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Some stable vector bundles with reducible theta divisor

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Abstract. Let C be a curve of genus g and L a line bundle of degree $2g$ on C. Let M_L be the kernel of the evaluation map $H^0(C, L) \otimes_C L \to L$. We show that when L is general enough, the rank g bundle M_L and its exterior powers are stable, but admit a reducible theta divisor.

Introduction

Let C be a curve of genus g, and E a vector bundle on C, of rank r; assume that the *slope* $\mu := \frac{1}{r}\deg E$ of E is an integer. Let J^ν be the translate of the Jacobian of C parametrizing line bundles of degree $\nu := g - 1 - \mu$ on C. We say that E *admits a theta divisor* if $H^0(E \otimes L) = 0$ for L general in J^ν. If this is the case, the locus

$$\Theta_E = \{L \in J^\nu \mid H^0(E \otimes L) \neq 0\}$$

has a natural structure of effective divisor in J^ν, the *theta divisor* of E. Its class in $H^2(J^\nu, \mathbf{Z})$ is $r\theta$, where $\theta \in H^2(J^\nu, \mathbf{Z})$ is the class of the principal polarization. This (generalized) theta divisor plays a key role in the recent work on vector bundles on curves – see for instance [B] for an overview.

If E admits a theta divisor, it is semi-stable (otherwise E contains a sub-bundle F of slope $> \mu$, and by Riemann-Roch $H^0(F \otimes L)$, and therefore $H^0(E \otimes L)$, is non-zero for all $L \in J^\nu$). The converse does not hold, at least in rank ≥ 4: Raynaud has constructed examples of stable vector bundles with no theta divisor [R]. Further examples have been constructed recently by Popa [P].

If E is semi-stable but not stable, its theta divisor (if it exists) is not integral: more precisely, E admits a filtration with stable quotients $E_1, \ldots, E_p$, and we have $\Theta_E = \Theta_{E_1} + \ldots + \Theta_{E_p}$. One may ask, conversely, if the reducibility of Θ_E implies that E is not stable. A counter-example has been given by Raynaud (unpublished), who constructed a rank 2 stable vector bundle on a curve of genus 3 with reducible theta divisor. Such an example can only occur on a special curve, because in rank 2 the divisor Θ_E characterizes the vector bundle E [B-V], and on a general curve

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the only reducible divisors on J^ν with cohomology class 2θ are the theta divisors of rank 2 decomposable bundles.

We describe in this note a counter-example of a different nature, namely a family of stable vector bundles of rank g which exist on any curve of genus g. They are defined by the exact sequence

$$0 \to M_L \longrightarrow H^0(C, L) \otimes_{\mathbf{C}} \mathcal{O}_C \xrightarrow{ev_L} L \to 0$$

where L is a line bundle generated by its global sections, and ev_L the evaluation map. These vector bundles have been intensively studied, notably by Green and Lazarsfeld (see in particular [L]), Paranjape and Ramanan [P-R], and more recently in [P] and [F-M-P]. In the latter paper the authors determine the theta divisor of M_K and of its exterior powers; we will take advantage of their result to do the same in the case of a line bundle L of degree $2g$ (so that M_L has rank g). We will prove (in a somewhat more precise form):

Theorem. *Let* C *be a non-hyperelliptic curve, and* L *a sufficiently general line bundle of degree* $2g$ *on* C*. The vector bundle* M_L *and its exterior powers* $\Lambda^2 M_L, \ldots,$ $\Lambda^{g-1} M_L$ *are stable and admit a reducible theta divisor.*

An interesting extra feature of our examples is that there exists a semi-stable, decomposable vector bundle on C with the same theta divisor as M_L; thus in rank ≥ 3 the divisor Θ_E does not characterize the bundle E any more.

Notation

We fix a curve C of genus g over the complex numbers; except in Remark 2 below, we assume throughout that C *is not hyperelliptic*. We denote by K its canonical bundle. For $d \in \mathbf{Z}$, we denote by J^d the translate of the Jacobian of C parametrizing line bundles of degree d on C, and by C_d the locus of effective divisor classes in J^d. If $p, q \in \mathbf{Z}$ the difference variety $C_p - C_q$ lies in J^{p-q}.

I. The theta divisor of E_L

Let L be a line bundle of degree $2g$ on the curve C. It is spanned by its global sections, so we have an exact sequence

$$0 \to M_L \longrightarrow H^0(L) \otimes_{\mathbf{C}} \mathcal{O}_C \longrightarrow L \to 0,$$

where M_L is a rank g vector bundle. We put $E_L := M_L^*$.

Though this will not be used in the sequel, let us recall the geometric interpretation of E_L. Let φ be the morphism of C into the projective space $\mathbf{P} := \mathbf{P}(H^0(L))$ defined by the linear system $|L|$; in view of the Euler exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}} \longrightarrow H^0(L)^* \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}}(1) \longrightarrow T_{\mathbf{P}} \to 0,$$

we have $E_L = \varphi^* T_{\mathbf{P}} \otimes L^{-1}$.

The vector bundle E_L has rank g and determinant L, hence slope 2.

Proposition 1. *a) The vector bundle* $\mathrm{E_L}$ *has a theta divisor*

$$\Theta_{\mathrm{E_L}} = (\mathrm{C}_{g-2} - \mathrm{C}) + \Theta_{\mathrm{L}\otimes\mathrm{K}^{-1}} \quad in\ \mathrm{J}^{g-3}.$$

b) $\mathrm{E_L}$ *is semi-stable; it is stable if and only if* L *is very ample.*

Proof. We will first compute set-theoretically the theta divisor $\Theta_{\mathrm{M_L}}$ of $\mathrm{M_L}$. By definition this is the set of line bundles $\mathrm{P} \in \mathrm{J}^{g+1}$ such that the multiplication map $m : \mathrm{H}^0(\mathrm{L}) \otimes \mathrm{H}^0(\mathrm{P}) \to \mathrm{H}^0(\mathrm{L} \otimes \mathrm{P})$ is not injective. Let us distinguish three cases:

(i) If $h^0(\mathrm{P}) > 2$ we have $\dim(\mathrm{H}^0(\mathrm{L}) \otimes \mathrm{H}^0(\mathrm{P})) > \dim \mathrm{H}^0(\mathrm{L} \otimes \mathrm{P})$, thus $\mathrm{P} \in \Theta_{\mathrm{M_L}}$.
(ii) Assume that $h^0(\mathrm{P}) = 2$ and that the pencil $|\mathrm{P}|$ has a base point. Both spaces $\mathrm{H}^0(\mathrm{L}) \otimes \mathrm{H}^0(\mathrm{P})$ and $\mathrm{H}^0(\mathrm{L} \otimes \mathrm{P})$ have the same dimension $2g + 2$. If m is injective, it is surjective, and the linear system $|\mathrm{L} \otimes \mathrm{P}|$ has a base point; this is impossible since $\deg(\mathrm{L} \otimes \mathrm{P}) = 3g + 1$. Thus we have again $\mathrm{P} \in \Theta_{\mathrm{M_L}}$.
(iii) Finally assume that $|\mathrm{P}|$ is a base-point free pencil. From the exact sequence

$$0 \to \mathrm{P}^{-1} \longrightarrow \mathrm{H}^0(\mathrm{P}) \otimes_{\mathbf{C}} \mathcal{O}_{\mathrm{C}} \longrightarrow \mathrm{P} \to 0$$

we get an exact sequence

$$0 \to \mathrm{H}^0(\mathrm{L} \otimes \mathrm{P}^{-1}) \longrightarrow \mathrm{H}^0(\mathrm{L}) \otimes_{\mathbf{C}} \mathrm{H}^0(\mathrm{P}) \xrightarrow{m} \mathrm{H}^0(\mathrm{L} \otimes \mathrm{P});$$

thus m is not injective in that case if and only if $\mathrm{H}^0(\mathrm{L} \otimes \mathrm{P}^{-1}) \neq 0$.

The line bundles P in case (i) and (ii) are exactly those which can be written $\mathrm{P}'(x)$, for some point x of C and some line bundle P' in J^g with $h^0(\mathrm{P}') \geq 2$; the ones in case (iii) are those of the form $\mathrm{L} \otimes \mathrm{P}'^{-1}$, with $\mathrm{P}' \in \Theta \subset \mathrm{J}^{g-1}$. Since $\Theta_{\mathrm{E_L}}$ is the image of $\Theta_{\mathrm{M_L}}$ by the isomorphism $\mathrm{P} \mapsto \mathrm{K} \otimes \mathrm{P}^{-1}$ of J^{g+1} onto J^{g-3}, we obtain (still set-theoretically) $\Theta_{\mathrm{E_L}} = (\mathrm{C}_{g-2} - \mathrm{C}) \cup \Theta_{\mathrm{L}\otimes\mathrm{K}^{-1}}$. Now $\mathrm{C}_{g-2} - \mathrm{C}$ is an irreducible divisor with cohomology class $(g-1)\theta$ (see e.g. [F-M-P], Prop. 3.7), and $\Theta_{\mathrm{L}\otimes\mathrm{K}^{-1}}$ is a (ordinary) theta divisor; since $\Theta_{\mathrm{E_L}}$ has cohomology class $g\theta$, we get the equality a).

Since $\mathrm{E_L}$ admits a theta divisor, it is semi-stable. Moreover, if $\mathrm{E_L}$ is not stable, its stable components are $\mathrm{L}' := \mathrm{L} \otimes \mathrm{K}^{-1}$ and a rank $(g-1)$ bundle. Thus L' is either a sub- or a quotient bundle of $\mathrm{E_L}$. The latter case cannot occur since $\mathrm{E_L}$ is generated by its global sections and L' is not. Now using the exact sequence

$$0 \to \mathrm{L}^{-1} \otimes \mathrm{L}'^{-1} \longrightarrow \mathrm{H}^0(\mathrm{L})^* \otimes_{\mathbf{C}} \mathrm{L}'^{-1} \longrightarrow \mathrm{E_L} \otimes \mathrm{L}'^{-1} \to 0$$

and Serre duality we see that $\mathrm{Hom}(\mathrm{L}', \mathrm{E_L})$ is zero if and only if the multiplication map $\mathrm{H}^0(\mathrm{L}) \otimes \mathrm{H}^0(\mathrm{L}) \to \mathrm{H}^0(\mathrm{L}^2)$ is surjective, that is, L is normally generated [G-L]. By [G-L], Thm. 1, this is the case if and only if L is very ample. □

Remarks. 1) If L is not very ample, we have $\mathrm{L} = \mathrm{K(D)}$, with D an effective divisor of degree 2. The snake lemma applied to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_K & \longrightarrow & H^0(K)\otimes\mathcal{O}_C & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_L & \longrightarrow & H^0(L)\otimes\mathcal{O}_C & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

provides an exact sequence $0 \to M_K \to M_L \to \mathcal{O}_C(-D) \to 0$; thus E_L is an extension of E_K by $\mathcal{O}_C(D)$. This extension is non-trivial, as we already observed that $\mathcal{O}_C(D)$ cannot be a quotient of E_L.

2) If C is hyperelliptic, the divisor $C_{g-2} - C$ is equal to Θ_H, where H is the hyperelliptic pencil on C. By specialization we get $\Theta_{E_L} = (g-1)\Theta_H + \Theta_{L\otimes K^{-1}}$. The line bundle L is not linearly normal [L-M], so E_L is not stable.

The difference variety $C_{g-2} - C$ is the theta divisor of the bundle E_K [P-R]; therefore:

Corollary 1. *Assume that* L *is very ample. The stable bundle* E_L *and the decomposable bundle* $E_K \oplus (L \otimes K^{-1})$ *have the same theta divisor.* □

The equality still holds of course when L is not very ample, but becomes immediate, since in that case the second bundle is the sum of the stable components of the first one.

In view of the results of [B-N-R], this corollary can be rephrased as follows. Let $\mathcal{SU}_C(g)$ be the moduli space of semi-stable rank g vector bundles on C with trivial determinant, and let $\mathcal{L}$ be the positive generator of $\mathrm{Pic}(\mathcal{SU}_C(g))$ (the *determinant bundle*). Let $B_{\mathcal{L}}$ be the base locus of the linear system $|\mathcal{L}|$.

Corollary 2. *The map* $\varphi_{\mathcal{L}} : \mathcal{SU}_C(g) - B_{\mathcal{L}} \longrightarrow \mathbf{P}(H^0(\mathcal{L}))$ *defined by the line bundle* $\mathcal{L}$ *is not injective.*

Indeed this map can be identified with the map which associates to a vector bundle its theta divisor [B-N-R]. Twisting E_L and $E_K \oplus (L\otimes K^{-1})$ by a line bundle λ on C with $\lambda^{-g} = L$, we get two different points of $\mathcal{SU}_C(g) - B_{\mathcal{L}}$ with the same image under $\varphi_{\mathcal{L}}$. □

II. The theta divisor of $\Lambda^p E_L$

We now consider the exterior power $\Lambda^p E_L$; this is a vector bundle of rank $\binom{g}{p}$ and slope $2p$, so its theta divisor, if it exists, lies in J^{g-1-2p}.

Proposition 2. *Let* $1 \le p \le g-1$. *If* L *is general enough, the vector bundle* $\Lambda^p E_L$ *is stable and admits a theta divisor*[1]

$$\Theta_{\Lambda^p E_L} = (C_{g-p-1} - C_p) + (C_{g-p} - C_{p-1} + K\otimes L^{-1}).$$

[1] The second term is the translate of $C_{g-p} - C_{p-1} \subset J^{g+1-2p}$ by the element $K\otimes L^{-1}$ of J^{-2}.

Proof. We first prove that $\Lambda^p E_L$ admits a theta divisor when L is general enough. Since this is an open property, it is sufficient to prove this for a particular choice of L: we take $L = K(D)$, with D an effective divisor of degree 2. The exact sequence

$$0 \to \mathcal{O}_C(D) \to E_L \to E_K \to 0$$

(Remark 1) gives rise to an exact sequence

$$0 \to \Lambda^{p-1}E_K\,(D) \to \Lambda^p E_L \to \Lambda^p E_K \to 0.$$

By [F-M-P] $\Lambda^p E_K$ and $\Lambda^{p-1}E_K$ admit a theta divisor, hence also so does $\Lambda^{p-1}E_K(D)$ for any divisor D. Since the three vector bundles in the exact sequence have the same slope $2p$, we see that $\Lambda^p E_L$ admits a theta divisor.

Let us now prove that the theta divisor Θ_{E_L}, when it exists, is given by the formula of the Proposition. The divisor $C_q - C_{g-1-q}$ has cohomology class $\binom{g-1}{q}\theta$ ([F-M-P], Prop. 3.7), so both sides of the formula have cohomology class $\binom{g}{p}\theta$. It suffices therefore to prove that each component of the right hand side is contained in $\Theta_{\Lambda^p E_L}$.

As in [P] and [F-M-P], we will use the following observation of Lazarsfeld [L]: if $x_1, \dots , x_{g-1}$ are generic points of C, there is an exact sequence

$$0 \to \bigoplus_{i=1}^{g-1} \mathcal{O}_C(x_i) \longrightarrow E_L \longrightarrow L(-\sum x_i) \to 0.$$

Put $F = \overset{g-1}{\underset{i=1}{\oplus}} \mathcal{O}_C(x_i)$. We have an exact sequence of exterior powers

$$0 \to \Lambda^p F \to \Lambda^p E_L \to \Lambda^{p-1}F \otimes L(-\sum x_i) \to 0,$$

that is,

$$0\to \bigoplus_{i_1<\ldots<i_p} \mathcal{O}_C(x_{i_1}+\ldots+x_{i_p}) \longrightarrow \Lambda^p E_L \longrightarrow \bigoplus_{j_1<\ldots<j_{g-p}} L(-x_{j_1}-\ldots-x_{j_{g-p}}) \to 0.$$

This gives:

- $H^0(\Lambda^p E_L(-x_1 - \ldots - x_p)) \neq 0$, hence the inclusion $C_{g-p-1} - C_p \subset \Theta_{\Lambda^p E_L}$;
- $H^0(\Lambda^p M_L \otimes L(-x_1 - \ldots - x_{g-p})) \neq 0$, hence $H^0(\Lambda^p M_L \otimes L(-D)) \neq 0$ for all D in $C_{g-p} - C_{p-1}$; by Serre duality this gives $H^0(\Lambda^p E_L \otimes K \otimes L^{-1}(D)) \neq 0$, hence the inclusion $C_{g-p} - C_{p-1} + K \otimes L^{-1} \subset \Theta_{\Lambda^p E_L}$.

It remains to prove that $\Lambda^p E_L$ is stable. Since L is generic, E_L is stable (Proposition 1), so $\Lambda^p E_L$ is *polystable* – that is, direct sum of stable bundles with the same slope $2p$. If $\Lambda^p E_L$ is not stable for L generic, it is decomposable for all values of L; we will see that this is not the case when L is of the form K(D), with D effective of degree 2. In that case we have by Remark 1 an exact sequence

$$0 \to \Lambda^{p-1}E_K\,(D) \longrightarrow \Lambda^p E_L \longrightarrow \Lambda^p E_K \to 0$$

where $\Lambda^{p-1}E_K$ (D) and $\Lambda^p E_K$ are stable with slope $2p$; if $\Lambda^p E_L$ is decomposable, this exact sequence splits. The following easy lemma shows that this is not the case, and thus concludes the proof of the Proposition. □

Lemma. *Let* X *be a scheme over a field of characteristic* 0*, and let*

$$(\mathcal{E}) \qquad 0 \to \mathrm{M} \to \mathrm{E} \to \mathrm{F} \to 0$$

be a non-split exact sequence of vector bundles on X*, with* $\operatorname{rk} \mathrm{M} = 1$*. The associated exact sequences*

$$(\mathbf{\Lambda}^p\mathcal{E}) \qquad 0 \to \mathbf{\Lambda}^{p-1}\mathrm{F} \otimes \mathrm{M} \longrightarrow \mathbf{\Lambda}^p\mathrm{E} \longrightarrow \mathbf{\Lambda}^p\mathrm{F} \to 0$$

do not split for $1 \le p \le \operatorname{rk} \mathrm{F}$.

Proof. Let $i : \mathrm{F}^* \otimes \mathrm{M} \to \mathcal{H}om(\mathbf{\Lambda}^p\mathrm{F}, \mathbf{\Lambda}^{p-1}\mathrm{F} \otimes \mathrm{M})$ be the linear map deduced from the interior product. A straightforward computation shows that the class of the extension $(\mathbf{\Lambda}^p\mathcal{E})$ in $\mathrm{H}^1(\mathrm{X}, \mathcal{H}om(\mathbf{\Lambda}^p\mathrm{F}, \mathbf{\Lambda}^{p-1}\mathrm{F} \otimes \mathrm{M}))$ is the image by $\mathrm{H}^1(i)$ of the class of the extension $(\mathcal{E})$ in $\mathrm{H}^1(\mathrm{X}, \mathrm{F}^* \otimes \mathrm{M})$. But in characteristic zero i admits a retraction $c^{-1}\rho$, where $c = \binom{\operatorname{rk}\mathrm{F}-1}{p-1}$ and $\rho : \mathcal{H}om(\mathbf{\Lambda}^p\mathrm{F}, \mathbf{\Lambda}^{p-1}\mathrm{F} \otimes \mathrm{M}) \to \mathrm{F}^* \otimes \mathrm{M}$ is the map deduced from the interior product $\mathbf{\Lambda}^p\mathrm{F}^* \otimes \mathbf{\Lambda}^{p-1}\mathrm{F} \to \mathrm{F}^*$. Thus $\mathrm{H}^1(i)$ is injective, and the lemma follows. □

As in section I this gives:

Corollary 1. *The vector bundles* $\mathbf{\Lambda}^p\mathrm{E}_\mathrm{L}$ *and* $\mathbf{\Lambda}^p\mathrm{E}_\mathrm{K} \oplus (\mathbf{\Lambda}^{p-1}\mathrm{E}_\mathrm{K} \otimes \mathrm{L} \otimes \mathrm{K}^{-1})$ *have the same theta divisor. In particular, the map* $\varphi_\mathcal{L} : \mathcal{SU}_\mathrm{C}(\binom{g}{p}) - \mathrm{B}_\mathcal{L} \longrightarrow \mathbf{P}(\mathrm{H}^0(\mathcal{L}))$ *defined by the line bundle* $\mathcal{L}$ *is not injective.* □

Let us conclude by a link with the main theme of [F-M-P], the so-called *minimal resolution conjecture* for the curve C embedded into $\mathbf{P}^g := \mathbf{P}(\mathrm{H}^0(\mathrm{L}))$. We have to refer to [F-M-P] for the statement of the conjecture, which is a bit technical. Let us just say that it describes, for all general finite subsets $\Gamma \subset \mathrm{C}$ of cardinality $\ge g+1$, the minimal graded resolution of the ideal I_Γ of Γ in the coordinate ring $\mathrm{S} = \mathbf{C}[\mathrm{X}_0, \ldots, \mathrm{X}_g]$ of $\mathbf{P}^g$. By Corollary 1.8 of [F-M-P], this conjecture holds if and only if each of the bundles $\mathbf{\Lambda}^p\mathrm{E}_\mathrm{L}$ admit a theta divisor. Thus:

Corollary 2. *The curve* C*, embedded into* $\mathbf{P}^g$ *by a general linear system of degree* $2g$*, satisfies the "minimal resolution conjecture" in the sense of* [F-M-P]. □

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