The Szpiro inequality for higher genus fibrations

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Abstract. Let $f : S \to B$ be a non-trivial family of semi-stable curves of genus g, N the number of critical points of f and s the number of singular fibres. We prove the inequality $N < (4g+2)(s+2g(B)-2)$.

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1. Introduction

The aim of this note is to prove the following result:

Proposition. *Let $f : S \to B$ be a non-trivial semi-stable fibration of genus $g \geq 2$, N the number of critical points of f and s the number of singular fibres. Then*

$$N < (4g+2)(s+2g(B)-2).$$

Recall that a semi-stable fibration of genus g is a surjective holomorphic map of a smooth projective surface S onto a smooth curve B, whose generic fibre is a smooth curve of genus g and whose singular fibres are allowed only ordinary double points; moreover we impose that each smooth rational curve contained in a fibre meets the rest of the fibre in at least 2 points (otherwise by blowing up non-critical points of f in a singular fibre we could arbitrarily increase N keeping s fixed).

The corresponding inequality $N \leq 6(s+2g(B)-2)$ in the case $g = 1$ has been observed by Szpiro; it was motivated by the case of curves over a number field, where an analogous inequality would have far-reaching consequences [S]. The higher genus case is considered in the recent preprint [BKP], where the authors prove the slightly weaker inequality $N \leq (4g+2)s$ for hyperelliptic fibrations over $\mathbf{P}^1$. Their method is topological, and in fact the result applies in the much wider context of symplectic Lefschetz fibrations. We will show that in the more restricted algebraic-geometric set-up, the proposition is a direct consequence of two classical inequalities in surface theory. It would be interesting to know whether the proof of [BKP] can be extended to non-hyperelliptic fibrations.

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2. Proof of the proposition

The main numerical invariants of a surface S are the square K_S^2 of the canonical bundle, the Euler–Poincaré characteristic $\chi(\mathcal{O}_S)$ and the topological Euler–Poincaré characteristic $e(S)$; they are linked by the Noether formula $12\,\chi(\mathcal{O}_S) = K_S^2 + e(S)$. For a semi-stable fibration $f : S \to B$ it has become customary to modify these invariants as follows. Let b be the genus of B, and $K_f = K_X \otimes f^*K_B^{-1}$ the relative canonical bundle of X over B; then we consider:

$$K_f^2 = K_X^2 - 8(b-1)(g-1)$$
$$\chi_f := \deg f_*(K_f) = \chi(\mathcal{O}_X) - (b-1)(g-1)$$
$$e_f := N = e(X) - 4(b-1)(g-1).$$

Observe that we have again $12\,\chi_f = K_f^2 + e_f$. We will use the Xiao inequality ([X], Theorem 2)

$$K_f^2 \geq \left(4 - \frac{4}{g}\right)\chi_f$$

and the "strict canonical class inequality" ([T], lemma 3.1)

$$K_f^2 < 2(g-1)(s+2b-2) \quad \text{for } s > 0.$$

Let us prove the proposition. If $s = 0$, we have $N = 0$ and $g(B) \geq 2$ (otherwise the fibration would be trivial), so the inequality of the proposition holds. Assume $s > 0$; the Xiao inequality gives

$$3g\,K_f^2 \geq 12(g-1)\,\chi_f = (g-1)(K_f^2 + e_f),$$

hence, using the strict canonical class inequality,

$$N = e_f \leq \frac{2g+1}{g-1}\,K_f^2 < (4g+2)(s+2b-2). \qquad \square$$

Example. We constructed in [B] a semi-stable genus 3 fibration over $\mathbf{P}^1$ with 5 singular fibres; each of these has 8 double points. Therefore

$$N = 40 \quad \text{and} \quad (4g+2)(s-2) = 42.$$

Remark. M. Kim pointed out to me that one gets a finer inequality by taking into account the dimension g_0 of the fixed part of the Jacobian fibration associated to f. Indeed one can deduce (with some work) from Arakelov's seminal paper [A] the inequality $\chi_f \leq \frac{1}{2}(g-g_0)(s+2b-2)$ (see also [D] for a Hodge-theoretical proof); combined with Xiao's inequality this gives

$$N \leq (4g+2)(s+2b-2)\left(1 - \frac{g_0}{g}\right)$$

(note however that the inequality is no longer strict).

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