Sp}(2r, \mathbf{C})$, which is the reason for the notation).

In terms of holonomy, this means that once a preferred complex structure has been chosen, a hyperkähler manifold can be characterized as a Kähler manifold with a parallel non-degenerate 2-form of type $(2, 0)$. As above this 2-form must be holomorphic, hence it is a (complex) *symplectic structure*, that is a closed¹, holomorphic, everywhere non-degenerate 2-form. Conversely, let X be a compact Kähler manifold of (complex) dimension $2r$, with a complex symplectic structure φ ; then X is a Calabi-Yau manifold (because φ^r does not vanish), hence admits a Ricci-flat metric, for which the form φ is parallel. If moreover we require the holomorphic 2-form φ to be unique up to a scalar, the holonomy of X is exactly $\mathrm{Sp}(r)$. We will call such a manifold *Kähler symplectic*, to emphasize that we have chosen a particular complex structure.

3.2. The two standard series

A typical example of a Kähler symplectic manifold is a K3 surface, that is a compact (simply-connected) Kähler surface with trivial canonical bundle. Note that in the statement of Berger's theorem I have deliberately chosen to view the group $\mathrm{SU}(2)$ as symplectic ($= \mathrm{Sp}(1)$) rather than unitary: we will see that the theory of K3 surfaces is an accurate model for the study of complex symplectic manifolds. For a long time no other example has been known, and it was even conjectured that such manifolds should not exist (see [Bo1]). In 1982 Fujiki gave an example in dimension 4, which I generalized in any dimension – in fact I constructed two series of examples [B1]. Let me explain these examples.

Start from a K3 surface S , with a holomorphic nonzero 2-form φ . The product S^r admits a natural symplectic form, namely $\mathrm{pr}_1^* \varphi + \dots + \mathrm{pr}_r^* \varphi$; but there are others, since we may take as well any expression $\lambda_1 \mathrm{pr}_1^* \varphi + \dots + \lambda_r \mathrm{pr}_r^* \varphi$ with $\lambda_1, \dots, \lambda_r$ in \mathbf{C}^* . A natural way to eliminate those is to ask for \mathfrak{S}_r -invariant 2-forms, which amounts to consider instead of S^r the symmetric product $S^{(r)} := S^r / \mathfrak{S}_r$.

¹ The closedness condition is automatic for compact Kähler manifolds.

Unfortunately this quotient is singular as soon as r is greater than 1; but it admits a nice desingularization, the *Douady space* $S^{[r]}$ which parameterizes the finite subspaces of S of length r (when S is projective this is known as the Hilbert scheme). We can view $S^{(r)}$ as the space of finite subsets $E \subset S$ with a positive multiplicity $m(p)$ assigned to each point p of E , in such a way that $\sum_{p \in E} m(p) = r$. The natural map $\varepsilon : S^{[r]} \rightarrow S^{(r)}$ which associates to a subspace Z of S its set of points counted with multiplicity turns out to be holomorphic; it induces an isomorphism on the open subset $S_o^{[r]}$ of $S^{[r]}$ parameterizing those subspaces which consist of r distinct points.

It is then easy to show that the 2-form $\text{pr}_1^* \varphi + \dots + \text{pr}_r^* \varphi$, which lives naturally on $S_o^{[r]}$, extends to a symplectic form on $S^{[r]}$, unique up to a scalar, and that $S^{[r]}$ is simply-connected. Moreover $S^{[r]}$ is Kähler as a consequence of a general result of Varouchas [Va]. In other words, *the Douady space $S^{[r]}$ is a $(2r)$ -dimensional irreducible symplectic manifold.*

We can perform the same construction starting from a 2-dimensional complex torus T : the Douady space $T^{[r]}$ is again symplectic, however it is not simply-connected. In fact it admits a smooth surjective map $S : T^{[r]} \rightarrow T$, which is the composite of $\varepsilon : T^{[r]} \rightarrow T^{(r)}$ and of the sum map $T^{(r)} \rightarrow T$. The fibre $K_{r-1} = S^{-1}(0)$ is a simply-connected, irreducible symplectic manifold of dimension $2r - 2$.

Thus we get two series of examples in each dimension. The first thing to look at, for algebraic geometers, is their deformations: there are some obvious ones obtained by deforming the surface S (or T), but it turns out that we get more than those. In fact, in the moduli space parameterizing all deformations of the manifolds we found, those of the form $S^{[r]}$ for some K3 surface S form a *hypersurface*, and similarly for K_r (this is, of course, for $r \geq 2$).

This is seen as follows. First of all, the universal deformation space of a symplectic manifold X is smooth, of dimension $\dim H^1(X, T_X)$. This is a general result for Calabi-Yau manifold, due to Tian and Todorov (see [T]); in the particular case of symplectic manifolds it had been proved earlier by Bogomolov [Bo1]. Since X is symplectic, the tangent sheaf T_X is isomorphic to Ω_X^1 , hence

$$\dim H^1(X, T_X) = \dim H^1(X, \Omega_X^1) = b_2(X) - 2 .$$

An easy computation gives $b_2(S^{[r]}) = b_2(S) + 1$ and $b_2(K_r) = b_2(T) + 1$ for $r \geq 2$, hence our assertion .

We will say that a symplectic manifold is *of type $S^{[r]}$* , or K_r , if it can be obtained by deformation of $S^{[r]}$, or K_r . As an example, we proved in [B-D] that the variety of lines contained in a smooth cubic hypersurface V of \mathbf{P}^5 is of type $S^{[2]}$, but it is not isomorphic to $S^{[2]}$ if V is general enough.

3.3. Other examples

Shortly after the two series were discovered, Mukai showed that they fit into an elegant construction which looks much more general [M]. He proved that the moduli space of stable vector bundles on a K3 or abelian surface S , with fixed rank and Chern classes, is smooth and admits a symplectic form. The idea is quite simple. The smoothness follows from a standard obstruction argument: one shows that the obstructions to deform E infinitesimally are the same as the obstructions to deform $\det E$, which vanish. Now the tangent space to the moduli space at E is $H^1(S, \mathcal{E}nd(E))$, and the symmetric form $(u, v) \mapsto \text{Tr } uv$ on $\mathcal{E}nd(E)$ gives rise to a skew-symmetric pairing

$$H^1(S, \mathcal{E}nd(E)) \otimes H^1(S, \mathcal{E}nd(E)) \longrightarrow H^2(S, \mathcal{O}_S) \cong \mathbf{C}$$

which is non-degenerate by Serre duality, and provides the required symplectic form.

If we want to exploit this construction to give new examples of symplectic manifolds, we need to fulfill the following requirements:

a) Our moduli space M should be compact. This is achieved by including in M *stable sheaves*, and choosing the polarization so that all semi-stable sheaves are actually stable. I refer for instance to [H-L] for the details.

b) M should be simply-connected, and satisfy $\dim H^0(M, \Omega_M^2) = 1$. This was proved in [OG1]. Observe that both properties are invariant by deformation, and also under birational equivalence. O'Grady deforms S to a surface S_e admitting an elliptic pencil, with a suitable polarization; then a detailed analysis shows that the moduli space is birational to $S_e^{[r]}$ for some r .

So M is a symplectic manifold, but the proof gives actually more: Huybrechts has proved that two birational symplectic manifolds are deformation of each other – we will discuss this in detail in 3.5. Therefore *the moduli space M is of type $S^{[r]}$* , and thus does not provide any new example.

When Huybrechts' result appeared, it implied that all known examples of Kähler symplectic manifolds were of type $S^{[r]}$ or K_r . Since then two new examples have been constructed by O'Grady ([OG2], [OG3]) of dimension 10 (resp. 6), by desingularizing a *singular* moduli space of vector bundles on a K3 (resp. an abelian surface). Unfortunately, it is shown in [KLS] that the moduli spaces considered by O'Grady are the only ones which admit a symplectic resolution.

It remains an intriguing and very interesting problem to construct more examples. As we will see in the next sections, we know a lot about the geometry of Kähler symplectic manifolds; it is somewhat embarrassing to have so few examples.

3.4. The period map

For K3 surfaces the theory of the period map gives us a fairly complete picture of the moduli space, thanks to the work of Shafarevich and Piatetski-Shapiro, Burns and Rapoport, Todorov, Looijenga, Siu – I refer to [B2] for a survey. The idea is to encode a K3 surface S by its Hodge decomposition (see App. B)

$$H^2(S, \mathbf{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} ,$$

which is determined by the position of the line $H^{2,0}$ in $H^2(S, \mathbf{C})$ (we have $H^{0,2} = \overline{H^{2,0}}$, and $H^{1,1}$ is the orthogonal of $H^{2,0} \oplus H^{0,2}$ for the intersection product). The point is that $H^2(S, \mathbf{C})$ depends only on the topology of S , while $H^{2,0}$ depends heavily on the complex structure: we have $H^{2,0} = \mathbf{C}\varphi$, where φ is the De Rham class of a non-zero holomorphic 2-form on S (unique up to a constant).

To be more precise, we denote by L a lattice isomorphic to $H^2(S, \mathbf{Z})$ (this is the unique even unimodular lattice of signature $(3, 19)$, but we will not need this). A *marked* K3 surface is a pair (S, σ) of a K3 S and a lattice isomorphism $\sigma : H^2(S, \mathbf{Z}) \rightarrow L$. The first (easy) result is that there is an analytic manifold \mathcal{M}_L which is a *fine moduli space* for marked K3's: that is, there is a universal family $u : \mathcal{U} \rightarrow \mathcal{M}_L$ of marked K3's over \mathcal{M}_L , such that any family $\mathcal{S} \rightarrow T$ of marked K3's is the pull-back of u through a classifying map $T \rightarrow \mathcal{M}_L$. The bad news is that \mathcal{M}_L is *not Hausdorff* – a rather surprising fact that we will explain later (3.5).

The advantage of working with \mathcal{M}_L is that we can now compare the Hodge structures of different surfaces. Given (S, σ) , we extend σ to an isomorphism $H^2(S, \mathbf{C}) \rightarrow L_{\mathbf{C}}$ and put²

$$\wp(S, \sigma) = \sigma(H^{2,0}) = \sigma([\varphi]) \in \mathbf{P}(L_{\mathbf{C}}) .$$

The map \wp is called the *period map*, for the following reason: choose a basis (e_1, \dots, e_{22}) of L^* , so that $L_{\mathbf{C}} = \mathbf{C}^{22}$. Put $\gamma_i = {}^t\sigma(e_i)$, viewed as an element of $H_2(S, \mathbf{Z})$; then

$$\wp(S, \sigma) = \left(\int_{\gamma_1} \varphi : \dots : \int_{\gamma_{22}} \varphi \right) \in \mathbf{P}^{21} ;$$

the numbers $\int_{\gamma_i} \varphi$ are classically called the “periods” of φ .

Since φ is holomorphic we have $\varphi \wedge \varphi = 0$ and $\int_S \varphi \wedge \bar{\varphi} > 0$. In other words, $\wp(S, \sigma)$ lies in the subvariety Ω_L of $\mathbf{P}(L_{\mathbf{C}})$, called the *period domain*, defined by

$$\Omega_L = \{[x] \in \mathbf{P}(L_{\mathbf{C}}) \mid x^2 = 0, x\bar{x} > 0\} .$$

² We denote as usual by $\mathbf{P}(V)$ the space of lines in a vector space V , and by $[v] \in \mathbf{P}(V)$ the line spanned by a nonzero vector v of V .

Theorem. – 1) $\wp : \mathcal{M}_L \rightarrow \Omega_L$ is étale and surjective.

2) If $\wp(S, \sigma) = \wp(S', \sigma')$, the surfaces S and S' are isomorphic.

Note that this does *not* say that \wp is an isomorphism (otherwise \mathcal{M}_L would be Hausdorff!): the same K3 with different markings can have the same period. There is a more precise statement which describes exactly the fibres of \wp (see for instance [P], p. 142, prop. 2).

Corollary. – Every K3 surface is a deformation of a projective one.

Proof: Write $\varphi = \alpha + i\beta$, with $\alpha, \beta \in H^2(S, \mathbf{R})$. The condition $[\varphi] \in \Omega_L$ translates as $\alpha^2 = \beta^2 > 0$, $\alpha.\beta = 0$. It follows that the classes $[\varphi]$ with $\alpha, \beta \in H^2(S, \mathbf{Q})$ are dense in Ω_L . The corresponding surfaces are dense in \mathcal{M}_L ; they have $H^{1,1} = (\mathbf{C}\alpha \oplus \mathbf{C}\beta)^\perp$ defined over \mathbf{Q} , hence they are projective (App. B). ■

Note that we only need an easy part of the theorem, namely the fact that \wp is étale.

We want to apply the same approach for any Kähler symplectic manifold X . We still have the Hodge decomposition

$$H^2(X, \mathbf{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} \quad \text{with } H^{2,0} = \mathbf{C}\varphi .$$

What seems to be lacking is the quadratic form, but in fact it is still there: I showed in [B1] that the point $[\varphi] \in \mathbf{P}(H^2(X, \mathbf{C}))$ must lie in a hyperquadric, which is rational over \mathbf{Q} ; this implies that there exists a canonical quadratic form $q : H^2(X, \mathbf{Z}) \rightarrow \mathbf{Z}$. It has the following properties (see [B1] and [H]):

- a) q is non-divisible, non-degenerate, of signature $(3, b_2 - 3)$;
- b) there exists a positive integer d_X such that $\alpha^{2r} = d_X q(\alpha)^r$ for all $\alpha \in H^2(X, \mathbf{Z})$;
- c) $q(\varphi) = 0$, and $q(\varphi + \bar{\varphi}) > 0$.

We can now mimic the K3 case. Let L be a lattice; we define as before the moduli space \mathcal{M}_L of pairs (X, σ) , where X is Kähler symplectic manifold and $\sigma : H^2(X, \mathbf{Z}) \rightarrow L$ a lattice isomorphism. We still have a natural structure of analytic (non-Hausdorff) manifold on \mathcal{M}_L (it is however no longer a fine moduli space in general). To each element (X, σ) of \mathcal{M}_L we associate

$$\wp(X, \sigma) = \sigma(H^{2,0}) = \sigma([\varphi]) \in \mathbf{P}(L_{\mathbf{C}}) .$$

As above, if we choose a basis (e_1, \dots, e_b) of L^* , the element $\wp(X, \sigma)$ is given by the “periods” $\int_{\gamma_i} \varphi$, with $\gamma_i = {}^t\sigma(e_i)$.

By property c) of q , $\wp(X, \sigma)$ lies in the subvariety Ω_L of $\mathbf{P}(L_{\mathbf{C}})$ defined by

$$\Omega_L = \{[x] \in \mathbf{P}(L_{\mathbf{C}}) \mid q(x) = 0, q(x + \bar{x}) > 0\} .$$

Theorem. – $\wp : \mathcal{M}_L \rightarrow \Omega_L$ is étale and surjective.

The fact that φ is étale follows from the (easy) computation of its tangent map. The much more delicate surjectivity has been proved by Huybrechts [H].

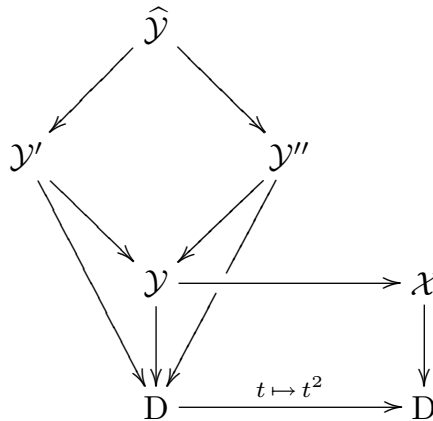
Using the easy part of the theorem and the same argument as for K3 surfaces we obtain:

Corollary .— *Every Kähler symplectic manifold is a deformation of a projective one.*

On the other hand, the answer to the Torelli problem is *negative*, at least for the manifolds K_r (3.2, see [N]). An interesting question here is to find some additional data which together with the Hodge structure determines the isomorphism type of the manifold.

3.4. Birational symplectic manifolds

The fact that the moduli space \mathcal{M}_L of marked K3 surfaces is non-Hausdorff goes back to a famous example of Atiyah [A]. Start with a family $f : \mathcal{X} \rightarrow \mathbb{D}$ of K3 surfaces over the unit disk, such that the total space \mathcal{X} is smooth, the surface \mathcal{X}_t is smooth for $t \neq 0$ and \mathcal{X}_0 has an ordinary double point s : near s we can find local coordinates (x, y, z) such that $f(x, y, z) = x^2 + y^2 + z^2$. Pull back f by the covering $t \mapsto t^2$ of the disk: we obtain a new family $\mathcal{Y} \rightarrow \mathbb{D}$, where now \mathcal{Y} has an ordinary double point $x^2 + y^2 + z^2 = t^2$. Blowing up s in \mathcal{Y} we get a smooth threefold $\widehat{\mathcal{Y}}$ with a smooth quadric Q as exceptional divisor; we can now blow down Q along each of its two rulings to get smooth threefolds \mathcal{Y}' , \mathcal{Y}'' , which are *small resolutions* of \mathcal{Y} : the singular point s has been blown-up to a line.



The two fibrations $\mathcal{Y}' \rightarrow \mathbb{D}$ and $\mathcal{Y}'' \rightarrow \mathbb{D}$ are smooth; their fibres at 0 are both isomorphic to the blow up of \mathcal{X}_0 at s . By construction they coincide above $\mathbb{D} - \{0\}$, but it is easily checked that the isomorphism does not extend over \mathbb{D} .

The local systems $H^2(\mathcal{Y}'_t, \mathbf{Z})_{t \in \mathbb{D}}$ and $H^2(\mathcal{Y}''_t, \mathbf{Z})_{t \in \mathbb{D}}$ are constant, and coincide over $\mathbb{D} - \{0\}$; choosing compatible trivializations we get two non-isomorphic families of marked K3 surfaces on \mathbb{D} , which coincide on $\mathbb{D} - \{0\}$. The corresponding maps

$D \rightarrow \mathcal{M}_L$ coincide on $D - \{0\}$, but take different values at 0 . In other words, the marked surfaces \mathcal{Y}'_0 and \mathcal{Y}''_0 give *non-separated* points in the moduli space \mathcal{M}_L (every neighborhood of one of these points contains the other one).

To explain the analogous construction for higher-dimensional symplectic manifolds, let us first describe, in the simplest possible case, the *elementary transformations* discovered by Mukai [M]. We start with a symplectic manifold X , of dimension $2r$, containing a submanifold P isomorphic to \mathbf{P}^r . The 2-form φ restricted to P vanishes (in fancy words, P is a Lagrangian submanifold); therefore we have a commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_P & \longrightarrow & T_{X|P} & \longrightarrow & N_{P/X} \longrightarrow 0 \\ & & \downarrow & & \downarrow \varphi & & \downarrow \\ 0 & \longrightarrow & N_{P/X}^* & \longrightarrow & \Omega_{X|P}^1 & \longrightarrow & \Omega_P^1 \longrightarrow 0 \end{array}$$

in which all vertical arrows are isomorphisms. In particular, $N_{P/X}$ is isomorphic to Ω_P^1 .

Now blow-up P in X :

$$\begin{array}{ccc} E & \hookrightarrow & \widehat{X} \\ \downarrow & & \downarrow \\ P & \hookrightarrow & X \end{array}$$

The exceptional divisor E is by definition the projective normal bundle³ $\mathbf{P}(N_{P/X})$, which by the above remark is isomorphic to the projective cotangent bundle $\mathbf{P}T^*(P)$; thus we can view E as the variety of pairs (p, h) with $p \in P$, $h \in P^*$ (the space of hyperplanes in P) and $p \in h$. This is clearly symmetric: E is also isomorphic to $\mathbf{P}T^*(P^*)$, and in fact, using a classical contractibility criterion (due to Fujiki and Nakano in this context), we can blow down E onto P^* and get a new symplectic manifold X' , called the elementary transform of X along P . The map $X \dashrightarrow X'$ is a typical example of a birational map between symplectic manifolds which is not an isomorphism. Note that it is not known whether X' is always Kähler.

Now suppose we deform X in a family $\mathcal{X} \rightarrow D$. We have an exact sequence of normal bundles

$$0 \rightarrow N_{P/X} \cong \Omega_P^1 \rightarrow N_{P/\mathcal{X}} \rightarrow N_{X/\mathcal{X}} \cong \mathcal{O}_P \rightarrow 0.$$

³ We use the standard differential-geometric notation: if F is a vector bundle on a variety B , we put $\mathbf{P}(F) = \cup_{b \in B} \mathbf{P}(F_b)$ (see footnote ²).

The class of this extension lives in $H^1(P, \Omega_P^1)$; a straightforward computation shows that it is the restriction of the tangent vector in the deformation space of X provided by the deformation $\mathcal{X} \rightarrow D$ (remember that this tangent vector belongs to $H^1(X, T_X) \cong H^1(X, \Omega_X^1)$). Choose \mathcal{X} so that this tangent vector does not vanish on P , for instance is a Kähler class in $H^1(X, \Omega_X^1)$. Then the above extension is the non-trivial Euler extension

$$0 \rightarrow \Omega_P^1 \rightarrow V^* \otimes_{\mathbf{C}} \mathcal{O}_P(-1) \rightarrow N_{\mathcal{X}/\mathcal{X}} = \mathcal{O}_P \rightarrow 0,$$

where $P = \mathbf{P}(V)$. So we get an isomorphism $N_{P/\mathcal{X}} \cong V^* \otimes_{\mathbf{C}} \mathcal{O}_P(-1)$. Thus if we blow-up P in \mathcal{X} , the exceptional divisor \mathcal{E} is isomorphic to $P \times P^*$. As before we can blow-down \mathcal{E} onto P^* and get a manifold \mathcal{X}' with a smooth map $\mathcal{X}' \rightarrow D$, whose fibre at 0 is isomorphic to X' . Again the two families coincide above $D - \{0\}$. Therefore if X' is Kähler, X and X' (with appropriate markings) give non-separated points in the moduli space \mathcal{M}_L .

This example, due to D. Huybrechts, was the point of departure of his investigation of birational symplectic manifolds. The outcome is:

Theorem (Huybrechts, [H]).— *Let X, X' be two birational Kähler symplectic manifolds. There exists smooth families $\mathcal{X} \rightarrow D$ and $\mathcal{X}' \rightarrow D$ which are isomorphic over $D - \{0\}$ and such that \mathcal{X}_0 is isomorphic to X and \mathcal{X}'_0 to X' .*

As before it follows that X and X' , with appropriate markings, give non-separated points in the moduli space \mathcal{M}_L . Conversely, Huybrechts also proves that if (X, σ) and (X', σ') are non-separated points in \mathcal{M}_L , the manifolds X and X' are birational.

Corollary.— *Two Kähler symplectic manifolds which are birational are diffeomorphic.*

It is interesting to compare this statement with the following result of Batyrev [Ba]:

Proposition.— *Two Calabi-Yau manifolds which are birational have the same Betti numbers.*

The proof is (of course) completely different: it proceeds by reduction to characteristic p . Note that the two Calabi-Yau manifolds need not be diffeomorphic, as shown by an example of Tian and Yau (see [F], example 7.7).

3.5. Further developments

Kähler symplectic manifolds have been much studied in the recent years; there are two directions which I would like to emphasize. The structure of the cohomology algebra has been studied by Verbitsky; we will follow the elegant presentation of Bogomolov [Bo2].

Proposition. — Let X be a Kähler symplectic manifold of dimension $2r$, and let A be the subalgebra of $H^*(X, \mathbf{Q})$ spanned by $H^2(X, \mathbf{Q})$. Then $H^*(X, \mathbf{Q}) = A \oplus A^\perp$, and A is the quotient of $\mathbf{S}^*H^2(X, \mathbf{Q})$ by the ideal spanned by the elements x^{r+1} for all $x \in H^2(X, \mathbf{Q})$ with $q(x) = 0$.

Let Q be the quadric $q(x) = 0$ in $H^2(X, \mathbf{C})$. Since the period map is étale (3.3), we know that there is an open subset V of Q such that any element of V is the class of a 2-form on X , holomorphic with respect to some complex structure on X . This implies $x^{r+1} = 0$ for $x \in V$, and therefore for all $x \in Q$ by analytic continuation.

The rest of the proof is purely algebraic. Given a vector space H over \mathbf{Q} with a non-degenerate quadratic form q , we consider the algebra $A_r(H, q)$ quotient of \mathbf{S}^*H by the ideal spanned by the elements x^{r+1} for all $x \in H$ with $q(x) = 0$. Using the representation theory of $O(H, q)$, one proves that $A_r(H, q)$ is a Gorenstein algebra; more precisely $A_r^{2r}(H, q)$ is one-dimensional, and the pairing $A_r^i(H, q) \times A_r^{2r-i}(H, q) \rightarrow A_r^{2r}(H, q) \cong \mathbf{Q}$ is non-degenerate for each i .

Put $H = H^2(X, \mathbf{Q})$. By the geometric property above we get a ring homomorphism $A_r(H, q) \rightarrow H^*(X, \mathbf{Q})$. Its kernel is an ideal of $A_r(H, q)$; if it is non-zero, it contains the minimum ideal $A_r^{2r}(H, q)$, so the map $\mathbf{S}^{2r}H \rightarrow H^{4r}(X, \mathbf{Q})$ is zero — which is impossible since $\omega^{2r} \neq 0$ for a Kähler class ω . Hence A is isomorphic to $A_r(H, q)$; since the restriction of the intersection form on $H^*(X, \mathbf{Q})$ to A is non-degenerate, we have $H^*(X, \mathbf{Q}) = A \oplus A^\perp$. ■

Another exciting recent development is the construction by Rozansky and Witten of invariants of 3-manifolds associated to any compact hyperkähler manifold ([R-W]; an account more readable for an algebraic geometer appears in [K]). By the advanced technology of 3-dimensional topology, defining such invariants amounts to associate a complex number (a “weight”) to each trivalent graph, in such a way that a certain identity, the so-called IHX relation, is satisfied. The weights associated by Rozansky and Witten to a hyperkähler manifold are sort of generalized Chern numbers, which certainly deserve further study. Some explicit computations have been done by Hitchin and Sawon [H-S].

4. Quaternion-Kähler manifolds

4.1. The twistor space

The group $Sp(1)$ is the group of quaternions of norm 1; it acts on \mathbf{H}^r by homotheties. Since \mathbf{H} is not commutative, it is *not* contained in the unitary group $Sp(r)$, but it of course commutes with $Sp(r)$. A manifold of dimension $4r$ is said to

be *quaternion-Kähler* if its holonomy subgroup is contained in $\mathrm{Sp}(r)\mathrm{Sp}(1) \subset \mathrm{SO}(4r)$. As usual our manifolds are assumed to be compact and simply-connected; since $\mathrm{Sp}(1)\mathrm{Sp}(1) = \mathrm{SO}(4)$ we always suppose $r \geq 2$.

Despite the terminology, which is unfortunate but classical, a quaternion-Kähler manifold has no natural complex structure: the group $\mathrm{Sp}(r)\mathrm{Sp}(1)$ is not contained in $\mathrm{U}(2n)$.

The complex structures I, J, K are not invariant under $\mathrm{Sp}(1)$, and therefore they do not correspond any more to parallel complex structures. What remains invariant, however, is the 3-dimensional space spanned by I, J and K ; it gives rise to a 3-dimensional parallel sub-bundle $E \subset \mathcal{E}nd(\mathrm{T}(M))$. The unit sphere bundle $Z \subset E$ is called the *twistor space* of M ; the fibre of $p : Z \rightarrow M$ at a point $m \in M$ is a sphere \mathbf{S}^2 of complex structures on $\mathrm{T}_m(M)$, as in the hyperkähler case. The link between quaternion-Kähler manifolds and algebraic geometry is provided by the following result of Salamon [S]:

Proposition. — *Z admits a natural complex structure, for which the fibres of p are complex rational curves.*

The construction of this complex structure is quite natural. Since E is parallel, it inherits from the Levi-Civita connection on $\mathrm{T}(M)$ a linear connection, which is compatible with the metric. It follows that the corresponding horizontal distribution (App. A) induces a horizontal distribution on the fibration $p : Z \rightarrow M$, that is a sub-bundle $H \subset \mathrm{T}(Z)$ which is supplementary to the vertical tangent bundle $\mathrm{T}(Z/M)$.

Let $z \in Z$, and let $m = p(z)$. The fibre $p^{-1}(m)$ is canonically isomorphic to the standard sphere \mathbf{S}^2 , and therefore the vertical tangent space $\mathrm{T}_z(Z/M)$ has a well-defined complex structure. The space H_z projects isomorphically onto $\mathrm{T}_m(M)$, on which z defines by definition a complex structure. The direct sum of these complex structures define a complex structure on $\mathrm{T}(Z) = \mathrm{T}(Z/M) \oplus H$. A non-trivial calculation shows that it is integrable. ■

As an example, for the quaternionic projective space $M = \mathbf{HP}^r$, the twistor space Z is \mathbf{CP}^{2r+1} ; the fibration $p : Z \rightarrow M$ is the natural quotient map $W/\mathbf{C}^* \rightarrow W/\mathbf{H}^*$, with $W = \mathbf{C}^{2r+2} - \{0\} = \mathbf{H}^{r+1} - \{0\}$. Its fibres are (complex projective) lines in \mathbf{CP}^{2r+1} .

The behaviour of the complex manifold Z depends heavily on the sign of the scalar curvature k of (M, g) . This turns out to be a constant: in fact, Berger proved that a n -dimensional quaternion-Kähler manifold (M, g) satisfies the Einstein condition $\mathrm{Ric}_g = \frac{k}{n}g$ (I refer to [Be], Ch. 14.D for a discussion of the proof). The case $k = 0$ gives the hyperkähler manifolds (§ 3). In the case $k < 0$ there seems to be no natural Kähler structure on Z ; actually no compact example is known. We will therefore concentrate on the case $k > 0$, where some nice geometry appears.

Let me recall that a (compact) manifold X is *Fano* if its anticanonical bundle K_X^{-1} is ample (App. B). We will call a quaternion-Kähler manifold *positive* if its scalar curvature is positive.

Proposition. — *If M is positive, Z is a Fano manifold and admits a Kähler-Einstein metric.*

The metric on Z is obtained in the same way as the complex structure, by putting together the standard metric of the sphere \mathbf{S}^2 on $T(Z/M)$ and the metric of M on H (with the appropriate normalization). ■

The space Z has one more property, namely a (holomorphic) *contact structure*. We will now explain what this is.

4.2. Contact structures

Let X be a complex manifold. A *contact structure* on X is a corank 1 sub-bundle H of the (holomorphic) tangent bundle $T(X)$, so that we have an exact sequence

$$0 \rightarrow H \longrightarrow T(X) \xrightarrow{\theta} L \rightarrow 0 ,$$

where L is a line bundle. Moreover the following equivalent properties must hold:

- a) The 2-form $d\theta$, restricted to H , is non-degenerate at each point⁴;
- b) $\dim(X)$ is odd, say $= 2r + 1$, and the form $\theta \wedge (d\theta)^r$ is everywhere $\neq 0$;
- c) The L -valued alternate form $(U, V) \mapsto \theta([U, V])$ on H is non-degenerate at each point.

Let L^\times be the complement of the zero section in L^* . The pull-back of the line bundle L to L^\times has a canonical trivialization, so $p^*\theta$ becomes a honest 1-form on L^\times . Put $\omega = d(p^*\theta)$. This 2-form is *equivariant* with respect to the natural action of \mathbf{C}^* on L^\times by homotheties, that is $\lambda^*\omega = \lambda\omega$ for every $\lambda \in \mathbf{C}^*$.

Proposition (“contactization”). — *The 2-form ω is a symplectic structure on L^\times . Conversely, any \mathbf{C}^* -equivariant symplectic 2-form on L^\times defines a unique contact form $\theta \in H^0(X, \Omega_X^1 \otimes L)$ such that $\omega = d(p^*\theta)$.*

The form ω is closed, and using b) above we see easily that it is non-degenerate. For the converse, consider the “Euler field” ξ on L^\times corresponding to the \mathbf{C}^* -action. The 1-form $i(\xi)\omega$ vanishes on ξ and is equivariant, therefore it is the pull-back of a form $\theta \in H^0(X, \Omega_X^1 \otimes L)$. Since ω is equivariant, its Lie derivative $L_\xi\omega$ equals ω ; using the Cartan formula $L_\xi = di(\xi) + i(\xi)d$ we find $\omega = d(p^*\theta)$. It is then an easy exercise to prove that θ is a contact form, using for instance condition a). ■

⁴ The form $d\theta$ is defined locally using a trivialization of L ; it is an easy exercise to check that conditions a) and b) do not depend on the choice of the trivialization.

Example.— Let M be a complex manifold, and $X = \mathbf{P}T^*(M)$ its (holomorphic) projective cotangent bundle. Recall that the cotangent bundle $T^*(M)$ has a canonical symplectic structure $\omega = d\eta$, where η is the tautological 1-form on $T^*(M)$: the value of η at a point (α, m) of $T^*(M)$ ($m \in M, \alpha \in T_m^*(M)$) is the pull-back of α by the projection $T^*(M) \rightarrow M$. By construction η is equivariant with respect to the action of \mathbf{C}^* on $T^*(M)$ by homotheties, and so is ω . By the proposition we see that η is the pull-back of a contact form on X .

Going back to quaternion-Kähler manifolds, the link with contact structures is provided by the following theorem. Part a) is due to Salamon [S], part b) to LeBrun [L].

Theorem (LeBrun, Salamon).— a) *The twistor space of a positive quaternion-Kähler manifold is a Fano contact manifold, admitting a Kähler-Einstein metric.*

b) *Conversely, a Fano contact manifold which admits a Kähler-Einstein metric is the twistor space of a positive quaternion-Kähler manifold.*

The key point is that the horizontal sub-bundle $H \subset T(Z)$ (4.1) is *holomorphic*; this is proved by a local computation, and so is the fact that H defines a contact structure.

Thus the classification of positive quaternion-Kähler manifolds is essentially reduced to a problem of Algebraic Geometry. We are now going to explain a conjecture describing this classification.

4.3. Homogeneous contact manifolds

We have already mentioned that the only known examples of positive quaternion-Kähler manifolds are symmetric. More precisely, for each simple compact Lie group K there exists a unique quaternion-Kähler symmetric quotient of K ; the corresponding twistor space is homogeneous under the complexification G of K . These spaces have been classified by Wolf [W]. The twistor spaces admit the following simple description:

Proposition.— *Let G be a complex simple Lie group, \mathfrak{g} its Lie algebra. There is a unique closed orbit $X_{\mathfrak{g}}$ for the adjoint action of G on $\mathbf{P}(\mathfrak{g})$; $X_{\mathfrak{g}}$ is a Fano manifold, and admits a G -invariant contact structure.*

Note that the closure in $\mathbf{P}(\mathfrak{g})$ of any adjoint orbit contains a closed orbit, necessarily equal to $X_{\mathfrak{g}}$. Hence $X_{\mathfrak{g}}$ is the smallest orbit in $\mathbf{P}(\mathfrak{g})$.

Proof: I will give the proof because it is quite simple, though it requires some knowledge of algebraic groups. Let X be a closed orbit in $\mathbf{P}(\mathfrak{g})$, and let v be a vector of \mathfrak{g} whose class $[v] \in \mathbf{P}(\mathfrak{g})$ belongs to X . Since X is projective, the stabilizer P of $[v]$ contains a Borel subgroup B of G ; this means that v is an

eigenvector of B in \mathfrak{g} . Since \mathfrak{g} is simple, the adjoint representation of G in \mathfrak{g} is irreducible, so B has exactly, up to a scalar, one eigenvector (“highest weight vector”) $v_B \in \mathfrak{g}$; thus X is the G -orbit of $[v_B]$. It does not depend on the particular choice of B because all Borel subgroups are conjugate.

The pull-back of $X_{\mathfrak{g}}$ in $\mathfrak{g} - \{0\}$ is an adjoint orbit of G ; using the Killing form we can view it as a coadjoint orbit in \mathfrak{g}^* . Every such orbit admits a symplectic form, the Kostant-Kirillov structure, which is \mathbf{C}^* -equivariant and G -invariant. Using contactization we see that $X_{\mathfrak{g}}$ carries a G -invariant contact structure. ■

For classical Lie algebras, the contact manifold $X_{\mathfrak{g}}$ and the corresponding quaternion-Kähler manifold $M_{\mathfrak{g}}$ are given below:

\mathfrak{g}	$X_{\mathfrak{g}}$	$M_{\mathfrak{g}}$
$\mathfrak{sl}(n)$	$\mathbf{PT}^*(\mathbf{P}^{n-1})$	$\mathbf{G}(2, \mathbf{C}^n)$
$\mathfrak{o}(n)$	$\mathbf{G}_{iso}(2, \mathbf{C}^n)$	$\mathbf{G}^+(4, \mathbf{R}^n)$
$\mathfrak{sp}(2n)$	\mathbf{CP}^{2n-1}	$\mathbf{G}(1, \mathbf{H}^n) = \mathbf{HP}^{n-1}$

We have described the map $X_{\mathfrak{sp}(2n)} \rightarrow M_{\mathfrak{sp}(2n)}$ in 4.1. $X_{\mathfrak{o}(n)}$ is the grassmannian of isotropic 2-planes in \mathbf{C}^n and $M_{\mathfrak{o}(n)}$ the grassmannian of oriented 4-planes in \mathbf{R}^n ; the map $X_{\mathfrak{o}(n)} \rightarrow M_{\mathfrak{o}(n)}$ associates to a 2-plane $P \subset \mathbf{C}^n$ the real part of $P \oplus \bar{P}$. As in 3.4 we view $X_{\mathfrak{sl}(n)} = \mathbf{PT}^*(\mathbf{P}^{n-1})$ as the space of flags $D \subset H \subset \mathbf{C}^n$, where D is a line and H a hyperplane; choosing a hermitian scalar product on \mathbf{C}^n , this is also the space of pairs of orthogonal lines in \mathbf{C}^n . The map $X_{\mathfrak{sl}(n)} \rightarrow M_{\mathfrak{sl}(n)}$ associates to such a pair the 2-plane that they span.

In view of the LeBrun-Salamon theorem (4.2), every positive quaternion-Kähler compact manifold will be symmetric if every Fano contact manifold admitting a Kähler-Einstein metric is homogeneous. It is tempting to be a little bit more optimistic and to conjecture:

(C) *Every Fano contact manifold is homogeneous.*

We will give some (weak) evidence for the conjecture. Let X be a compact complex manifold, of dimension $2r + 1$, with a contact structure

$$0 \rightarrow H \rightarrow T(X) \xrightarrow{\theta} L \rightarrow 0 .$$

The form $\theta \wedge (d\theta)^r$ defines a nowhere vanishing section of $K_X \otimes L^{r+1}$; therefore we have $K_X \cong L^{-r-1}$, and X is Fano if and only if L is ample.

Proposition.— *Let X be a Fano contact manifold. If the line bundle L is very ample, X is homogeneous, and more precisely isomorphic to $X_{\mathfrak{g}}$ for some simple Lie algebra \mathfrak{g} .*

Proof: Let G be the group of automorphisms of X preserving the contact structure; its Lie algebra \mathfrak{g} consists of the vector fields V on X such that $[V, H] \subset H$. Let us prove that the space of global vector fields $H^0(X, T(X))$ is the direct sum of \mathfrak{g} and $H^0(X, H)$. Let V be a vector field on X . The map $W \mapsto \theta([V, W])$ from H to L is \mathcal{O}_X -linear, hence by property *c*) of contact structures (4.2), there exists a unique vector field V' in H such that $\theta([V, W]) = \theta([V', W])$ for all W in H . This means that $[V - V', W]$ belongs to H , that is that $V - V'$ belongs to \mathfrak{g} . Writing $V = V' + (V - V')$ provides the required direct sum decomposition.

The map $V \mapsto V'$ provides a \mathbf{C} -linear retraction of the inclusion of sheaves $H \hookrightarrow T(M)$; therefore the exact sequence

$$0 \rightarrow H \longrightarrow T(X) \xrightarrow{\theta} L \rightarrow 0$$

splits as a sequence of sheaves of vector spaces (*not* of \mathcal{O}_X -modules). In particular, the sequence

$$0 \rightarrow H^0(X, H) \longrightarrow H^0(X, T(X)) \xrightarrow{\theta} H^0(X, L) \rightarrow 0$$

is exact, and θ induces an isomorphism of \mathfrak{g} onto $H^0(X, L)$. This isomorphism is equivariant with respect to the action of G .

We will therefore identify $H^0(X, L)$ with \mathfrak{g} . The diagram of App. B becomes:

$$\begin{array}{ccc} L^\times & \xrightarrow{\mu} & \mathfrak{g}^* \\ \downarrow p & & \downarrow \gamma \\ X & \xrightarrow{\varphi} & \mathbf{P}(\mathfrak{g}^*) . \end{array}$$

Let $V \in \mathfrak{g}$. The action of G on L defines a canonical lift \tilde{V} of the vector field V to L^\times . By construction we have $\langle \mu, V \rangle = \eta(\tilde{V})$, where η is the 1-form $p^*\theta$ on L^\times (4.2). Since η is preserved by G , the Lie derivative $L_{\tilde{V}}\eta$ vanishes. By the Cartan homotopy formula, this implies

$$\langle d\mu, V \rangle = d(i(\tilde{V})\eta) = -i(\tilde{V})\omega ,$$

where $\omega := d\eta$ is the symplectic form on L^\times (this relation means by definition that μ is a *moment map* for the action of G on the symplectic manifold L^\times).

For $\xi \in L^\times$, $v \in T_\xi(L^\times)$, this formula reads $\langle T_\xi(\mu) \cdot v, V \rangle = \omega(v, \tilde{V}(\xi))$. When V runs in \mathfrak{g} , the vectors $\tilde{V}(\xi)$ span the tangent space to the orbit $G\xi$ at ξ ; thus

the kernel of $T_\xi(\mu)$ is the orthogonal of $T_\xi(G\xi)$ with respect to ω . In particular, if $T_\xi(\mu)$ is injective, the orbit $G\xi$ is open, and therefore *the orbit of $x = p(\xi)$ is open in X* .

Now if L is very ample, μ is an embedding, hence all the orbits of G are open – this is possible only if G acts transitively on X . Since X is projective this implies that G is semi-simple, so we can identify \mathfrak{g}^* with \mathfrak{g} , and $\varphi(X)$ with a closed adjoint orbit in $\mathbf{P}(\mathfrak{g})$. It follows easily that \mathfrak{g} is simple and $\varphi(X) = X_{\mathfrak{g}}$. ■

This result is improved in [B3], at the cost of assuming the Lie algebra \mathfrak{g} *reductive* – this is not too serious since it is always the case if X admits a Kähler-Einstein metric. The main result of [B3] is:

Theorem. – *Let X be a Fano contact manifold, such that:*

a) *The rational map $\varphi_L : X \dashrightarrow \mathbf{P}(H^0(X, L)^*)$ is generically finite (that is, $\dim \varphi_L(X) = \dim X$);*

b) *The Lie algebra \mathfrak{g} of infinitesimal contact automorphisms of X is reductive.*

Then \mathfrak{g} is simple, and X is isomorphic to $X_{\mathfrak{g}}$.

Idea of the proof: In view of the above proof, a) implies that G has an open orbit in L^\times . The image of this orbit in \mathfrak{g} (identified with \mathfrak{g}^* thanks to b)) is invariant by homotheties; this implies that it is a nilpotent orbit (if a matrix N is conjugate to λN for every $\lambda \in \mathbf{C}^*$, we have $\text{Tr } N^p = 0$ for each p , so N is nilpotent). Thus *the image of φ is the closure of a nilpotent orbit in $\mathbf{P}(\mathfrak{g})$* . Then a detailed study of nilpotent orbits leads to the result. ■

4.4. Further developments

More generally, we can ask which projective varieties admit contact structures. We have seen two examples, the projective cotangent bundles $\mathbf{P}T^*(M)$ (4.2) and the homogeneous spaces $X_{\mathfrak{g}}$ (4.3). Remarkably, *conjecture (C) implies that they are the only ones*, thanks to the following result:

Theorem ([KPSW] + [Dm]). – *A projective contact manifold is either the projective cotangent bundle of some projective manifold, or a Fano manifold with $b_2 = 1$.*

The result is proved in [KPSW] under the assumption that the canonical bundle is not nef; then Demailly proved that this is always the case for a contact manifold, as a consequence of a more general integrability theorem.

Appendix A

Connections

Let M be a differentiable manifold, E a vector bundle on M , $\mathcal{D}if^1(E)$ the vector bundles of differential operators of order ≤ 1 on E . A *connection* on E is a linear map $\nabla : T(M) \rightarrow \mathcal{D}if^1(E)$ which satisfies the Leibnitz rule

$$\nabla_V(fs) = f\nabla_V(s) + (Vf)s$$

for any vector field V , function f and section s of E defined over some open subset of M .

The connection extends naturally to the various tensor, symmetric or exterior powers of E , covariant or contravariant. For instance, if b is a bilinear form on E and u an endomorphism of E , we have

$$\begin{aligned}\nabla_V(b)(s, t) &= Vb(s, t) - b(\nabla_V s, t) - b(s, \nabla_V t) \\ \nabla_V(u)(s) &= \nabla_V(u(s)) - u(\nabla_V s)\end{aligned}$$

for any local sections s, t of E . We say that a section s of E (or of one of its associated tensor bundles) is *parallel* if $\nabla_V s = 0$ for any vector field V on M .

Let $f : M' \rightarrow M$ be a differentiable map. There exists a natural connection $f^*\nabla$ on f^*E , characterized by the condition $(f^*\nabla)_{V'}(f^*s) = f^*(\nabla_V s)$ for any section s of E and vector fields V on M , V' on M' such that f projects V' onto V . In particular, for any path $\gamma : [0, 1] \rightarrow M$, we get a connection on γ^*E , or equivalently a first order differential operator $\nabla_{d/dt}$ of γ^*E . Let $p = \gamma(0)$ and $q = \gamma(1)$; given a vector $v_p \in E_p$, there exists a unique section $t \mapsto v(t)$ of γ^*E such that $\nabla_{d/dt} v(t) = 0$ and $v(0) = v_p$. The map $v_p \mapsto v(1)$ defines the *parallel transport* isomorphism $\varphi_\gamma : E_p \rightarrow E_q$. Observe that a section s of E is parallel if and only if $\varphi_\gamma(s(p)) = s(q)$ for every path γ (this implies $s(\gamma(t)) = v(t)$, hence $\nabla_{\dot{\gamma}(t)} s = 0$).

The tangent vector $\dot{v}(0) \in T_{v_p}(E_p)$ is said to be *horizontal*; it is easy to show that the horizontal vectors form a sub-bundle H of $T(M)$, the *horizontal distribution* of ∇ , which is a supplement of the vertical sub-bundle $T(E/M)$.

Suppose now $E = T(M)$. The connection is said to be *symmetric* (or torsion-free) if $\nabla_V W - \nabla_W V = [V, W]$ for any vector fields V, W on M . Let g be a Riemannian metric on M ; a simple-minded computation shows that *there exists a unique symmetric connection ∇ on $T(M)$ for which g is parallel*. It is called the Levi-Civita connection of (M, g) .

Appendix B

Ample line bundles, Hodge theory

Ample line bundles

Let X be a compact complex manifold and L a line bundle on X ; we suppose $H^0(X, L) \neq 0$. For $x \in X$, let $\varphi_L(x)$ denote the subspace of global sections of L which vanish at x . It is either equal to $H^0(X, L)$ or to a hyperplane in $H^0(X, L)$. In the first case x belongs to the *base locus* B_L of L , that is the subvariety of the common zeros of all sections of L . The map $x \mapsto \varphi_L(x)$ defines a morphism $X - B_L \rightarrow \mathbf{P}(H^0(X, L))^*$, which we consider as a rational map $X \dashrightarrow \mathbf{P}(H^0(X, L))^*$. We say that L is *very ample* if φ_L is an embedding (this implies in particular $B_L = \emptyset$); it amounts to say that there is an embedding of X into some projective space \mathbf{P} such that L is the restriction of the tautological line bundle $\mathcal{O}_{\mathbf{P}}(1)$. We say that L is *ample* if some (positive) power of L is very ample.

Consider the dual line bundle $p: L^* \rightarrow X$. To any $\xi \in L^*$ associate the linear form $\mu(\xi): s \mapsto \langle s(p(\xi)), \xi \rangle$ on $H^0(X, L)$. We have a commutative diagram

$$\begin{array}{ccc} L^* & \xrightarrow{\mu_L} & H^0(X, L)^* \\ \downarrow p & & \downarrow \\ X & \xrightarrow{\varphi_L} & \mathbf{P}(H^0(X, L)^*) . \end{array}$$

Hodge decomposition

Let X be a compact Kähler manifold. Recall that a differentiable form on X is of type (p, q) if it can be written in any system of local coordinates (z_1, \dots, z_n) as a sum of forms $a(z, \bar{z}) dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$. We denote by $H^{p,q} \subset H^{p+q}(X, \mathbf{C})$ the subspace of De Rham cohomology classes of forms of type (p, q) ; we have $H^{q,p} = \overline{H^{p,q}}$. The fundamental result of Hodge theory is the Hodge decomposition

$$H^n(X, \mathbf{C}) = \bigoplus_{p+q=n} H^{p,q} ,$$

together with the canonical isomorphisms $H^{p,q} \xrightarrow{\sim} H^q(X, \Omega_X^p)$. In particular,

$$H^2(X, \mathbf{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} ,$$

with $H^{2,0} \cong H^0(X, \Omega_X^2)$, embedded into $H^2(X, \mathbf{C})$ by associating to a holomorphic form its De Rham class.

To any hermitian metric g on X is associated a real 2-form ω of type $(1, 1)$, the *Kähler form*, defined by $\omega(V, W) = g(V, JW)$ for any real vector fields V, W ;

the metric is Kähler if ω is closed. Then its class in $H^2(X, \mathbf{C})$ is called a Kähler class. The Kähler classes form an open cone in $H_{\mathbf{R}}^{1,1} := H^{1,1} \cap H^2(X, \mathbf{R})$.

Let L be a line bundle on X . The Chern class $c_1(L) \in H^2(X, \mathbf{C})$ is integral, that is comes from $H^2(X, \mathbf{Z})$, and belongs to $H^{1,1}$. Conversely, any integral class in $H^{1,1}$ is the Chern class of some line bundle on X (Lefschetz theorem).

If L is very ample, its Chern class is the pull-back by φ_L of the Chern class of $\mathcal{O}_{\mathbf{P}}(1)$, which is a Kähler class, and therefore $c_1(L)$ is a Kähler class. More generally, if L is ample, some multiple of $c_1(L)$ is a Kähler class, hence also $c_1(L)$. Conversely, the celebrated Kodaira embedding theorem asserts that *a line bundle whose Chern class is Kähler is ample*. As a corollary, we see that *any compact Kähler manifold X with $H^0(X, \Omega_X^2) = 0$ is projective*: we have $H^2(X, \mathbf{C}) = H^{1,1}$, hence the cone of Kähler classes is open in $H^2(X, \mathbf{R})$. Therefore it contains integral classes; by the above results such a class is the first Chern class of an ample line bundle, hence X is projective. More generally, the same argument shows that X is projective whenever the subspace $H^{1,1}$ of $H^2(X, \mathbf{C})$ is defined over \mathbf{Q} .

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