VECTOR BUNDLES AND THETA FUNCTIONS ON CURVES OF GENUS 2 AND 3

By Arnaud Beauville

Abstract. Let $\mathcal{SU}_C(r)$ be the moduli space of vector bundles of rank r and trivial determinant on a curve C. A general E in $\mathcal{SU}_C(r)$ defines a divisor Θ_E in the linear system $|r\Theta|$, where Θ is the canonical theta divisor in $\mathrm{Pic}^{g-1}(C)$. This defines a rational map θ: $\mathcal{SU}_C(r) \dashrightarrow |r\Theta|$, which turns out to be the map associated to the determinant bundle on $\mathcal{SU}_C(r)$ (the positive generator of $\mathrm{Pic}(\mathcal{SU}_C(r))$). In genus 2 we prove that this map is generically finite and dominant. The same method, together with some classical work of Morin, shows that in rank 3 and genus 3 the theta map is a finite morphism – in other words, every vector bundle in $\mathcal{SU}_C(3)$ admits a theta divisor.

Introduction. Let C be a smooth projective complex curve, of genus $g \geq 2$. The moduli space $\mathcal{SU}_C(r)$ of semi-stable vector bundles of rank r on C, with trivial determinant, is a normal projective variety, which can be considered as a nonabelian analogue of the Jacobian variety JC. It is actually related to JC by the following construction, which goes back (at least) to [N-R]. Let J^{g-1} be the translate of JC parameterizing line bundles of degree $g-1$ on C, and $\Theta \subset J^{g-1}$ the canonical theta divisor. For $E \in \mathcal{SU}_C(r)$, consider the locus

$$\Theta_E := \{L \in J^{g-1} \mid H^0(C, E \otimes L) \neq 0\}.$$

Then either $\Theta_E = J^{g-1}$, or Θ_E is in a natural way a divisor in J^{g-1}, belonging to the linear system $|r\Theta|$. In this way we get a rational map

$$\theta\colon\ \mathcal{SU}_C(r) \dashrightarrow |r\Theta|$$

which is the most obvious rational map of $\mathcal{SU}_C(r)$ in a projective space: it can be identified to the map $\varphi_{\mathcal{L}}$: $\mathcal{SU}_C(r) \dashrightarrow \mathbb{P}(H^0(\mathcal{SU}_C(r), \mathcal{L})^*)$ given by the global sections of the determinant bundle $\mathcal{L}$, the positive generator of the Picard group of $\mathcal{SU}_C(r)$ [B-N-R].

For $r = 2$ the map θ is an embedding if C is not hyperelliptic [vG-I]. We consider in this paper the higher rank case, where very little is known. The first part is devoted to the case $g = 2$. There a curious numerical coincidence occurs, namely

$$\dim \mathcal{SU}_C(r) = \dim |r\Theta| = r^2 - 1.$$

Manuscript received June 7, 2004.
American Journal of Mathematics 128 (2006), 607–618.

For $r = 2$ θ is an isomorphism [N-R]; for $r = 3$ it is a double covering, ramified along a sextic hypersurface which is the dual of the "Coble cubic" [O]. For $r \geq 4$ it is no longer a morphism [R], and this makes its analysis much more delicate. We will prove:

THEOREM A. *For a curve C of genus* 2, *the map* θ: $\mathcal{SU}_C(r) \dashrightarrow |r\Theta|$ *is generically finite* (or, equivalently, dominant). *It admits some fibers of dimension* $\geq [\frac{r}{2}] - 1$.

Our method is to consider the fibre of θ over a reducible element of $|r\Theta|$ of the form $\Theta + \Delta$, where Δ is general in $|(r-1)\Theta|$. The main point is to show that this fibre restricted to the stable locus of $\mathcal{SU}_C(r)$ is finite. The other elements of the fibre are the classes of the bundles $\mathcal{O}_C \oplus F$, with $\Theta_F = \Delta$; reasoning by induction on r we may assume that there are finitely many such F, and this gives the first assertion of the theorem (§1). The second one is obtained by considering the restriction of θ to the subspace of symplectic vector bundles (§2).

The method is not, in principle, restricted to genus 2 curves – but the geometry in higher genus becomes much more intricate. In the second part of the paper (§3) we will apply it to rank 3 bundles in genus 3. Our result is:

THEOREM B. *Let C be a curve of genus* 3. *The map* θ: $\mathcal{SU}_C(3) \to |3\Theta|$ *is a finite morphism.*

This means that a semi-stable vector bundle of rank 3 on C has always a theta divisor; or alternatively (see e.g. [B1]), that the linear system $|\mathcal{L}|$ on $\mathcal{SU}_C(3)$ is base point free.

This is not a big surprise since the result is already known for a *generic* curve of genus 3 [R]. We believe, however, that the method is more interesting than the result itself. In fact we translate the problem into a question of classical projective geometry: what are the continuous families of planes in $\mathbb{P}^5$ such that any two planes of the family intersect? It turns out that this question has been completely (and beautifully) solved by Morin [M]. Translating back his result into the language of vector bundles we get a complete list of the stable rank 3 bundles E of degree 0 such that $\Theta_E \supset \Theta$ (Theorem 3.1 below). Theorem B follows as a corollary.

Acknowledgments. I am very much indebted to C. Ciliberto for pointing out the paper of Morin and for making it accessible to me.

Notations. Throughout the paper we will work with a complex curve C (smooth, projective, connected), of genus g. If E is a vector bundle on C, we will write $H^0(E)$ for $H^0(C, E)$, and $h^0(E)$ for its dimension.

1. Genus 2: the generic finiteness. In this section we assume $g = 2$. The first part of Theorem A follows from a slightly more precise result:

PROPOSITION 1.1. *Let Δ be a general divisor in $|(r-1)\Theta|$. The fibre $\theta^{-1}(\Theta+\Delta)$ is finite and nonempty.*

1.2. We will prove the proposition by induction on r. Let $[E_0] \in \theta^{-1}(\Theta+\Delta)$. If it is not stable, it is the class of a direct sum $\bigoplus\limits_i E_i$, so that $\Theta_{E_0} = \sum_i \Theta_{E_i}$; thus $[E_0]$ is the class in $\mathcal{SU}_C(r)$ of $\mathcal{O}_C \oplus F$ for some $F \in \mathcal{SU}_C(r-1)$ with $\Theta_F = \Delta$. By the induction hypothesis there exists only finitely many such F, and there exists at least one.

Thus we can assume that E_0 is stable. Let $E := E_0^* \otimes K_C$. We have $h^0(E) = r$ by Riemann-Roch and the stability of E_0. The inclusion $\Theta = C \subset \Theta_{E_0}$ means that $h^0(E_0(p)) \geq 1$ for all $p \in C$, or equivalently by Serre duality $h^0(E(-p)) \geq 1$; this implies that the subsheaf F of E generated by the global sections of E has rank $< r$. Moreover if p does not belong to Δ, it is a smooth point of Θ_{E_0}, and thus satisfies $h^0(E(-p)) = 1$ (see e.g. [L], §V); therefore $\operatorname{rk} F = r-1$ (otherwise we would have $h^0(E(-p)) \geq h^0(F(-p)) \geq 2$).

1.3. Let Z be a component of the locus of stable bundles E of rank r and determinant $K^{\otimes r}$ with the property that $H^0(E)$ spans a subsheaf of rank $r-1$ of E. We will prove the inequality $\dim Z \leq \dim|(r-1)\Theta|$. It implies that the general fibre of $\theta\colon Z \dashrightarrow \Theta + |(r-1)\Theta|$ is finite (possibly empty), so the Proposition follows.

Let E be a general element of Z, and let F be the subsheaf of E spanned by $H^0(E)$. Put $L = \det F$ and $d = \deg F = \deg L$; we have an exact sequence

$$0 \to L^{-1} \longrightarrow H^0(E) \otimes_{\mathbb{C}} \mathcal{O}_C \longrightarrow F \to 0,$$

hence a linear map $H^0(E)^* \to H^0(L)$. Let $s = r - \dim H^0(C, F^*)$ be the rank of that map. Then $F = \mathcal{O}_C^{r-s} \oplus G$, where G is a vector bundle of rank $s-1$ with $h^0(G) = s$, $h^0(G^*) = 0$, which fits into an exact sequence

$$0 \to L^{-1} \longrightarrow \mathcal{O}_C^s \longrightarrow G \to 0.$$

The quotient $\mathcal{M} = E/F$ is the direct sum of a line bundle M and a torsion sheaf $\mathcal{T}$. We have $c_1(M) + c_1(\mathcal{T}) = rc_1(K_C) - c_1(L)$, and this formula determines M once $\mathcal{T}$ and L are given. We denote by t the length of $\mathcal{T}$.

1.4. To summarize, we have associated to a general bundle E in Z integers s, d, t and

- a line bundle L of degree d, and a s-dimensional subspace $V \subset H^0(C, L)$ generating L; from these data we define G as the cokernel of the natural map $L^{-1} \to V^* \otimes \mathcal{O}_C$, and put $F := \mathcal{O}_C^{r-s} \oplus G$;

• a torsion sheaf $\mathcal{T}$ of length t and an extension

$$(\mathcal{E}) \qquad 0 \to F \longrightarrow E \longrightarrow M \oplus \mathcal{T} \to 0,$$

where the line bundle M is determined by $c_1(M) = rc_1(K_C) - c_1(L) - c_1(\mathcal{T})$.

The integers s, d, t are bounded: we have $s \leq r$, $t \leq 2r - d$, and $d < 2(r-1)$ by the stability of E. Observe also that $d \geq 3$: indeed L is generated by its global sections, and cannot be isomorphic to K_C since otherwise F would contain a copy of K_C, contradicting the stability of E.

The data $(L, V, \mathcal{T}, \mathcal{E})$ are parameterized by a variety dominating Z; we will bound its dimension. The line bundle L depends on 2 parameters. We have $h^0(L) = d - 1$ since $d \geq 3$, therefore the subspace $V \subset H^0(L)$ depends on $s(d-1-s)$ parameters. The torsion sheaf $\mathcal{T}$ depends on t parameters. Over the variety parameterizing these data we build a vector bundle with fibre $\operatorname{Ext}^1(\mathcal{M}, F)$, with $\mathcal{M} = M \oplus \mathcal{T}$, M and F being determined as above. The group $\operatorname{Aut}(\mathcal{M}) \times \operatorname{Aut}(F)$ acts on $\operatorname{Ext}^1(\mathcal{M}, F)$, with the group $\mathbb{C}^*$ of homotheties of $\mathcal{M}$ and F acting in the same way; in fact, since the middle term of the extensions we are interested in is stable, the stabilizer of a general extension class is $\mathbb{C}^*$. This gives a bound

$$\dim Z \leq 2 + s(d-1-s) + t + \dim \operatorname{Ext}^1(\mathcal{M}, F) - \dim \operatorname{Aut}(\mathcal{M}) - \dim \operatorname{Aut}(F) + 1.$$

Let us estimate the dimensions which appear in the right hand side. We have $\operatorname{Hom}(M, F) \subset \operatorname{Hom}(M, E) = 0$ because E is stable, hence by Riemann-Roch

$$\dim \operatorname{Ext}^1(M \oplus \mathcal{T}, F) = (r-1)(2r+1) - dr.$$

The group $\operatorname{Aut}(F) = \operatorname{Aut}(\mathcal{O}_C^{r-s} \oplus G)$ contains the group of matrices $\begin{pmatrix} u & 0 \\ v & \lambda \end{pmatrix}$, with $u \in \operatorname{Aut}(\mathcal{O}_C^{r-s})$, $v \in \operatorname{Hom}(\mathcal{O}_C^{r-s}, G)$, $\lambda \in \mathbb{C}^*$; this group has dimension

$$(r-s)^2 + s(r-s) + 1 = r(r-s) + 1.$$

The group $\operatorname{Aut}(\mathcal{T})$ has dimension at least t, so similarly $\operatorname{Aut}(\mathcal{M})$ has dimension $\geq 2t + 1$. We get finally:

$$\begin{aligned} \dim Z &\leq 2 + s(d-1-s) + t + (r-1)(2r+1) - dr - r(r-s) - 2t - 1 \\ &= (r-1)^2 - 1 - (d-1-s)(r-s) - t \end{aligned}$$

Since $d - 1 = h^0(L) \geq s$, this implies $\dim Z \leq (r-1)^2 - 1 = \dim |(r-1)\Theta|$ as required. □

2. Symplectic bundles. Let C be a curve of genus $g \geq 2$, and r a positive integer. The moduli space $\mathcal{SU}_C(r)$ has a natural involution D: $E \mapsto E^*$. Let ι be the involution $L \mapsto K_C \otimes L^{-1}$ of J^{g-1}. The diagram

$$\begin{tikzcd} \mathcal{SU}_C(r) \arrow[r, "D"] \arrow[d, dashed, "\theta"'] & \mathcal{SU}_C(r) \arrow[d, dashed, "\theta"] \\ |r\Theta| \arrow[r, "\iota^*"] & |r\Theta| \end{tikzcd}$$

is commutative.

Assume now that r is even. Let $\mathcal{S}p_C(r)$ be the moduli space of semi-stable symplectic bundles of rank r on C. This is a normal connected projective variety, with a forgetful morphism to $\mathcal{SU}_C(r)$, which is an embedding on the stable locus. It is contained in the fixed locus of D, thus its image under θ is contained in the fixed locus of ι^*.

This fixed locus is described for instance in [B-L], ch. 4, §6 (up to a translation from JC to J^{g-1}). The involution ι^* acts linearly on $|r\Theta|$ and has 2 fixed spaces $|r\Theta|^+$ and $|r\Theta|^-$: a symmetric divisor in $|r\Theta|$ is in $|r\Theta|^+$ (resp. $|r\Theta|^-$) if and only if its multiplicity at any theta-characteristic $\kappa \in J^{g-1}$ is even (resp. odd). The dimension of $|r\Theta|^{\pm}$ is $\frac{1}{2}(r^g \pm 2^g) - 1$.

Proposition 2.1. θ: $\mathcal{SU}_C(r) \dashrightarrow |r\Theta|$ *induces a rational map from* $\mathcal{S}p_C(r)$ *to* $|r\Theta|^+$.

Proof. Since $\mathcal{S}p_C(r)$ is connected, it suffices to find one semi-stable bundle E which admits a symplectic form, and such that $\Theta_E \in |r\Theta|^+$. We take $E = F \oplus F^*$ with the standard alternate form, where $F \in \mathcal{SU}_C(r/2)$ admits a theta divisor. Then $\Theta_E = \Theta_F + \iota^*\Theta_F$. Thus if $\kappa \in J^{g-1}$ is a theta-characteristic, we have $\operatorname{mult}_\kappa(\Theta_E) = 2\operatorname{mult}_\kappa(\Theta_F)$, hence $\Theta_E \in |r\Theta|^+$. □

Let us go back to the case $g = 2$.

Proposition 2.2. *If C has genus 2, some fibres of* θ: $\mathcal{SU}_C(r) \dashrightarrow |r\Theta|$ *have dimension* $\geq [\frac{r}{2}] - 1$.

Proof. If r is even, θ induces a rational map θ_{sp}: $\mathcal{S}p_C(r) \dashrightarrow |r\Theta|^+$ (Prop. 2.1). We have

$$\dim \mathcal{S}p_C(r) = \frac{1}{2}r(r+1), \qquad \dim |r\Theta|^+ = \frac{r^2}{2} + 1,$$

hence the fibres have dimension $\geq \frac{r}{2} - 1$.

If r is odd, consider the bundle $E \oplus \mathcal{O}_C$, for E general in $\mathcal{S}p_C(r-1)$; by what we have just seen θ is defined at E, and its fibre at E has dimension $\geq \frac{r-1}{2} - 1$. □

Remark 2.3. The *degree* of θ_r: $\mathcal{SU}_C(r) \dashrightarrow |r\Theta|$ grows exponentially with r: indeed the commutative diagram

$$\begin{tikzcd} \mathcal{SU}_C(r) \times \mathcal{SU}_C(s) \arrow[r, "\oplus"] \arrow[d, dashed, "\theta_r \times \theta_s"'] & \mathcal{SU}_C(r+s) \arrow[d, dashed, "\theta_{r+s}"] \\ |r\Theta| \times |s\Theta| \arrow[r, "+"] & |(r+s)\Theta| \end{tikzcd}$$

shows that $\deg \theta_{r+s} \geq \deg \theta_r \cdot \deg \theta_s$. Since $\deg \theta_3 = 2$, we obtain $\deg \theta_r \geq 2^{[r/3]}$ (we expect the actual value to be much higher).

3. Genus 3, rank 3. It is more natural to express our result in this case for bundles of degree zero (but arbitrary determinant). For such a bundle E the locus Θ_E is defined by the same formula as before; it is either equal to J^{g-1}, or to an effective divisor algebraically equivalent to $r\Theta$.

Recall also that if L is a line bundle on C generated by its global sections, the *evaluation bundle* Q_L is defined through the exact sequence

$$0 \to Q_L^* \longrightarrow H^0(L) \otimes_{\mathbb{C}} \mathcal{O}_C \longrightarrow L \to 0 \ ;$$

it has rank $h^0(L) - 1$ and determinant L.

THEOREM 3.1. *Let C be a curve of genus* 3, *and E_0 a stable vector bundle of rank* 3 *and degree* 0 *on C, such that $\Theta_{E_0} \supset \Theta$. Then C is not hyperelliptic, and E_0 is one of the following bundles:*

(a) *The vector bundles $E_N := Q_{K\otimes N} \otimes N^{-1}$, for $N \in J^2 - \Theta$;*

(b) *The vector bundle $\mathcal{E}nd_0(Q_K)$ of traceless endomorphisms of Q_K.*

Conversely, the bundles in (a) *and* (b) *are stable and admit a theta divisor which contains Θ.*

Since the condition $\Theta_E = J^2$ implies $\Theta_E \supset \Theta$, it follows that all stable vector bundles of rank 3 and degree 0 admit a theta divisor; in particular, the map θ: $\mathcal{SU}_C(3) \to |3\Theta|$ is a morphism. Since $\theta^*\mathcal{O}(1) = \mathcal{L}$ is ample, this morphism is finite: this implies Theorem B of the introduction.

The proof of Theorem 3.1 will occupy the rest of this section. Let E_0 be a stable bundle of rank 3 and degree 0 on C with $\Theta_{E_0} \supset \Theta$. We will deal mainly with its Serre dual $E := E_0^* \otimes K_C$. It has slope 4, degree 12 and satisfies $h^1(E) = h^0(E_0) = 0$ by stability of E_0, so that $h^0(E) = 6$ by Riemann-Roch. We first establish some properties of E that will be needed later on.

LEMMA 3.2. *Any rank* 2 *sub-bundle F of E satisfies $h^0(F) \leq 4$.*

Proof. Assume $h^0(F) \geq 5$. Let A be a sub-line bundle of F of maximal degree; this degree is ≥ 2 (since $h^0(F(-p-q) \geq 1$ for $p, q \in C$) and ≤ 3 by the stability of E. Let $B := F/A$; again by stability of E we have $\deg(F) \leq 7$, hence $\deg(B) \leq 5$. This gives

$$5 \leq h^0(F) \leq h^0(A) + h^0(B) \leq 2 + 3 = 5,$$

hence $h^0(A) = 2$, $h^0(B) = 3$; moreover the class of the extension

$$0 \to A \longrightarrow F \longrightarrow B \to 0$$

must be nonzero (because E cannot contain a line bundle of degree ≥ 4), but must go to zero under the canonical map

$$\mathrm{Ext}^1(B, A) \longrightarrow \mathrm{Hom}(H^0(B), H^1(A)).$$

In particular this map cannot be injective; equivalently its transpose, the multiplication map

$$H^0(K \otimes A^{-1}) \otimes H^0(B) \longrightarrow H^0(K \otimes A^{-1} \otimes B)$$

cannot be surjective. Now we distinguish two cases:

(a) If $\deg(A) = 3$, we must have $A = K_C(-p)$ for some $p \in C$, and $B = K_C$. But then the multiplication map $H^0(\mathcal{O}_C(p)) \otimes H^0(K_C) \xrightarrow{\sim} H^0(K_C(p))$ is an isomorphism.

(b) If $\deg(A) = 2$, C is hyperelliptic and A is the hyperelliptic line bundle on C (that is, $h^0(A) = \deg A = 2$). If $B = K_C$, the multiplication map $H^0(A) \otimes H^0(K_C) \to H^0(A \otimes K_C)$ is surjective. So we must have $\deg(B) = 5$. By the base point free pencil trick, the multiplication map $H^0(A) \otimes H^0(B) \to H^0(A \otimes B)$ is surjective if and only if $H^1(B \otimes A^{-1}) = 0$, that is, $H^0(K \otimes A \otimes B^{-1}) = 0$. This fails only if $B \cong K(q)$ for some $q \in C$. But in that case B, and therefore also F, are not globally generated. The subsheaf F' of F spanned by $H^0(F)$ has $h^0(F') = 5$, $\deg(F') \leq 6$, and this is impossible by the previous analysis. □

LEMMA 3.3. *Let p, q be general points of C. Then $h^0(E(-p)) = 3$ and $h^0(E(-p-q)) = 1$.*

Proof. If $h^0(E(-p)) \geq 4$ for all $p \in C$, the global sections of E span a sub-bundle F of rank ≤ 2 with $h^0(F) = 6$. This is impossible by Lemma 3.2. Similarly if $h^0(E(-p-q)) \geq 2$ for all q, the global sections of $E(-p)$ span a sub-line bundle L of $E(-p)$ with $h^0(L) = 3$, hence $\deg L \geq 4$, contradicting the stability of E. □

Thus the spaces $\mathbb{P}(H^0(E(-p)))$ form a one-dimensional family of planes in $\mathbb{P}(H^0(E)) \cong \mathbb{P}^5$ with the property that any two of them intersect. This situation has been thoroughly analyzed by Morin [M].

THEOREM. (Morin) *Any irreducible family of planes in $\mathbb{P}^5$ such that any two planes of the family intersect is contained in one of the following families:*

(e1) *The planes passing trough a given point.*

(e2) *The planes contained in a given hyperplane.*

(e3) *The planes intersecting a given plane along a line.*

(g1) *One of the family of generatrices of a smooth quadric in $\mathbb{P}^5$.*

(g2) *The family of planes cutting down a smooth conic on the Veronese surface.*

(g3) *The family of planes in $\mathbb{P}^5$ tangent to the Veronese surface.*

3.4. The elementary cases. We will first show that our family of planes cannot satisfy one of the elementary conditions (e1) to (e3).

(e1) This would mean that there exists a non-zero section $s \in H^0(E)$ which vanishes at each point of C, a contradiction.

(e2) In that case there exists a hyperplane H in $H^0(E)$ such that $H^0(E(-p)) \subset H$ for all p in C. It follows that H span a sub-bundle F of E of rank ≤ 2, with $h^0(F) \geq 5$; this contradicts Lemma 3.2.

(e3) In that case there exists a 3-dimensional subspace W in $H^0(E)$ such that $\dim W \cap H^0(E(-p)) \geq 2$ for all p in C. This implies that W spans a sub-line bundle L of E with $h^0(L) \geq 3$, contradicting the stability of E.

3.5. The geometric cases. Suppose now that our family of planes $\mathbb{P}(H^0(E(-p))) \subset \mathbb{P}(H^0(E))$ is contained in one of the families (g1) to (g3). We put $V := H^0(E)$ and consider the map $g\colon C \to \mathbb{G}(3, V)$ which associates to a general point p of C the subspace $H^0(E(-p))$ of V. This map is defined by the sub-bundle E' of E spanned by $H^0(E)$; that is, the universal exact sequence on $\mathbb{G}(3, V)$

$$0 \to N \longrightarrow V \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow Q \to 0$$

pulls back to the exact sequence

$$0 \to N_C \longrightarrow V \otimes \mathcal{O}_C \longrightarrow E' \to 0$$

on C, where $(N_C)_p = H^0(E(-p))$ for p general in C. The Morin theorem tells us that g factors as

$$g\colon\ C \xrightarrow{f} \mathbb{P}^r \hookrightarrow \mathbb{G}(3, V),$$

where $r = 2$ or 3 and $\mathbb{P}^r$ is embedded in $\mathbb{G}(3, V)$ as described in (g1) to (g3). Conversely if this holds, the vector bundle $E' = g^*Q$ has the property that $h^0(E'(-p-q)) \geq 1$ for all p, q in C.

We will now analyze each of these cases and deduce from this the possibilities for E. We put $L := f^*\mathcal{O}_{\mathbb{P}^r}(1)$.

(g1) *Planes in a quadric*. Let U be a 4-dimensional vector space, and $V = \mathsf{\Lambda}^2 U$. The equation $v \wedge v = 0$ for $v \in V$ defines a smooth quadric $\mathcal{Q}$ in $\mathbb{P}(V)$. The subvariety of $\mathbb{G}(3, V)$ parameterizing planes contained in $\mathcal{Q}$ has two components, which are exchanged under the automorphism group of $\mathcal{Q}$. One of these is the image of the map $\mathbb{P}^3 = \mathbb{P}(U^*) \to \mathbb{G}(3, V)$ which maps the hyperplane $H \subset U$ to the 3-plane $\mathsf{\Lambda}^2 H \subset \mathsf{\Lambda}^2 U = V$. The Euler exact sequence

$$0 \to \Omega^1_{\mathbb{P}^3}(1) \longrightarrow U \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

gives rise to an exact sequence

$$0 \to \mathsf{\Lambda}^2\big(\Omega^1_{\mathbb{P}^3}(1)\big) \longrightarrow \mathsf{\Lambda}^2 U \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^3} \longrightarrow \Omega^1_{\mathbb{P}^3}(2) \to 0$$

which is the pull back to $\mathbb{P}^3$ of the universal exact sequence on $\mathbb{G}(3, V)$.

Thus $E' \cong f^*\Omega^1_{\mathbb{P}^3}(2)$; the Euler exact sequence twisted by $\mathcal{O}_{\mathbb{P}^3}(1)$ pulls back to

$$0 \to E' \longrightarrow U \otimes_{\mathbb{C}} L \longrightarrow L^{\otimes 2} \to 0.$$

This implies $\det E' \cong L^{\otimes 2}$, hence $\deg L \leq 6$. On the other hand the condition $h^0(E'(-p-q)) \geq 1$ for all p, q in C implies $h^0(L) \geq 3$ and therefore $\deg L \geq 4$. The map $U \to H^0(L)$ must then be injective, because otherwise a copy of L would inject into E', contradicting the stability of E. This gives $h^0(L) \geq 4$; the only possibility is $\deg L = 6$ and $h^0(L) = 4$, hence $E' = E$ and $U = H^0(L)$. Thus E is isomorphic to $Q_L^* \otimes L$, where Q_L is the evaluation bundle of L. This vector bundle is analyzed in [B2]: it always admits a theta divisor, and it is stable if and only if C is not hyperelliptic and L is very ample, that is, $L = K_C \otimes N$ with $\deg N = 2$, $h^0(N) = 0$. Dualizing we find $E_0 = Q_{K \otimes N} \otimes N^{-1}$; this gives case (a) of the theorem.

(g2) *Secant planes to the Veronese surface*. Let U be a 3-dimensional vector space, and $V = \mathsf{S}^2 U$. The Veronese surface S is the image of the map $u \mapsto u^2$ from $\mathbb{P}(U)$ into $\mathbb{P}(V)$. The family of planes which cut S along a conic is the image of the map $\mathbb{P}^2 = \mathbb{P}(U^*) \to \mathbb{G}(3, V)$ which maps a 2-plane $H \subset U$ to $\mathsf{S}^2 H \subset \mathsf{S}^2 U$. The pull back to $\mathbb{P}^2$ of the universal exact sequence on $\mathbb{G}(3, V)$ is the sequence

$$0 \to \mathsf{S}^2(\Omega^1_{\mathbb{P}^2}(1)) \longrightarrow V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1)^3 \to 0$$

obtained by taking the symmetric square of the Euler exact sequence on $\mathbb{P}^2$.

Thus E' is isomorphic to $L^{\oplus 3}$. Since E is stable this implies $\deg L \leq 3$, while the inequality $h^0(E'(-p-q)) \geq 1$ imposes $h^0(L) \geq 3$, a contradiction.

(g3) *Tangent planes to the Veronese surface*. Consider again the Veronese surface S, image of the square map $\mathbb{P}(U) \to \mathbb{P}(V)$. The projective tangent bundle of S in $\mathbb{P}(V)$ is $\mathbb{P}_S(\widetilde{T}_S)$, where $\widetilde{T}_S$ appears in the extension

$$0 \to \mathcal{O}_S \longrightarrow \widetilde{T}_S \longrightarrow T_S \to 0$$

with class $c_1(\mathcal{O}_{\mathbb{P}(V)}(1)_{|S}) \in H^1(S, \Omega_S^1)$; the Euler exact sequence provides an isomorphism $\widetilde{T}_S \cong U \otimes \mathcal{O}_{\mathbb{P}(U)}(1)$. Similarly we have an extension $\widetilde{T}_{\mathbb{P}(V)}$ of $T_{\mathbb{P}(V)}$ by $\mathcal{O}_{\mathbb{P}(V)}$ and an isomorphism $\widetilde{T}_{\mathbb{P}(V)} \cong V \otimes \mathcal{O}_{\mathbb{P}(V)}(1)$. These bundles fit into a normal exact sequence

$$0 \to \widetilde{T}_S \longrightarrow \widetilde{T}_{\mathbb{P}(V)|S} \longrightarrow N_{S/\mathbb{P}(V)} \to 0,$$

that is, after a twist by $\mathcal{O}_S(-2)$,

$$0 \to U \otimes \mathcal{O}_S(-1) \longrightarrow V \otimes \mathcal{O}_S \longrightarrow N_{S/\mathbb{P}(V)}(-2) \to 0,$$

which is the pull back to S of the universal exact sequence on $\mathbb{G}(3, V)$. Recall that the second fundamental form gives an isomorphism $N_{S/\mathbb{P}^5} \cong \mathsf{S}^2 T_S$ (see for instance [G-H]).

Thus $E' = \mathsf{S}^2 f^*(T_{\mathbb{P}^2}(-1))$. This gives $\det E' = L^{\otimes 3}$, hence $\deg L \leq 4$. On the other hand we have $h^0(L) \geq 3$: otherwise the image of C in $\mathbb{P}^5$ is a conic $c \subset S$, and all tangent planes to S along c meet the plane of c along a line, so that we are in case (e3). Therefore $L = K_C$, $E = E'$. The Euler exact sequence shows that $f^*(T_{\mathbb{P}^2}(-1))$ is isomorphic to the evaluation bundle Q_K of K_C, so that $E \cong \mathsf{S}^2 Q_K$. Using the canonical isomorphism $\mathsf{S}^2 F \otimes (\det F)^{-1} \xrightarrow{\sim} \mathcal{E}nd_0(F)$ for a rank 2 bundle F we get $E_0 \cong \mathcal{E}nd_0(Q_K)$.

The vector bundles Q_K, and therefore $\mathcal{E}nd_0(Q_K)$, are semi-stable. If C is hyperelliptic, Q_K is isomorphic to $H \oplus H$, where H is the hyperelliptic line bundle, hence $\mathcal{E}nd_0(Q_K) \cong \mathcal{O}_C^{\oplus 3}$.

Assume now that C is not hyperelliptic; then Q_K is stable [P-R]. If $\mathcal{E} := \mathcal{E}nd_0(Q_K)$ is not stable, it admits as sub- or quotient sheaf a line bundle of degree 0; this means that there exists a nonzero homomorphism $Q_K \to Q_K \otimes M$, with $M \in JC$, which must be an isomorphism because Q_K is stable. Taking determinants gives $M^{\otimes 2} \cong \mathcal{O}_C$. Since C is not hyperelliptic M cannot be written $\mathcal{O}_C(p - q)$ with $p, q \in C$; therefore $h^0(Q_K \otimes M) = 0$ [P-R], so that $Q_K \otimes M$ cannot be isomorphic to Q_K.

It remains to prove that $\mathcal{E}$ admits a theta divisor. What we have proved so far is that $\mathcal{E}$ is the only stable rank 3 vector bundle of degree 0 which might possibly satisfy $\Theta_{\mathcal{E}} = J^2$. But if this was the case, all the vector bundles $\mathcal{E} \otimes M$, for $M \in JC$, should have the same property—an obvious contradiction. □

Remarks 3.6. In what follows we assume that C is not hyperelliptic.

(a) It follows from Thm. 3.1 that there are 37 stable bundles $E_0 \in \mathcal{SU}_C(3)$ with $\Theta_{E_0} \supset \Theta$, namely $\mathcal{E}nd_0(Q_K)$ and the bundles E_κ where κ is an even theta-characteristic. These bundles appear already in [P], in a somewhat disguised form: one can show indeed that E_κ is isomorphic to $\mathcal{E}nd_0(A(\kappa, L, x))$, where $A(\kappa, L, x)$ is the *Aronhold bundle* defined in [P] (up to a twist, this bundle depends only on κ).

(b) The theta divisor of E_N is determined in [B2]: it is equal to $\Theta + \Delta_N$, where Δ_N is the translate by N of the divisor $C - C$ in JC. The theta divisor of $\mathcal{E}nd_0(Q_K)$ is $\Theta + \Xi$, where Ξ is an interesting canonical element of $|2\Theta|$. One can show that the trace of Ξ on $\Theta \cong \mathsf{S}^2C$ is the locus of divisors $p+q$ such that the residual intersection points of C with the line $\langle p, q\rangle$ are harmonically conjugate with respect to p, q (here we view C as a plane quartic).

(c) Let $X \subset |3\Theta|$ be the closed subvariety of divisors of the form $\Theta + \Theta_E$ for some E in $\mathcal{SU}_C(2)$. It follows from Theorem 3.1 and the above remarks that the fibre of θ: $\mathcal{SU}_C(3) \to |3\Theta|$ over a general point of X is reduced to one element, while $\theta^{-1}(\Theta + \Delta_\kappa)$, for κ an even theta-characteristic, has 2 elements, namely E_κ and $\mathcal{O}_C \oplus (Q_K \otimes \kappa^{-1})$. From general principles this implies that the variety $\theta(\mathcal{SU}_C(3))$ is *not normal* at the 36 points $\Theta + \Delta_\kappa$ (see for instance [EGA], 15.5.3).

It seems plausible that θ is generically injective. One possible approach would be to prove that its tangent map is injective at $\mathcal{O}_C \oplus E$ for some E in $\mathcal{SU}_C(2)$, perhaps in the spirit of [vG-I].

(d) Assume that the Néron-Severi group of JC has rank 1—this holds if C is general enough. Then a reducible divisor in $|3\Theta|$ must contain a translate of Θ. We thus deduce from Theorem 3.1 that the stable vector bundles of rank 3 and degree 0 on C which admit a reducible theta divisor are those of the form $E_N \otimes M$ or $\mathcal{E}nd_0(Q_K) \otimes M$, for $M \in JC$.

INSTITUT UNIVERSITAIRE DE FRANCE, LABORATOIRE J.-A. DIEUDONNÉ, UMR 6621 DU CNRS, UNIVERSITÉ DE NICE, PARC VALROSE, F-06108 NICE CEDEX 02, FRANCE

REFERENCES

[B1] A. Beauville, Vector bundles on curves and generalized theta functions: recent results and open problems, *Current Topics in Complex Algebraic Geometry*, MSRI Publications, vol. 28, Cambridge University Press, 1995, pp. 17–33.

[B2] ———, Some stable vector bundles with reducible theta divisors, *Manuscripta Math.* **110** (2003), 343–349.

[B-N-R] A. Beauville, M. S. Narasimhan, and S. Ramanan, Spectral curves and the generalised theta divisor, *J. Reine Angew. Math.* **398** (1989), 169–179.

[B-L] C. Birkenhake and H. Lange, *Complex Abelian Varieties*, *Grundlehren Math. Wiss.*, vol. 302, Springer-Verlag, Berlin, 1992.

[EGA] A. Grothendieck, *Éléments de géométrie algébrique*, *IV, Inst. Hautes Études Sci. Publ. Math.* **32** (1967).

[G-H] P. Griffiths and J. Harris, Algebraic geometry and local differential geometry, *Ann. Sci. École Norm. Sup. (4)* **12** (1979), 355–452.

[L] Y. Laszlo, Un théorème de Riemann pour les diviseurs thêta sur les espaces de modules de fibrés stables sur une courbe, *Duke Math. J.* **64** (1991), 333–347.

[M] U. Morin, Sui sistemi di piani a due a due incidenti, *Atti Ist. Veneto* **89** (1930), 907–926.

[N-R] M. S. Narasimhan and S. Ramanan, Moduli of vector bundles on a compact Riemann surface, *Ann. of Math.* (2) **89** (1969), 14–51.

[O] A. Ortega, On the moduli space of rank 3 vector bundles on a genus 2 curve and the Coble cubic, *J. Algebraic Geom.* **14** (2005), 327–356.

[P-R] K. Paranjape and S. Ramanan, On the canonical ring of a curve, *Algebraic Geometry and Commutative Algebra*, vol. II, Kinokuniya, Tokyo, 1988, pp. 503–516.

[P] C. Pauly, Self-duality of Coble's quartic hypersurface and applications, *Michigan Math. J.* **50** (2002), 551–574.

[R] M. Raynaud, Sections des fibrés vectoriels sur une courbe, *Bull. Soc. Math. France* **110** (1982), 103–125.

[vG-I] B. van Geemen and E. Izadi, The tangent space to the moduli space of vector bundles on a curve and the singular locus of the theta divisor of the Jacobian, *J. Algebraic Geom.* **10** (2001), 133–177.