Invent. math. (1999)
Digital Object Identifier (DOI) 10.1007/s002229900043

Inventiones mathematicae

Symplectic singularities

Arnaud Beauville

DMI – École Normale Supérieure (UMR 8553 du CNRS), 45 rue d'Ulm,
F-75230 Paris Cedex 05, France

Oblatum 16-III-1999 & 2-IX-1999 / Published online: 29 November 1999

Introduction

We introduce in this paper a particular class of rational singularities, which we call *symplectic*, and classify the simplest ones. Our motivation comes from the analogy between rational Gorenstein singularities and Calabi-Yau manifolds: a compact, Kähler manifold of dimension n is a Calabi-Yau manifold if it admits a nowhere vanishing n-form, while a normal variety V of dimension n has rational Gorenstein singularities[1] if its smooth part V_{reg} carries a nowhere vanishing n-form, with the extra property that its pull-back in any resolution $X \to V$ extends to a holomorphic form on X. Among Calabi-Yau manifolds an important role is played by the symplectic (or hyperkähler) manifolds, which admit a holomorphic, everywhere non-degenerate 2-form; by analogy we say that a normal variety V has *symplectic singularities* if V_{reg} carries a closed symplectic 2-form whose pull-back in any resolution $X \to V$ extends to a holomorphic 2-form on X. Note that this last condition is automatic if the singular locus of V has codimension ≥ 4 [F], in particular for isolated singularities of dimension > 2.

We will look for the simplest possible isolated symplectic singularities $o \in V$, namely those whose projective tangent cone is smooth: this means that blowing up o in V provides a resolution of V with a smooth exceptional divisor. Examples of such singularities are obtained as follows. Each simple complex Lie algebra has a smallest non-zero nilpotent orbit $\mathcal{O}_{\min}$ for the adjoint action; its closure $\overline{\mathcal{O}}_{\min} = \mathcal{O}_{\min} \cup \{0\}$ has a symplectic singularity at 0, isomorphic to the cone over the smooth variety $\mathbf{P}\mathcal{O}_{\min} := \mathcal{O}_{\min}/\mathrm{C}^*$. In particular its projective tangent cone is smooth (it is isomorphic to $\mathbf{P}\mathcal{O}_{\min}$).

Our main result is the converse:

Theorem.– *Let* (V, o) *be a germ of isolated symplectic singularity, whose projective tangent cone is smooth. Then* (V, o) *is analytically isomorphic to the germ* $(\overline{\mathcal{O}}_{\min}, 0)$ *for some simple complex Lie algebra*.

[1] also called canonical singularities of index 1.

The key point of the proof is the fact that the homogeneous space $\mathbf{P}\mathcal{O}_{\min}$ carries a holomorphic *contact structure* (inherited from the symplectic structure of $\mathcal{O}_{\min}$). Given a resolution $X \to V$ with a smooth exceptional divisor E, we show that the extension to X of the symplectic form has a residue on E which defines a contact structure. We then deduce from [B1] that E is isomorphic to some $\mathbf{P}\mathcal{O}_{\min}$, and we conclude with a classical criterion of Grauert.

We discuss in §4 whether a classification of isolated symplectic singularities makes sense. Each such singularity gives rise to many others by considering its quotient by a finite group; to get rid of those we propose to consider only isolated symplectic singularities with trivial local fundamental group. The singularities $(\overline{\mathcal{O}}_{\min}, 0)$ have this property when the Lie algebra is not of type C_l; it is certainly desirable to find more examples.

1. Definition and basic properties

We consider algebraic varieties over C (our results extend readily to the analytic category). We will say that a holomorphic 2-form on a smooth variety is *symplectic* if it is closed and non-degenerate at every point. A *resolution* of an algebraic variety V is a proper, birational morphism $f : X \to V$ where X is smooth.

Definition 1.1.– *A variety has a symplectic singularity at a point if this point admits an open neighborhood* V *such that:*

a) V *is normal;*
b) *The smooth part* V_{reg} *of* V *admits a symplectic* 2*-form* φ*;*
c) *For any resolution* $f : X \to V$*, the pull back of* φ *to* $f^{-1}(V_{\text{reg}})$ *extends to a holomorphic* 2*-form on* X.

We will mostly consider a symplectic singularity as a germ (V, o) – in which case we will always assume that V satisfies the conditions a) to c).

(1.2) A result of Flenner [F] guarantees that condition c) holds when $\operatorname{codim} \operatorname{Sing}(V) \geq 4$. We chose to impose it in all cases in order to get uniform results.

As for rational singularities it is enough to check c) for one particular resolution: this follows easily from the fact that two given resolutions of V are dominated by a common resolution.

Proposition 1.3.– *A symplectic singularity is rational Gorenstein.*

Proof: We keep the notation of Definition 1.1 and put $\dim V = 2r$. The form φ^r generates the line bundle $\omega_{V_{\text{reg}}}$, and for any resolution $X \to V$ extends to a holomorphic form on X; this implies that V has rational Gorenstein singularities [R]. □

The following remark shows that isolated symplectic singularities of dimension > 2 are *not* local complete intersections:

Proposition 1.4.– *Let* V *be a variety with symplectic singularities which is locally a complete intersection. Then the singular locus of* V *has codimension* ≤ 3*.*

Proof: We can realize locally V as a complete intersection in some smooth variety S. The exact sequence

$$0 \to N^*_{V/S} \longrightarrow \Omega^1_{S|V} \longrightarrow \Omega^1_V \to 0$$

provides a length 1 locally free resolution of Ω^1_V. We can assume codim Sing(V) ≥ 3; by the Auslander-Buchsbaum theorem and the fact that V is Cohen-Macaulay, the depth of Ω^1_V at every point of Sing(V) is ≥ 2. It follows that Ω^1_V is a reflexive sheaf, so the isomorphism $\Omega^1_{V_{reg}} \to T_{V_{reg}}$ defined by a symplectic 2-form on V_{reg} extends to an isomorphism $\Omega^1_V \to T_V$. Combining the resolution of Ω^1_V and its dual we get an exact sequence

$$0 \to N^*_{V/S} \longrightarrow \Omega^1_{S|V} \longrightarrow T_{S|V} \xrightarrow{u} N_{V/S} ,$$

where the support of the cokernel T^1 of u is exactly Sing(V). Using the Auslander-Buchsbaum theorem again we get $\dim(T^1) = \dim \mathrm{Sing}(V) \geq \dim(V) - 3$. □

2. Examples

(2.1) In dimension 2, the symplectic singularities are the rational double points (that is, the A-D-E singularities).

(2.2) Any product of varieties with symplectic singularities has again symplectic singularities.

(2.3) *Quotient singularities*

The following result will provide us with a large list of symplectic singularities:

Proposition 2.4.– *Let* V *be a variety with symplectic singularities,* G *a finite group of automorphisms of* V*, preserving a symplectic* 2*-form on* V_{reg}*. Then* V/G *has symplectic singularities.*

Proof: We first observe that the fixed locus F_g in V_{reg} of any element $g \neq 1$ in G is a symplectic subvariety of V_{reg} ([Fu], Prop. 2.6), and therefore has codimension ≥ 2. Let $V^o := V_{reg} - \bigcup_{g\neq 1} F_g$. The symplectic 2-form on V^o descends to a symplectic 2-form φ^o on V^o/G; since the complement of V^o/G in V/G has codimension ≥ 2, φ^o extends to a symplectic 2-form φ on

$(V/G)_{reg}$. Let $g : Y \to V/G$ be a resolution of V/G; by taking a resolution of $Y \times_{(V/G)} V$ we get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & V \\ \big\downarrow & & \big\downarrow \\ Y & \xrightarrow{g} & V/G \end{array}$$

where f is a resolution of V. Then $g^*\varphi$ is a meromorphic 2-form on Y, whose pull back to X is holomorphic. By an easy local computation, this implies that $g^*\varphi$ is holomorphic. □

(2.5) This applies for instance when V is a finite-dimensional symplectic vector space, and G a finite subgroup of Sp(V). If we impose moreover that the non trivial elements of G have all their eigenvalues $\neq 1$, then V/G has an isolated symplectic singularity. As J. Wahl pointed out to me, a complete (and rather lengthy) list of such finite subgroups can be deduced from [Wo], thm. 7.2.18 (if $\dim(V) = 2$ we get the well-known list of finite subgroups of SL(V), the corresponding quotient singularities being the rational double points). The simplest case is obtained when $G = \{\pm \mathrm{Id}_V\}$; the quotient V/G is then isomorphic to the cone over the Veronese embedding of $\mathbf{P}(V)$ into $\mathbf{P}(S^2V)$. In particular, the projective tangent cone at the singular point of V/G is isomorphic to $\mathbf{P}(V)$. It will follow from our Theorem and from §4 below that for all other isolated symplectic quotient singularities V/G, the projective tangent cone at the singular point is not smooth.

Proposition 2.4 also applies to the symmetric products $V^{(p)} = V^p/\mathfrak{S}_p$: if the variety V has symplectic singularities, so does $V^{(p)}$.

(2.6) *Nilpotent orbits*

Let $\mathfrak{g}$ be a simple complex Lie algebra and $\mathcal{O} \subset \mathfrak{g}$ a nilpotent orbit (for the adjoint action)[2]. Then *the normalization of the closure of $\mathcal{O}$ in $\mathfrak{g}$ has symplectic singularities*. This is due to Panyushev [P], who uses it to prove that this variety has rational Gorenstein singularities. The point is that $\mathcal{O}$ can be identified with a coadjoint orbit using the Killing form, and therefore carries the Kostant-Kirillov symplectic 2-form.

In particular, the Lie algebra $\mathfrak{g}$ contains a unique (non-zero) minimal nilpotent orbit $\mathcal{O}_{\min}$, which is contained in the closure of all non-zero nilpotent orbits. The closure $\overline{\mathcal{O}}_{\min} = \mathcal{O}_{\min} \cup \{0\}$ is normal, and has an isolated symplectic singularity at 0.

This singularity can be described as follows. The orbit $\mathcal{O}_{\min}$ is stable by homotheties; the quotient $\mathbf{P}\mathcal{O}_{\min} := \mathcal{O}_{\min}/\mathbf{C}^*$ is a smooth, closed subvariety of $\mathbf{P}(\mathfrak{g})$. The variety $\overline{\mathcal{O}}_{\min}$ is the cone over $\mathbf{P}\mathcal{O}_{\min} \subset \mathbf{P}(\mathfrak{g})$. This means that we have a resolution $f : L^{-1} \to \overline{\mathcal{O}}_{\min}$, where L is the restriction of $\mathcal{O}_{\mathbf{P}(\mathfrak{g})}(1)$

[2] A general reference for nilpotent orbits is [C-M].

to $\mathbf{P}\mathcal{O}_{\min}$, and f contracts to 0 the zero section E of L^{-1}. In this situation f is the blow up of 0 in $\overline{\mathcal{O}}_{\min}$, and the exceptional divisor E, isomorphic to $\mathbf{P}\mathcal{O}_{\min}$, is the projective tangent cone to 0 in $\overline{\mathcal{O}}_{\min}$.

For instance, let V be a finite-dimensional symplectic vector space; the Lie algebra $\mathfrak{sp}(V)$ can be identified with S^2V, in such a way that $\mathcal{O}_{\min}$ (resp. $\overline{\mathcal{O}}_{\min}$) is the image of $V - \{0\}$ (resp. V) by the map $v \mapsto v \cdot v$. In other words, $\overline{\mathcal{O}}_{\min}$ is isomorphic to $V/\{\pm 1\}$ (see (2.5)) and $\mathbf{P}\mathcal{O}_{\min}$ to $\mathbf{P}(V)$.

3. Characterization of minimal orbits singularities

(3.1) This section is devoted to the proof of the theorem stated in the introduction. So we let (V, o) be an isolated symplectic singularity, $f : X \to V$ the blow up of the maximal ideal of o in V, and E the exceptional divisor. By construction E is isomorphic to the projective tangent cone to V at o; we assume that it is smooth. Since E is a Cartier divisor in X it follows that X is smooth.

We denote by i the embedding of E in X, and put $L := i^*\mathcal{O}_X(-E)$. By the standard properties of the blow up the line bundle L on E is *very ample*.

(3.2) Let $\dim V = 2r$. We can assume that $V - \{o\}$ carries a symplectic 2-form which extends to a holomorphic 2-form φ on X; we have $\operatorname{div}(\varphi^r) = kE$ for some integer $k \geq 0$. The adjunction formula gives $K_E = L^{-k-1}$, so that E is a Fano manifold. This implies $H^0(E, \Omega_E^p) = 0$ for each $p \geq 1$, and in particular $i^*\varphi = 0$.

Let $e \in E$. Since φ is closed, we can write $\varphi = d\alpha$ in a neighbourhood U of e in X, where α is a 1-form on U such that $i^*\alpha$ is closed. Shrinking U if necessary we can write $i^*\alpha = d(i^*g)$ for some function g on U; replacing α by $\alpha - dg$ we may assume $i^*\alpha = 0$. If $u = 0$ is a local equation of E in U, this means that α is of the form $u\,\tilde{\theta} + h\,du$, where $\tilde{\theta}$ is a 1-form and h a function on U; replacing α by $\alpha - d(hu)$ and $\tilde{\theta}$ by $\tilde{\theta} - dh$ we arrive at $\alpha = u\tilde{\theta}$ and

$$\varphi = du \wedge \tilde{\theta} + u\, d\tilde{\theta}\ .$$

This gives $\varphi^r = ru^{r-1}du \wedge \tilde{\theta} \wedge (d\tilde{\theta})^{r-1} + u^r(d\tilde{\theta})^r$. Thus the order of vanishing k of φ^r along E is $\geq r-1$; the crucial point of the proof is the equality $k = r-1$. We need an easy lemma:

Lemma 3.3.– *Let* X *be a smooth closed submanifold of a projective space* $\mathbf{P}^N$, *of degree* ≥ 2. *Then* $H^0(X, \bigwedge^p T_X(-p)) = 0$ *for* $0 < p < \dim(X)$, *and for* $p = \dim(X)$ *except if* X *is a hyperquadric.*

Proof: When X is a hyperquadric our assertion is equivalent to $H^0(X, \Omega_X^q(q)) = 0$ for $0 < q < \dim(X)$, which can be checked by a direct computation (see for instance [K], thm. 3). We assume $\deg(X) \geq 3$.

The case $p = 1$ follows from a more general result of Wahl ([W], see remark below). Let $p \geq 2$; we use induction on the dimension of X, the case of curves being clear. Let H be a smooth hyperplane section of X; the exact sequence

$$0 \to T_H \longrightarrow T_{X|H} \longrightarrow \mathcal{O}_H(1) \to 0$$

gives rise to exact sequences

$$0 \to \textstyle\bigwedge^p T_H(-p) \longrightarrow \bigwedge^p T_{X|H}(-p) \longrightarrow \bigwedge^{p-1} T_H(-(p-1)) \to 0.$$

By the induction hypothesis we conclude that $H^0(H, \bigwedge^p T_{X|H}(-p))$ is zero. Thus a section of $H^0(X, \bigwedge^p T_X(-p))$ must vanish on any smooth hyperplane section of X, and therefore vanishes identically. □

Remark 3.4.– Wahl's result is rather easy in our situation: using the exact sequence

$$0 \to H^0(X, T_X(-1)) \longrightarrow H^0(X, T_{\mathbf{P}^N}(-1)_{|X}) \longrightarrow H^0(X, N_{X/\mathbf{P}^N}(-1))$$

and the isomorphism $\mathbf{C}^{N+1} \xrightarrow{\sim} H^0(X, T_{\mathbf{P}^N}(-1)_{|X})$ deduced from the Euler exact sequence, we see that a nonzero element of $H^0(X, T_X(-1))$ corresponds to a point $p \in \mathbf{P}^N$ such that all projective tangent spaces $\mathbf{P}T_x(X)$, for x in X, pass through p. This is easily seen to be impossible, for instance by induction on dim(X).

It seems natural to conjecture that the statement of the lemma extends to the more general situation considered in [W], namely that $H^0(X, \bigwedge^p T_X \otimes L^{-p}) = 0$ *for* $p > 0$ *whenever* L *is ample, except if* $(X, L) = (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$, *with* $n \geq p$, *or* $(X, L) = (Q_p, \mathcal{O}_{Q_p}(1))$.

(3.5) We now prove the equality $k = r - 1$. If $E = \mathbf{P}^{2r-1}$ and $L = \mathcal{O}_{\mathbf{P}^{2r-1}}(1)$, V is smooth; if $E = \mathbf{P}^1$ and $L = \mathcal{O}_{\mathbf{P}^1}(2)$, V is a surface with an ordinary double point. We exclude these two cases. The perfect pairing $\Omega^1_X \otimes \Omega^{2r-1}_X \to K_X$ provides an isomorphism $\Omega^{2r-1}_X \cong T_X \otimes K_X$; thus exterior product with φ^{r-1} gives a linear map $\Omega^1_X \to T_X(kE)$, which is an isomorphism outside E (it is the inverse of the isomorphism defined by φ). This map may vanish on E, say with order $k - j$ ($j \leq k$), so that we get a map $\lambda : \Omega^1_X \to T_X(jE)$ whose restriction to E is nonzero. Observe that $\det \lambda$ is a section of $\mathcal{O}_X(2(rj - k)E)$ which is nonzero outside E, hence $k \leq rj$ and in particular $j \geq 0$.

We have a diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & \Omega^1_{X|E} & \longrightarrow & \Omega^1_E & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \lambda_{|E}} & & & & \\ 0 & \longrightarrow & T_E \otimes L^{-j} & \longrightarrow & T_{X|E} \otimes L^{-j} & \longrightarrow & L^{-j-1} & \longrightarrow & 0\,. \end{array}$$

Since $j \geq 0$ we have $\mathrm{Hom}(L, L^{-j-1}) = \mathrm{Hom}(\Omega^1_E, L^{-j-1}) = \mathrm{Hom}(L, T_E \otimes L^{-j}) = 0$ by Lemma 3.3. Thus $\lambda_{|E}$ factors through a map

$\mu : \Omega^1_E \to T_E \otimes L^{-j}$; since λ is antisymmetric μ is antisymmetric, that is, comes from an element of $H^0(E, \bigwedge^2 T_E \otimes L^{-j})$.

Since $\lambda_{|E}$ is non-zero, Lemma 3.3 implies $j \leq 1$, hence $k \leq r$. Moreover if $k = rj$, det λ does not vanish, hence λ and therefore $\lambda_{|E}$ are isomorphisms; but this is impossible because $\lambda_{|E}$ vanishes on the sub-bundle $L \subset \Omega^1_{X|E}$. Thus we have $k < rj$, and therefore $j = 1$ and $k = r - 1$.

(3.6) Going back to the local computation of (3.2), we observe that the form $\theta := i^*\tilde{\theta}$ is defined globally as a section of $\Omega^1_E \otimes L$: it is the image of $\varphi \in H^0(X, \Omega^2_X(\log E)(-E))$ by the residue map $\Omega^2_X(\log E)(-E) \to \Omega^1_E \otimes \mathcal{O}_X(-E)_{|E}$. We now know that the $(2r)$-form $du \wedge \tilde{\theta} \wedge (d\tilde{\theta})^{r-1}$ on U does not vanish, so the twisted $(2r-1)$-form $\theta \wedge (d\theta)^{r-1} \in H^0(E, K_E \otimes L^r)$ does not vanish. This means, by definition, that θ is a *contact structure* on the Fano manifold E. The classification of Fano contact manifolds is an interesting problem, with important applications to Riemannian geometry (see for instance [L] or [B2]). Here we have one more information, namely that the line bundle L is *very* ample; this implies that E is isomorphic to one of the homogeneous contact manifolds $\mathbf{P}\mathcal{O}_{\min}$ ([B1], cor. 1.8).

(3.7) It remains to show that the embedding of E in X is isomorphic, in some open neighbourhood of E, to the embedding of the zero section in the line bundle $L^{-1} \to E$. By a criterion of Grauert [G], it is sufficient to prove that the spaces $H^1(E, T_E \otimes L^k)$ and $H^1(E, L^k)$ are zero for $k \geq 1$. Since E is a Fano manifold, the second assertion follows from the Kodaira vanishing theorem; since the tangent bundle of E is spanned by its global sections, the first one follows from the Griffiths vanishing theorem ([Gr], Theorem G).

4 Local fundamental group

(4.1) In view of (2.3) it seems hopeless to classify all isolated symplectic singularities: there are too many quotient singularities, already in dimension 4. One way to get around this problem is to consider only singularities with *trivial local fundamental group*. We briefly recall the definition: if (V, o) is an isolated singularity, we can find a fundamental system $(V_n)_{n\geq 1}$ of neighbourhoods of o such that V_q is a deformation retract of V_p for $q \geq p$; the group $\pi_1(V_n)$, which is independant of n and of the particular fundamental system, is called the local fundamental group of V at o and denoted $\pi_1^o(V)$ (for a canonical definition one should be more careful about base points, but this is irrelevant here).

If (V, o) is a quotient of an isolated singularity (W, ω) by a finite group G acting on W with ω as only fixed point, we have an exact sequence

$$0 \to \pi_1^{\omega}(W) \longrightarrow \pi_1^o(V) \longrightarrow G \to 0$$

(in particular $\pi_1^o(V) = G$ if W is smooth of dimension ≥ 2). Conversely, to each surjective homomorphism of $\pi_1^o(V)$ onto a finite group G corresponds an isolated singularity (W, ω) with an action of G fixing only ω such that $W/G \cong V$; if (V, o) is a symplectic singularity, so is (W, ω). Therefore a first step in a possible classification is to study isolated symplectic singularities with trivial local fundamental group. It turns out that the singularities $(\overline{\mathcal{O}}_{\min}, 0)$ are of this type (with one exception):

Proposition 4.2.– *Let $\mathfrak{g}$ be a simple complex Lie algebra, and $\mathcal{O}_{\min} \subset \mathfrak{g}$ its minimal nilpotent orbit. Then $\pi_1^0(\overline{\mathcal{O}}_{\min}) = 0$ except if $\mathfrak{g}$ is of type C_r $(r \geq 1)$; in that case $\pi_1^0(\overline{\mathcal{O}}_{\min}) = \mathbf{Z}/(2)$, and the corresponding double covering of $\overline{\mathcal{O}}_{\min}$ is smooth.*

Proof: Consider the resolution $f : L^{-1} \to \overline{\mathcal{O}}_{\min}$ (2.6); denote by $E \subset L^{-1}$ the zero section. Let D be a tubular neighbourhood of E in L^{-1}, and $D^* = D - E$. Since the homogeneous space $\mathbf{P}\mathcal{O}_{\min}$ is simply-connected, the homotopy exact sequence of the fibration $f : D^* \to \mathbf{P}\mathcal{O}_{\min}$ reads

$$H_2(\mathbf{P}\mathcal{O}_{\min}, \mathbf{Z}) \xrightarrow{\partial} \mathbf{Z} \longrightarrow \pi_1(D^*) \to 0 ,$$

where the map ∂ corresponds to the Chern class $c_1(L^{-1}) \in H^2(\mathbf{P}\mathcal{O}_{\min}, \mathbf{Z})$.

Put $\dim \mathbf{P}\mathcal{O}_{\min} = 2r-1$. Since $K_{\mathbf{P}\mathcal{O}_{\min}} = L^{-r}$, the class $c_1(L)$ is primitive unless $\mathbf{P}\mathcal{O}_{\min} = \mathbf{P}^{2r-1}$, which occurs exactly when $\mathfrak{g}$ is of type C_r (see [B1]). Assume this is not the case. The homotopy exact sequence gives $\pi_1(D^*) = 0$; since the pull back of any neighbourhood of 0 in $\overline{\mathcal{O}}_{\min}$ contains a tubular neighbourhood of E, this implies $\pi_1^0(\overline{\mathcal{O}}_{\min}) = 0$.

If $\mathfrak{g}$ is of type C_r the same argument gives $\pi_1^0(\overline{\mathcal{O}}_{\min}) = \mathbf{Z}/(2)$; actually we have seen in (2.6) that $\overline{\mathcal{O}}_{\min}$ is isomorphic to the quotient of $\mathbf{C}^{2r}$ by the involution $v \mapsto -v$. □

(4.3) It would be interesting to find more examples of isolated symplectic singularities with trivial local fundamental group, and also examples with *infinite* local fundamental group.

Acknowledgements. I am grateful to J. Wahl for useful discussions.

References

[B1] A. Beauville: Fano contact manifolds and nilpotent orbits, Comment. Math. Helv. **73**, 566–583 (1998)

[B2] A. Beauville: Riemannian Holonomy and Algebraic Geometry. Preprint math.AG/9902110

[C-M] D. Collingwood, W. McGovern: Nilpotent orbits in semi-simple Lie algebras. Van Nostrand Reinhold Co., New York (1993)

[F] H. Flenner: Extendability of differential forms on non-isolated singularities. Invent. math. **94**, 317–326 (1988)

[Fu] A. Fujiki: On primitively symplectic compact Kähler *V*-manifolds of dimension 4. Classification of algebraic and analytic manifolds. PM **39**, 71–250; Birkhäuser (1983)

[G] H. Grauert: Über Modifikationen und exzeptionelle analytische Mengen. Math. Ann. **146**, 331–368 (1962)

[Gr] P. Griffiths: Hermitian differential geometry, Chern classes, and positive vector bundles. Global Analysis (Papers in Honor of K. Kodaira), 185–251; Univ. Tokyo Press (1969)

[K] Y, Kimura: On the hypersurfaces of Hermitian symmetric spaces of compact type. Osaka J. Math. **16**, 97–119 (1979)

[L] C. Le Brun: Fano manifolds, contact structures, and quaternionic geometry. Int. J. Math. **6**, 419–437 (1995)

[P] D.I. Panyushev: Rationality of singularities and the Gorenstein properties of nilpotent orbits. Functional Anal. Appl. **25**, 225–226 (1991)

[R] M. Reid: Canonical 3-folds. Journées de Géometrie Algébrique d'Angers (1979), 273–310; Sijthoff & Noordhoff (1980)

[W] J. Wahl: A cohomological characterization of $\mathbf{P}^n$. Invent. math. **72**, 315–322 (1983)

[Wo] J.A. Wolf: Spaces of constant curvature, 3rd ed. Publish or Perish, Boston (1974)