# Partial differential equations and spectral theory

# Introduction

This course will be an introduction to spectral methods in linear partial differential equations. Many of the partial differential equations coming from physics have a similar form. Consider for instance the heat equation

$$\partial_t u = \Delta u, \tag{1}$$

the wave equation

$$\partial_t^2 u = \Delta u, \tag{2}$$

or the Schrödinger equation

$$i\partial_t u = (-\Delta + V)u, \tag{3}$$

where $V \in C^\infty(\mathbb{R}^n; \mathbb{R})$.

In these equations, the left-hand side is rather simple, involving one or two derivatives in time of $u$, while the right-hand side involves a differential operator applied to $u$. A good way of finding solutions to equations having such a structure is to use separation of variables, namely, to look for solutions of the form

$$u(t, x) = v(t)w(x). \tag{4}$$

For example, if one looks for a solution of (1) of the form (4), then an easy computation shows that there must exist $\lambda \in \mathbb{R}$ such that we have

$$v' = \lambda v \tag{5}$$

$$\Delta w = \lambda w, \tag{6}$$

so that $u(t, x) = e^{\lambda t} w(x)$.

Solving equation (5) is easy for any $\lambda$, but solving equation (6) is not always trivial: the lambdas for which solutions exist will depend on the space of functions with which we are working. For instance, if we work in $\mathbb{R}^n$, we may want to restrict ourselves to functions that are $L^2$, or if we work in a domain $\Omega$, we may want to consider only functions that vanish at the boundary of $\Omega$...

At first glance, the separation of variables may look like a naive method, giving only very special solutions of equations like (1), (2) or (3). This is not the case: we will show that, working in well-chosen functional spaces, any initial condition may be "decomposed" as a linear combination of eigenfunctions. For instance, in $\mathbb{R}^d$, any function can be decomposed into plane waves, which are eigenfunctions of the Laplacian: this is the point of Fourier transform, which we now recall.

## Review of Fourier transform

### The Schwartz space and its dual

If $\alpha \in \mathbb{N}^n$ is a multi-index, with $|\alpha| := \alpha_1 + \alpha_2 + ... + \alpha_n$ and if $x \in \mathbb{R}^n$, we write $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$. If $u \in C^{|\alpha|}(\mathbb{R}^n)$, we also write

$$\partial^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$$

**Definition 0.1.** *The Schwartz space is*

$$\mathcal{S}(\mathbb{R}^n) := \{u \in C^\infty(\mathbb{R}^n); \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta u(x)| < \infty \text{ for all } \alpha, \beta \in \mathbb{N}^n\}.$$

For any $\alpha, \beta \in \mathbb{N}^n$, we define the seminorm $|\cdot|_{\alpha,\beta}$ on $\mathcal{S}$ by

$$|u|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta u(x)|.$$

**Definition 0.2.** *Let $u_j, u \in \mathcal{S}$. We say that $u_j \to u$ in $\mathcal{S}$ if*

$$|u_j - u|_{\alpha,\beta} \to 0 \text{ for all } \alpha, \beta \in \mathbb{N}^n.$$

**Definition 0.3.** *The space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ is the topological dual of $\mathcal{S}$. In other words, $T \in \mathcal{S}'$ if $T : \mathcal{S} \to \mathbb{C}$ is linear, and $u_j \to u$ in $\mathcal{S}$ implies that $T(u_j) \to T(u)$.*

### Definition and properties of the Fourier Transform

If $u \in \mathcal{S}(\mathbb{R}^n)$, we define its Fourier transform $\mathcal{F}u : \mathbb{R}^n \to \mathbb{C}$ by

$$(\mathcal{F}u)(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) \mathrm{d}x$$

and its inverse Fourier transform $\mathcal{F}^{-1}u : \mathbb{R}^n \to \mathbb{C}$ by

$$(\mathcal{F}^{-1}u)(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} u(x) \mathrm{d}x.$$

**Theorem 0.1** (Plancherel's formula)**.** *Let $u \in \mathcal{S}(\mathbb{R}^n)$. Then $\mathcal{F}u$ and $\mathcal{F}^{-1}u$ belong to $L^2(\mathbb{R}^n)$, and we have*

$$\|u\|_{L^2(\mathbb{R}^n)} = \|\mathcal{F}u\|_{L^2(\mathbb{R}^n)} = \|\mathcal{F}^{-1}u\|_{L^2(\mathbb{R}^n)}. \tag{7}$$

Thanks to Plancherel's formula, $\mathcal{F}$ and $\mathcal{F}^{-1}$ can be uniquely defined as unitary operators from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. By duality, the Fourier transform can also be extended to $\mathcal{S}'(\mathbb{R}^d)$.

The Fourier transform has the following properties:

**Theorem 0.2.** *Let $u, v \in \mathcal{S}(\mathbb{R}^n)$. We have*

1. $$\int_{\mathbb{R}^n} u(x)\overline{v}(x)\mathrm{d}x = \int_{\mathbb{R}^n} (\mathcal{F}u)(\xi)\overline{(\mathcal{F}v)(\xi)}\mathrm{d}\xi.$$

2. $$\mathcal{F}^{-1}(\mathcal{F}u) = u.$$

3. *For any multi-index $\alpha \in \mathbb{N}^n$, we have*
$$\mathcal{F}(\partial^\alpha u) = (i\xi)^\alpha \mathcal{F}(u)$$

4. *If $u, v \in \mathcal{S}(\mathbb{R}^n)$, we have*
$$\mathcal{F}(u * v) = (2\pi)^{n/2}(\mathcal{F}u)(\mathcal{F}v).$$

5. *Let $\theta > 0$ and $f_\theta(x) := e^{-|x|^2/(2\theta)}$. We then have for all $\xi \in \mathbb{R}^n$*
$$(\mathcal{F}f_\theta)(\xi) = \theta^{n/2} e^{-\theta\frac{|\xi|^2}{2}}.$$

## How to use these lecture notes

The sections of these lecture notes are divided in three categories:

- The essential ones are indicated by a ♡. You should master them all perfectly, and, if you don't understand something about them, send me an e-mail at maxime.ingremeau@unice.fr.
- The complementary ones are indicated by ♣. They are important, and you should know them for the exam, but you can skip them at first reading, because they are not essential for the rest of the course.
- The hardest ones are indicated by ♠, and you don't need to know them perfectly for the exam. However, you should read them at some point, because they contain interesting results and perspectives, and because they contain enlightning applications of the results of the other sections.

## References

These lecture notes were largely inspired by the lecture notes of Konstantin Pankrashkin [4], but I also borrowed from many other sources. Will you be able to find them all ?

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# Chapter 1

# Spectral theory for elliptic equations in bounded domains

## 1.1 Sobolev spaces (♡)

Some good introductory references on Sobolev spaces are [1] and [3] (and many others). For Sobolev spaces in Lipschitz domains, there are fewer references (one can see, for instance, [2]).

### Definition of Sobolev spaces

Let $\Omega$ be an open subset of $\mathbb{R}^n$. For any $k \in \mathbb{N}$, we define the spaces $H^k(\Omega)$ as follows:

$$H^k(\Omega) := \{f \in L^2(\Omega) \text{ such that } \forall \alpha \in \mathbb{N}^n, |\alpha| \leq k, \text{ we have } \partial^\alpha f \in L^2(\Omega)\}.$$

In this definition, the derivatives are taken in the sense of distributions. For instance, $f \in H^1(\Omega)$ if and only if there exists $g_1, ..., g_n \in L^2(\Omega)$ such that for any $\varphi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega f \partial_i \varphi = -\int_\Omega g_i \varphi \quad \forall i = 1, ..., n.$$

The spaces $H^k(\Omega)$ can be equipped with the scalar product

$$\langle f, g \rangle_{H^k(\Omega)} := \int_\Omega f\overline{g} + \sum_{1 \leq |\alpha| \leq k} \int_\Omega \partial^\alpha f \overline{\partial^\alpha g}.$$

Equipped with this scalar product, $H^k(\Omega)$ is a separable Hilbert space. The associated norm is

$$\|f\|_{H^k(\Omega)} := \|f\|_{L^2(\Omega)} + \sum_{\alpha; 1 \leq |\alpha| \leq k} \|\partial^\alpha f\|_{L^2(\Omega)}.$$

### Sobolev embeddings when $\Omega = \mathbb{R}^n$

**Theorem 1.1** (Sobolev embeddings in $\mathbb{R}^n$)**.** *Let $n \geq 2$, $k \in \mathbb{N}$.*

- *If $2k < n$, then $H^k(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ with a continuous embedding, for any $p \in \left[2; \frac{2n}{n-2k}\right]$.*
- *If $2k = n$, then $H^k(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ with a continuous embedding for any $p \in [2, \infty)$.*
- *If $2k > d$, then $H^k(\mathbb{R}^d) \subset C_b^\ell(\mathbb{R}^d)$, with a continuous embedding, where $\ell$ is the only integer such that $0 \leq \ell < k - \frac{n}{2} < \ell + 1$.*

Here, $C_b^\ell(\mathbb{R}^d)$ is the set of functions in $C^\ell(\mathbb{R}^d)$ which are bounded, as well as all its derivatives of order $\leq \ell$.

**Regularity of $\Omega$** Let $j \in \mathbb{N}$. We say that a function $f$ is $C^{j,1}$ if it is differentiable $j$ times, and its $j$-th derivatives are Lipschitz continuous.

**Definition 1.1.** *Let $n \geq 2$, and $j \in \mathbb{N}$. An open set $\Omega \subset \mathbb{R}^n$ is said to be a $C^{j,1}$ domain if, for any $x_0 \in \partial\Omega$, we may find a hyperplane $H \subset \mathbb{R}^n$ with normal $\vec{n}$, numbers $h, r > 0$ and a function $f : H \to \mathbb{R}$ which is $C^{j,1}$ such that, if we set*

$$C = \{x + y\vec{n} \in \mathbb{R}^n; x \in H, |x - x_0| \leq r, -h < y < h\},$$

*we have*

$$\Omega \cap C = \{x + y\vec{n} \in \mathbb{R}^n; x \in H, \ |x - x_0| < r, \ -h < y < g(x)\}$$
$$(\partial\Omega) \cap C = \{x + y\vec{n} \in \mathbb{R}^n; x \in H, \ |x - x_0| < r, \ y = g(x)\}$$

*A $C^{0,1}$ domain will be called a* Lipschitz *domain.*

Most of the domains we will consider will be smooth (that is to say, $\partial\Omega$ will be a smooth submanifold of $\mathbb{R}^n$). However, we also want to consider Lipschitz domains so as to treat the case of polygons, which appear naturally in numerical simulations, and in some theoretical questions.

**Extending functions**

**Theorem 1.2.** *Let $n \geq 2$, $k \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Then there exists a bounded operator $T : H^k(\Omega) \to H^k(\mathbb{R}^n)$ such that, for any $f \in H^k(\Omega)$, we have*

$$Tf_{|\Omega} \equiv f.$$

**Traces on the boundary** If $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain, then its boundary $\partial\Omega$ can be naturally endowed with a measure.

**Theorem 1.3.** *Let $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. There exists a unique continuous operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ such that, if $f \in H^1(\Omega) \cap C(\overline{\Omega})$, we have $\gamma f = f_{|\partial\Omega}$. More generally, if $\Omega \subset \mathbb{R}^n$ is a $C^{k,1}$ domain, there exists a unique continuous operator $\gamma^{(k)} = (\gamma_0^{(k)}, ..., \gamma_{k-1}^{(k)}) : H^k(\Omega) \to \left(L^2(\partial\Omega)\right)^k$ such that, if $f \in H^k(\Omega) \cap C^{k-1}(\overline{\Omega})$, we have*

$$\gamma^{(k)} f = (f_{|\partial\Omega}, \partial_n f_{|\partial\Omega}, ..., \partial_n^{k-1} f_{|\partial\Omega}),$$

*where $\partial_n = n \cdot \nabla$ denotes the derivative in the direction of the outgoing normal to $\partial\Omega$.*

**Theorem 1.4** (Green's identities)**.** *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain $u \in H^2(\Omega)$, $v \in H^1(\Omega)$. We have*

$$\langle \Delta u, v \rangle_{L^2(\Omega)} = -\langle \nabla u, \nabla v \rangle_{L^2(\Omega)} + \langle \gamma_1^{(2)} u, \gamma_0^{(1)} v \rangle_{L^2(\Omega)}.$$

*If furthermore, $v \in H^2(\Omega)$, we have*

$$\langle \Delta u, v \rangle_{L^2(\Omega)} - \langle \Delta v, u \rangle_{L^2(\Omega)} = \langle \gamma_1^{(2)} u, \gamma_0^{(1)} v \rangle_{L^2(\Omega)} - \langle \gamma_1^{(2)} v, \gamma_0^{(1)} u \rangle_{L^2(\Omega)}.$$

We denote by $C_c^\infty(\Omega)$ the set of smooth functions in $\Omega$ whose support is strictly included in $\Omega$.

**Definition 1.2.** *The space $H_0^k(\Omega)$ is the closure of $C_c^\infty(\Omega)$ for the $H^k(\Omega)$ norm.*

From Theorem 1.3, we deduce the following proposition.

**Proposition 1.1.** *Let $\Omega \subset \mathbb{R}^n$ be a $C^{k,1}$ domain. Then $H_0^k(\Omega) = \mathrm{Ker}\ \gamma^{(k)}$.
In particular, if $\Omega$ is a Lipschitz domain, and if $f \in C(\overline{\Omega}) \cap H^1(\Omega)$. Then $f \in H_0^1(\Omega)$ if and only if $f$ vanishes on $\partial\Omega$.*

**Sobolev embeddings in compact domains** Recall that a linear operator $T$ between two Banach spaces $E$ and $F$ is *compact* if the image of any bounded set of $E$ by $T$ is a compact set of $F$ for the strong topology.

**Theorem 1.5** (Sobolev embeddings in $\mathbb{R}^n$)**.** *Let $n \geq 2$, $k \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain.*

- *If $2k < n$, then $H^k(\Omega) \subset L^p(\Omega)$ with a continuous embedding, for any $p \in \left[1; \frac{2n}{n-2k}\right]$. This embedding is compact if $p \in \left[1; \frac{2n}{n-2k}\right)$.*
- *If $2k = n$, then $H^k(\Omega) \subset L^p(\Omega)$ with a compact embedding for any $p \in [1, \infty)$.*
- *If $2k > d$, then $H^k(\Omega) \subset C^\ell(\overline{\Omega})$, with a compact embedding, where $\ell$ is the only integer such that $0 \leq \ell < k - \frac{n}{2} < \ell + 1$.*

Note that the embedding of $H^1(\Omega)$ in $L^2(\Omega)$ is always compact. In particular, if $(u_n)$ converges weakly to $u$ in $H^1(\Omega)$, then $(u_n)$ converges strongly to $u$ in $L^2(\Omega)$.

**Proposition 1.2** (Poincaré's inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set bounded in one direction. Then there exists $C > 0$ such that for any $f \in H^1_0(\Omega)$, we have*

$$\int_\Omega |f|^2 \leq C \int_\Omega |\nabla f|^2.$$

This proposition implies that, on $H^1_0(\Omega)$, the norm $\|f\|_{H^1(\Omega)}$ and the norm $\|f\|_{H^1_0(\Omega)} := \|\nabla f\|_{L^2(\Omega)}$ are equivalent.

## 1.2 Elliptic equations (♡)

Let $\Omega$ be a bounded Lipschitz domain, and let $V \in C(\Omega)$. Consider the Dirichlet problem

$$\begin{cases} -\Delta u + Vu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

A classical (or strong) solution of (1.1) is a function $u \in C^2(\overline{\Omega})$ satisfying the first PDE pointwise, and the Dirichlet condition on the boundary. This implies that $f \in C(\Omega)$. However, we will want to consider (1.1) for less regular $f$, typically for $f \in L^2(\Omega)$. To do so, we need to consider *weak solutions* of (1.1).

**Definition 1.3.** *Let $\Omega$ be a Lipschitz domain, let $V \in C(\Omega)$, and let $f \in L^2(\Omega)$. A* weak solution *of (1.1) is a function $u \in H^1_0(\Omega)$ which satisfies*

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega V u \varphi = \int_\Omega f \varphi, \quad \forall \varphi \in H^1_0(\Omega).$$

Note that, if $u$ is a strong solution of (1.1), then it is also a weak solution of (1.1). Furthermore, if $u \in C^2(\overline{\Omega})$ is a weak solution of (1.1), then $u$ is a strong solution of (1.1).

**Remark 1.1.** *The notion of weak solution of (1.1) still makes sense when $V \in L^\infty(\Omega)$, and when $f$ is in the topological dual of $H^1_0$.*

We will also consider the Neumann problem

$$\begin{cases} -\Delta u + Vu = f & \text{in } \Omega \\ \partial_n u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

**Definition 1.4.** *Let $\Omega$ be a Lipschitz domain, let $V \in C(\Omega)$, and let $f \in L^2(\Omega)$. A* weak solution *of (1.2) is a function $u \in H^1(\Omega)$ which satisfies*

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega V u \varphi = \int_\Omega f \varphi, \quad \forall \varphi \in H^1(\Omega).$$

**Proposition 1.3.** *Let $\Omega$ be a Lipschitz domain, let $V \in C(\Omega)$, $V \geq 0$, and let $f \in L^2(\Omega)$.*

1. *The problem (1.1) admits a unique weak solution.*
2. *Suppose furthermore that there exists $c > 0$ such that $V \geq c$. Then the problem (1.2) admits a unique weak solution.*

*Proof.* 1. Consider the symmetric bilinear map

$$B(u, v) = \int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega V u \varphi.$$

We have

$$B(u, u) \leq \int_\Omega |\nabla u|^2 + \|V\|_{L^\infty} \int_\Omega |u|^2 \leq C \|u\|^2_{H^1(\Omega)} \leq C' \|u\|^2_{H^1_0(\Omega)}$$

for some $C, C' > 0$, by Poincaré's inequality . On the other hand, we have

$$B(u, u) \geq \int_\Omega |\nabla u|^2 = \|u\|_{H^1_0(\Omega)}.$$

Therefore, $B$ defines a scalar product on $H^1_0$ which is equivalent to the usual one. Now, $u \mapsto \int_\Omega fu$ defines a continuous linear map on $H^1_0(\Omega)$, by Poincaré's inequality. By Riesz representation theorem, there exists a unique $v \in H^1_0(\Omega)$ such that for all $u \in H^1_0(\Omega)$, we have $B(u, v) = \int_\Omega uf$. The result follows for the Dirichlet problem.

2. The proof is similar for the Neumann problem, by considering the same bilinear map, but acting on $H^1(\Omega)$. The assumption that $V \geq c > 0$ is used to say that

$$B(u, u) \geq \int_\Omega |\nabla u|^2 + c \int_\Omega |u|^2 \geq c' \|u\|^2_{H^1(\Omega)}.$$

□

Note that the second part of the proposition is false when $V \equiv 0$. For instance, when $f \equiv 0$, any constant function is a solution to the Neumann problem, so we do not have uniqueness.

### Elliptic regularity

**Theorem 1.6** (Interior elliptic regularity)**.** *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, and let $\omega \Subset \Omega$. Let $V \in L^\infty(\Omega)$. There exists $C > 0$ such that, if $f \in L^2(\Omega)$ and if $u$ is a weak solution of (1.1) or of (1.2), then $u \in H^2(\omega)$, and we have*

$$\|u\|_{H^2(\omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big). \tag{1.3}$$

*Furthermore, if $f, V \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.*

Before proving the theorem, let us recall the *Niremberg translation* method. If $g$ is a function on $\mathbb{R}^n$ and if $h \in \mathbb{R}^n$, we write

$$(\tau_h g)(x) := g(x + h)$$
$$D_h g := \frac{\tau_h g - g}{|h|}.$$

Note that we have

$$D_h(g_1 g_2) = (D_h g_1)\tau_h g_2 + g_1 D_h g_2.$$

Let $\omega$ is an open set of $\mathbb{R}^d$. If $f, g \in L^2(\omega)$, and if $fg \equiv 0$ in a neighbourhood of size $h$ of $\partial\omega$, we have

$$\begin{aligned}\int_\omega f D_h g &= \frac{1}{|h|}\int_\omega f\tau_h g - \frac{1}{|h|}\int_\omega fg \\ &= \frac{1}{|h|}\int_{\omega+h} g\tau_{-h} f - \frac{1}{|h|}\int_\omega fg \\ &= \int_\omega g D_{-h} f\end{aligned} \tag{1.4}$$

**Lemma 1.1.** *Let $\Omega \subset \mathbb{R}^n$, $\omega \Subset \Omega$ be open sets. Suppose that $g \in L^p(\Omega)$, and that there exists a constant $C > 0$ such that for all $h \in \mathbb{R}^n$ with $|h|$ small enough, we have*

$$\|D_h g\|_{L^2(\omega)} \leq C.$$

*Then we have $g \in H^1(\omega)$, and $\|\nabla g\|_\omega \leq Cn$.*

*Proof of the lemma.* We want to bound the quantity $\|\nabla g\|_{L^2(\omega)} = \sup_{\varphi \in C_c^\infty(\omega)} \int u\nabla\varphi$.

By the assumption and Hölder's inequality, we have $\int_\omega (D_h u)\varphi \leq C\|\varphi\|_{L^2}(\omega)$.

Now, let $\varphi \in C_c^\infty(\omega)$. We extend it by zero in $\Omega$. For $h$ small enough, we have

$$\begin{aligned}C\|\varphi\|_{L^2(\omega)} &\geq \int_\Omega (D_h u)\varphi \\ &= \int_\Omega u(D_{-h}\varphi).\end{aligned}$$

Taking $h = te_j$, where $e_j$ is the $j$-th vector in the canonical basis of $\mathbb{R}^n$, and let $t \to 0$. We obtain that $\int u\partial_j\varphi \leq C\|\varphi\|_{L^2(\omega)}$. The result follows. ☐

**Lemma 1.2.** *There exists $C > 0$ such that for all $\varphi \in H^1(\Omega)$, we have*

$$\|D_h\varphi\|_{L^2} \leq C\|\nabla\varphi\|_{L^2}.$$

*Proof.* The proof is left as an exercise. ☐

*Proof of the theorem.* First of all, let us consider the case when $V \equiv 0$.

Let $\chi \in C_c^\infty(\Omega)$, with $\chi \geq 0$ and $\chi \equiv 1$ on $\omega$. Using (1.4), we have for $|h|$ small enough

$$\begin{aligned}\int_\Omega fD_{-h}\big(\chi^2 D_h u\big) &= \int_\Omega \big(-\Delta u\big)D_{-h}\big(\chi^2 D_h u\big) \\ &= \sum_{1\leq j\leq n}\int_\Omega (\partial_j u)\partial_j D_{-h}\big(\chi^2 D_h u\big) \\ &= \sum_{1\leq j\leq n}\int_\Omega \Big[\chi^2\big|\partial_j D_h u\big|^2 + 2\chi\partial_j\chi\big(D_h u\big)\big(D_h\partial_j u\big)\Big].\end{aligned}$$

Using the inequality $ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$, valid for any $a, b \in \mathbb{R}$, $\varepsilon > 0$, we deduce that

$$\begin{aligned}\int_\Omega \Big|2\chi\partial_j\chi\big(D_h u\big)\big(D_h\partial_j u\big)\Big| &\leq \varepsilon\int_\Omega \chi^2\big|\partial_j D_h u\big|^2 + \frac{1}{\varepsilon}\int_\Omega |\partial_j\chi|^2\big|D_h u\big|^2 \\ &\leq \varepsilon\int_\Omega \chi^2\big|\partial_j D_h u\big|^2 + \frac{C}{\varepsilon}\int_\Omega D_h u\big|^2.\end{aligned}$$

Therefore, we deduce that there exists $C > 0$ such that

$$\int_\Omega \chi^2|D_h\nabla u|^2 \leq C\int_\Omega |D_h u|^2 + C\int_\Omega |fD_{-h}\big(\chi^2 D_h u\big)|$$

Using Lemma 1.2 several times and noting that $\|\nabla u\|_{L^2(\Omega)} = \|f\|_{L^2(\Omega)}$, we have that

$$\begin{aligned}\int_\Omega \chi^2 |D_h \nabla u|^2 &\leq C \int_\Omega |D_h u|^2 + C\Big| \int_\Omega f D_{-h}(\chi^2)\tau_{-h}\big(D_h u\big)\Big| + C\Big| \int_\Omega f\chi^2 D_h D_{-h} u\Big| \\ &\leq C \int_\Omega |\nabla u|^2 + 2C\Big| \int_\Omega \tau_h(\chi) f D_{-h}(\chi)\tau_{-h}\big(D_h u\big)\Big| + C\|f\|_{L^2}\|\chi D_h \nabla u\|_{L^2} \\ &\leq C \int_\Omega |D_h u|^2 + C'\|f\|_{L^2}\|\chi D_h u\|_{L^2(\Omega)} + C\|f\|_{L^2}\|\chi D_h \nabla u\|_{L^2} \\ &\leq C\|f\|^2_{L^2(\Omega)} + C'\|f\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} + \varepsilon C\varepsilon\|\chi D_h \nabla u\|^2_{L^2(\Omega)} + \frac{C}{4\varepsilon}\|f\|^2_{L^2(\Omega)}.\end{aligned}$$

Taking $\varepsilon$ small enough, we deduce that

$$\|\chi D_h \nabla u\|_{L^2(\Omega)} \leq C'''\|f\|_{L^2(\Omega)}$$

for some $C''' > 0$. The proof of (1.3) then follows from Lemma 1.1 when $V = 0$. When $V \neq 0$, (1.3) follows from the result when $V = 0$, by replacing $f$ by $f - Vu$.

The second part of the statement is proven by an easy induction. □

**Theorem 1.7** (Elliptic regularity up to the boundary). *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain, $V \in L^\infty(\Omega)$. There exists $C > 0$ such that, if $f \in L^2(\Omega)$ and if $u$ is a weak solution of (1.1) or of (1.2), then $u \in H^2(\Omega)$, and we have*

$$\|u\|_{H^2(\Omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big). \tag{1.5}$$

*Furthermore, if $f, V \in C^\infty(\overline{\Omega})$, then $u \in C^\infty(\overline{\Omega})$.*

*Proof.* Admitted. One can see [1, 6.3.2] for a proof. □

## 1.3 Spectral theory for compact self-adjoint operators (♡)

Let $\mathcal{H}$ be a separable Hilbert space. Recall that a continuous linear operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ is called

- *compact* if for any $A \subset \mathcal{H}$ closed, $T(A)$ is a compact subset of $\mathcal{H}$;
- *self-adjoint* if, for all $v, w \in \mathcal{H}$, we have
$$\langle Tv, w\rangle = \langle v, Tw\rangle;$$
- *positive* if $\langle Tv, v\rangle \geq 0$ for all $v \in \mathcal{H}$.

**Theorem 1.8** (Spectral decomposition of compact self-adjoint operators). *Let $T$ be a compact self-adjoint operator on $\mathcal{H}$. We may find a sequence of real numbers $(\lambda_k)$ going to zero, called the* eigenvalues *of $T$, and a sequence of vectors $v_k \in \mathcal{H}$ forming a Hilbert basis of $\mathcal{H}$, such that*

$$Tv_k = \lambda_k v_k.$$

Note that if a compact self-adjoint operator is positive, then all its eigenvalues are positive.

**Theorem 1.9** (The min-max formula). *Let $T$ be a positive compact self-adjoint operator on $\mathcal{H}$, and let $\lambda_k$ be its eigenvalues, in decreasing order: $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N \geq 0$. We have*

$$\lambda_1 = \sup_{v\in\mathcal{H}, v\neq 0} \frac{\langle v, Tv\rangle}{\|v\|^2}.$$

*More generally, we have*

$$\lambda_k = \inf_{u_1,\dots,u_{k-1}\in\mathcal{H}} \sup_{\substack{v\in\mathcal{H}, v\neq 0\\ v\perp Vect(u_1,\dots,u_{k-1})}} \frac{\langle v, Tv\rangle}{\|v\|^2}.$$

*Proof.* Let us write

$$\mu_k := \inf_{u_1,\dots,u_{k-1}\in\mathcal{H}} \sup_{\substack{v\in\mathcal{H},v\neq 0\\ v\perp Vect(u_1,\dots,u_{k-1})}} \frac{\langle v, Tv\rangle}{\|v\|^2}$$

and show that $\mu_k = \lambda_k$.

Let us choose a Hilbert basis of eigenvectors $(v_k)$ as in Theorem 1.8, and define $V_k := \mathrm{Vect}(v_1, ..., v_k)$. If $v \in V_k$, we may write $v = \sum_{i=1}^k \alpha_i v_i$, so that

$$\frac{\langle v, Tv\rangle}{\|v\|^2} = \frac{\sum_{i=1}^k |\alpha_i|^2\lambda_i}{\sum_{i=1}^k |\alpha_i|^2} \geq \lambda_k.$$

Since, for any $u_1, .., u_k \in \mathcal{H}$, we may find $v \in V_k \cap \mathrm{Vect}(u_1, ..., u_k)^\perp$, $v \neq 0$ we deduce that $\mu_k \geq \lambda_k$.

Now, taking $u_1 = v_1$, ... $u_{k-1} = v_{k-1}$, we have that

$$\mu_k \leq \sup_{\substack{v\in\mathcal{H},v\neq 0\\ v\perp V_{k-1}}} \frac{\langle v, Tv\rangle}{\|v\|^2}.$$

But any $v \in \mathcal{H}$ perpendicular to $V_{k-1}$ can be written as $v = \sum_{i=k}^{+\infty} \alpha_i v_i$, so that

$$\frac{\langle v, Tv\rangle}{\|v\|^2} = \frac{\sum_{i=k}^\infty |\alpha_i|^2\lambda_i}{\sum_{i=k}^\infty |\alpha_i|^2} \leq \lambda_k.$$

We deduce that $\mu_k \leq \lambda_k$, and hence, $\mu_k = \lambda_k$. □

## 1.4 Diagonalisation of the Dirichlet and Neumann Laplacian in a bounded domain (♡)

**Theorem 1.10.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain.*

1. *There exists a sequence $0 < \lambda_1^D \leq \lambda_2^D \leq ... \leq \lambda_k^D \leq ...$, and an orthonormal basis $(\varphi_k)_{k\geq 1}$ of $L^2(\Omega)$, with $\varphi_k \in C^\infty(\Omega)$ satisfying*

$$\begin{cases} -\Delta\varphi_k = \lambda_k^D \varphi_k & \text{in } \Omega \\ \varphi_k = 0 & \text{on } \partial\Omega \end{cases}$$

*in the weak sense.*

2. *There exists a sequence $0 = \lambda_1^N \leq \lambda_2^N \leq ... \leq \lambda_k^N \leq ...$, and an orthonormal basis $(\psi_k)_{k\geq 1}$ of $L^2(\Omega)$, with $\psi_k \in C^\infty(\Omega)$ satisfying*

$$\begin{cases} -\Delta\psi_k = \lambda_k^N \psi_k. & \text{in } \Omega \\ \partial_n\psi_k = 0 & \text{on } \partial\Omega \end{cases}$$

*in the weak sense.*

Actually, using Theorem 1.7, we can show that if the domain $\Omega$ is regular enough, the functions $\varphi_k$ and $\psi_k$ will be strong solutions of their elliptic equations.

The set $\{\lambda_i^{D/N}, i \in \mathbb{N}\}$ is called the *spectrum* of the Dirichlet/Neumann Laplacian. We will see a more general definition of the spectrum of an operator in the next chapter.

*Proof.* 1. If $f \in L^2(\Omega)$, let $Tf$ be the unique solution in $H_0^1(\Omega)$ to the equation $-\Delta(Tf) = f$, given by Proposition 1.3. The operator $T$ is then bounded. Now, let $\iota$ denote the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$. The operator $\tilde{T} := (\iota T) : L^2(\Omega) \longrightarrow L^2(\Omega)$ is then a compact operator.

Let us check that $\tilde{T}$ is self-adjoint. Let $f, g \in L^2(\Omega)$. We have

$$\begin{aligned}\int_\Omega (\tilde{T}f)\cdot g &= \int_\Omega (\nabla\tilde{T}f)\cdot\nabla\tilde{T}g \quad \text{since } \tilde{T}g \text{ is a weak solution}\\ &= \int_\Omega (\tilde{T}g)\cdot f \quad \text{since } \tilde{T}f \text{ is a weak solution.}\end{aligned}$$

The operator $\tilde{T}$ is thus self-adjoint. Therefore, by Theorem 1.8, there exists a non-increasing sequence of numbers $(u_k)_{k\geq 1}$ going to zero, and an orthonormal basis $(\varphi_k)_{k\geq 1}$ of $L^2(\Omega)$, such that $\tilde{T}\varphi_k = u_k\varphi_n$.

Now, we have

$$\int_\Omega f\tilde{T}f = -\int_\Omega (\Delta\tilde{T}f)\tilde{T}f = \int_\Omega |\nabla\tilde{T}f|^2 \geq 0.$$

We deduce from this that the $u_k$ are all $\geq 0$. Furthermore, if $f \neq 0$, then $\int_\Omega f\tilde{T}f > 0$, so that the $u_n$ are all strictly positive.

From the equation $\tilde{T}\varphi_k = u_k\varphi_k$, we deduce that

$$-\Delta\varphi_k = \frac{1}{u_k}\varphi_k.$$

We thus set $\lambda_k^D = \frac{1}{u_k}$. The functions $\varphi_k$ are in $C^\infty(\Omega)$, by Theorem 1.7. This concludes the proof of the first point.

2. The proof of the second point is similar, by taking $Tf$ to be the unique weak solution of $(-\Delta + 1)(Tf) = f$. Note that we have $\lambda_1^N = 0$, because constant functions $\psi$ satisfy $-\Delta\psi = 0$, and the correct boundary conditions. □

**Example 1.1** (The Dirichlet Laplacian in a rectangle)**.** *Let $\Omega = (0,a)\times(0,b)$. We look for solutions of*

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

*of the form $\varphi(x,y) = f(x)g(y)$. We thus have*

$$-f''(x)g(y) - f(x)g''(y) = \lambda f(x)g(y),$$

*so that we must formally have $-\lambda = \frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)}$.*

*Therefore, we look for $f$ and $g$ such that $f''(x) = \lambda_1 f(x)$, $f(0) = f(a) = 0$, $g''(x) = \lambda_2 g(x)$, $g(0) = g(b) = 0$, and then take $\lambda = \lambda_1 + \lambda_2$.*

*We must therefore have $f(x) = \sin\left(\frac{n\pi x}{a}\right)$, $g(y) = \sin\left(\frac{p\pi y}{b}\right)$, $n,p\in\mathbb{N}$.*

*By the theory of Fourier series, the functions $\left(\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{p\pi y}{b}\right)\right)_{n,p\in\mathbb{N}}$ form an orthogonal basis of $L^2(\Omega)$.*

*Therefore, the Dirichlet spectrum of $-\Delta$ in $\Omega$ is*

$$\left\{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{p\pi}{b}\right)^2; n,p\in\mathbb{N}\right\},$$

*and the multiplicity of each eigenvalue is the number of different choices of $(n,p)\in\mathbb{N}^2$ giving the same eigenvalue.*

Adapting the proof of Theorem 1.9, we obtain the following characterization of the Dirichlet and Neumann eigenvalues:

**Theorem 1.11** (The max-min formula for the Laplacian)**.** *Let $\Omega\subset\mathbb{R}^d$ be a Lipschitz domain, and let $(\lambda_k^D)_{k\in\mathbb{N}}$ and $(\lambda_k^N)_{k\in\mathbb{N}}$ be as in Theorem 1.10. We have*

$$\lambda_k^D = \sup_{u_1,\ldots,u_{k-1}\in L^2(\Omega)} \inf_{\substack{v\in H_0^1(\Omega), v\neq 0 \\ v\perp Vect(u_1,\ldots,u_{k-1})}} \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^2(\Omega)}}$$

$$\lambda_k^N = \sup_{u_1,\ldots,u_{k-1}\in L^2(\Omega)} \inf_{\substack{v\in H^1(\Omega), v\neq 0 \\ v\perp Vect(u_1,\ldots,u_{k-1})}} \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^2(\Omega)}}$$

## 1.5 Dependence of the eigenvalues on the domain (♣)

In this section, if $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain, we will denote by $\lambda_k(\Omega)$ and $\mu_k(\Omega)$ the Dirichlet/Neumann eigenvalues given in Theorem 1.10, to emphasize their dependence on the domain $\Omega$.

**Lemma 1.3.** *Let $\Omega, \Omega'$ be Lipschitz domains with $\Omega \subset \Omega'$. Then, for every $k \in \mathbb{N}$, we have*

$$\lambda_k(\Omega') \leq \lambda_k(\Omega).$$

*Proof.* We use the max-min formula, extending functions in $H_0^1(\Omega)$ to functions in $H_0^1(\Omega')$. □

Note that no such monotonicity holds for Neumann eigenvalues. However, we have the following result.

**Lemma 1.4.** *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, and let $\Omega_1, \Omega_2 \subset \Omega$ be Lipschitz domains such that $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$, and $\Omega_1 \cap \Omega_2 = \emptyset$. Then for any $k \in \mathbb{N}$, we have*

$$\lambda_k^N(\Omega_1 \cup \Omega_2) \leq \lambda_k^N(\Omega).$$

*Proof.* With the hypotheses we made, we have $L^2(\Omega) = L^2(\Omega_1 \cup \Omega_2)$, and $H^1(\Omega) \subset H^1(\Omega_1 \cup \Omega_2)$. We therefore have

$$\begin{aligned}
\lambda_k^N(\Omega_1 \cup \Omega_2) &= \sup_{u_1,\dots,u_{k-1} \in L^2(\Omega_1 \cup \Omega_2)} \inf_{\substack{v \in H^1(\Omega_1 \cup \Omega_2), v \neq 0 \\ v \perp Vect(u_1,\dots,u_{k-1})}} \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^2(\Omega)}} \\
&\leq \sup_{u_1,\dots,u_{k-1} \in L^2(\Omega)} \inf_{\substack{v \in H^1(\Omega), v \neq 0 \\ v \perp Vect(u_1,\dots,u_{k-1})}} \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^2(\Omega)}} = \lambda_k^N(\Omega).
\end{aligned}$$

□

Putting together Lemmas 1.3 and 1.4, we have that for any $k \in \mathbb{N}$

$$\lambda_k^N(\Omega_1 \cup \Omega_2) \leq \lambda_k^N(\Omega) \leq \lambda_k^D(\Omega) \leq \lambda_k^D(\Omega_1 \cup \Omega_2).$$

We end this discussion with a continuity result for the Dirichlet eigenfunctions.

**Proposition 1.4.** *Let $(\Omega_j)_{j \in \mathbb{N}}$ be an increasing sequence of Lipschitz domains of $\mathbb{R}^d$, and suppose that $\Omega = \cup_{j=1}^{\infty} \Omega_j$ is a Lipschitz domain. Then for any $k \in \mathbb{N}$,*

$$\lambda_k^D(\Omega_j) \longrightarrow \lambda_k^D(\Omega).$$

This result is particularly important from a numerical point of view. In practice, a domain $\Omega$ is approximated by simpler shapes, for instance, polygons. This proposition says that, from a spectral point of view, the approximation will be good if the polygons are all included in $\Omega$.

*Proof.* Lemma 1.3 tells us that, for any $k \in \mathbb{N}$, $\lambda_k^D(\Omega) \leq \lambda_k^D(\Omega_j)$. Take $k \in \mathbb{N}$, $\varepsilon > 0$, and let us show that there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$, $\lambda_k^D(\Omega_j) \leq \lambda_k^D(\Omega) + \varepsilon$.

Let $u_1, \dots, u_k \in H_0^1(\Omega)$ denote the mutually orthogonal first $k$ eigenvalues of the Dirichlet Laplacian on $\Omega$, with eigenvalues $\lambda_1^D(\Omega), \dots, \lambda_k^D(\Omega)$, and $U = \text{Vect}(u_1, \dots, u_k)$. We have, for any $u \in U$,

$$\|\nabla u\|_{L^2(U)} \leq \lambda_k^D(\Omega) \|u\|_{L^2(\Omega)}.$$

Using the density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$, we can approximate the functions $u_1, \dots, u_k$ by functions $v_1, \dots, v_k \in C_c^\infty(\Omega)$ such that $v_1, \dots, v_k$ are linearly independent, and $\|\nabla v\|_{L^2(U)} \leq \big(\lambda_k^D(\Omega) + \varepsilon\big) \|v\|_{L^2(\Omega)}$ for all $v \in \text{Vect}(v_1, \dots, v_k)$. Let $K$ be a compact set containing all the supports of the $v_1, \dots, v_k$. By assumption, there exists $j_0$ such that for all $j \geq j_0$, $K \subset \Omega_j$.

Now, let $j \geq j_0$, and let $w_1, ..., w_{k-1} \in L^2(\Omega_j)$. There exists a function $v \in \text{Vect}(v_1, ..., v_k)$ which is orthogonal to $w_1, ..., w_{k-1}$ in $L^2(\Omega_j)$. Therefore, we have

$$\inf_{\substack{v\in H_0^1(\Omega_j), v\neq 0\\ v\perp Vect(w_1,...,w_{k-1})}} \frac{\|\nabla v\|^2_{L^2(\Omega_j)}}{\|v\|^2_{L^2(\Omega_j)}} \leq \lambda_k^D(\Omega) + \varepsilon,$$

and we may conclude the proof using the max-min principle. □

## 1.6 Weyl's law (♠)

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, and let

$$\mathcal{N}_\Omega^{D/N}(\lambda) = \sharp\{k \in \mathbb{N}; \lambda_k^{D/N}(\Omega) \leq \lambda\}.$$

The aim of this section is to prove the following result.

**Theorem 1.12** (Weyl's law)**.** *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. We have*

$$\lim_{\lambda\to\infty} \frac{\mathcal{N}_\Omega^D(\lambda)}{\lambda^{d/2}} = \frac{\omega_d}{(2\pi)^d}\text{Vol}(\Omega),$$

*where $\omega_d$ is the voume of the d-dimensional unit ball.*

Note that Weyl's law also holds for Neumann eigenvalues, but the proof is harder. Weyl's law was proven by Hermann Weyl in 1911. It can be considered as the first result of *semiclassical analysis*, i.e., result about the behaviour of $(\lambda_k(\Omega), \varphi_k)$ as $k \to \infty$. Although most results in semiclassical analysis involve complicated properties of the open set $\Omega$ (for instance, involving the billiard dynamics in $\Omega$), Weyl's law involves only the volume of $\Omega$, and its dimension. Refinements of Weyl's law exist in certain cases, involving the area of $\partial\Omega$.

*Proof.* **Step 1: Weyl's law holds for a rectangular box**
This step is proven by recurrence on the dimension. The result is trivial when $d = 1$. Let $d \geq 2$. Let $\Omega = \prod_{i=1}^d (0, a_i)$. We have seen that the Dirichlet spectrum of $\Omega$ is made of the numbers

$$\Big\{\sum_{i=1}^d \Big(\frac{\pi n_i}{a_i}\Big)^2; n_1, ..., n_d \in \mathbb{N}\Big\},$$

while the Neumann spectrum is made of the same numbers, but where the $n_i$ are also allowed to take value 0. Therefore, if we write

$$\mathcal{D}(\lambda) := \{(x_1, ..., x_d) \in [0, \infty)^d; \sum_{i=1}^d \Big(\frac{\pi x_i}{a_i}\Big)^2 \leq \lambda\},$$

we have

$$\mathcal{N}_\Omega^D(\lambda) = \sharp\mathcal{D}(\lambda) \cap \mathbb{N}^d$$
$$\mathcal{N}_\Omega^N(\lambda) = \sharp\mathcal{D}(\lambda) \cap (\mathbb{N} \cup \{0\})^d.$$

If $(n_1, ..., n_d) \in \mathcal{D}(\lambda) \cap \mathbb{N}^d$, then the unit cube $[n_1 - 1; n_1] \times ... \times [n_d - 1, n_d]$ is included in $\mathcal{D}(\lambda)$. There are exactly $\mathcal{N}_\Omega^D(\lambda)$ such cubes, which are all disjoint. Therefore, we have

$$\text{Vol}(\mathcal{D}(\lambda)) \geq \mathcal{N}_\Omega^D(\lambda).$$

On the other hand, we have

$$\mathcal{D}(\lambda) \subset \bigcup_{(n_1,...,n_d)\in\mathcal{D}(\lambda)\cap(\mathbb{N}\cup\{0\})^d} [n_1, n_1 + 1] \times ... \times [n_d, n_d + 1],$$

so that

$$\text{Vol}(\mathcal{D}(\lambda)) \leq \mathcal{N}_\Omega^N(\lambda).$$

Finally, note that

$$\mathcal{N}_\Omega^N(\lambda) - \mathcal{N}_\Omega^D(\lambda) \leq \sharp\mathbb{N}^d \cap \{(x_1, ..., x_d) \in [0,\infty)^d; \sum_{i=1}^d \left(\frac{\pi x_i}{a_i}\right)^2 \leq \lambda; \exists i \in \{1, ..., d\}, x_i = 0\},$$

which is the cardinal of a union of $d$ intersections of $(d-1)$- dimensional ellipsoids with $\mathbb{N}^{d-1}$. By the recurrence hypothesis, this is $O(\lambda^{d-1})$.

Therefore, we have

$$\mathrm{Vol}(\mathcal{D}(\lambda)) \leq \mathcal{N}_\Omega^N(\lambda) \leq \mathcal{N}_\Omega^D(\lambda) + O(\lambda^{d-1}) \leq \mathrm{Vol}(\mathcal{D}(\lambda)) + O(\lambda^{d-1}).$$

Therefore we have

$$\mathcal{N}_\Omega^N(\lambda) \sim \mathcal{N}_\Omega^D(\lambda) \sim \mathrm{Vol}(\mathcal{D}(\lambda)) = \lambda^{d/2} \frac{\omega_d}{(2\pi)^d} \prod_{i=1}^d a_i,$$

which concludes the first step.

**Step 2: Weyl's law holds for finite unions of rectangular boxes**

Let $\Omega_1, ..., \Omega_m$ be disjoint open rectangular boxes, and suppose that there exists $\Omega$ a Lipschitz open set such that $\overline{\Omega} = \overline{\Omega_1 \cup ... \cup \Omega_m}$. Let us show that Weyl's law holds for $\Omega$. We have

$$\begin{aligned}
\frac{\mathcal{N}_{\Omega_1}^D(\lambda) + ... + \mathcal{N}_{\Omega_m}^D(\lambda)}{\lambda^{d/2}} &= \frac{\mathcal{N}_{(\Omega_1 \cup ... \cup \Omega_m)}^D(\lambda)}{\lambda^{d/2}} \\
&\leq \frac{\mathcal{N}_\Omega^D(\lambda)}{\lambda^{d/2}} \qquad \text{by Lemma 1.3} \\
&\leq \frac{\mathcal{N}_\Omega^N(\lambda)}{\lambda^{d/2}} \\
&\leq \frac{\mathcal{N}_{(\Omega_1 \cup ... \cup \Omega_m)}^N(\lambda)}{\lambda^{d/2}} \qquad \text{by Lemma 1.4} \\
&= \frac{\mathcal{N}_{\Omega_1}^N(\lambda) + ... + \mathcal{N}_{\Omega_m}^N(\lambda)}{\lambda^{d/2}},
\end{aligned}$$

and we conclude this step by taking the limit $\lambda \to \infty$, using the previous step, and noting that $\mathrm{Vol}(\Omega) = \mathrm{Vol}(\Omega_1) + ... + \mathrm{Vol}(\Omega_m)$.

**Step 3: End of the proof**

For any $\varepsilon > 0$, we find two open sets $\Omega_\varepsilon$ and $\Omega'_\varepsilon$ as in the previous step, such that $\Omega_\varepsilon \subset \Omega \subset \Omega'_\varepsilon$, and $\mathrm{Vol}(\Omega'_\varepsilon \backslash \Omega_\varepsilon) < \varepsilon$.

Using the monotonicity properties of the eigenvalues, we have

$$\frac{\mathcal{N}_{\Omega_\varepsilon}^D(\lambda)}{\lambda^{d/2}} \leq \frac{\mathcal{N}_\Omega^D(\lambda)}{\lambda^{d/2}} \leq \frac{\mathcal{N}_{\Omega'_\varepsilon}^D(\lambda)}{\lambda^{d/2}} \leq \frac{\mathcal{N}_{\Omega'_\varepsilon}^N(\lambda)}{\lambda^{d/2}}.$$

Now, by the previous step, we may find $\lambda_\varepsilon$ such that for all $\lambda \geq \lambda_\varepsilon$, we have

$$\begin{aligned}
\frac{\mathcal{N}_{\Omega_\varepsilon}^D(\lambda)}{\lambda^{d/2}} &\geq \frac{\omega_d}{(2\pi)^d}\mathrm{Vol}(\Omega_\varepsilon) - \varepsilon \geq \frac{\omega_d}{(2\pi)^d}\mathrm{Vol}(\Omega) - 2\varepsilon \\
\frac{\mathcal{N}_{\Omega'_\varepsilon}^N(\lambda)}{\lambda^{d/2}} &\leq \frac{\omega_d}{(2\pi)^d}\mathrm{Vol}(\Omega'_\varepsilon) + \varepsilon \leq \frac{\omega_d}{(2\pi)^d}\mathrm{Vol}(\Omega) + 2\varepsilon.
\end{aligned}$$

We therefore have, for $\lambda$ large enough,

$$\frac{\omega_d}{(2\pi)^d}\mathrm{Vol}(\Omega) - 2\varepsilon \leq \frac{\mathcal{N}_\Omega^D(\lambda)}{\lambda^{d/2}} \leq \frac{\omega_d}{(2\pi)^d}\mathrm{Vol}(\Omega) + 2\varepsilon,$$

which concludes the proof of the theorem. □

# Chapter 2

# Operators in Hilbert spaces

In this chapter, we fix a separable Hilbert space $\mathcal{H}$, and we will denote by Id the identity operator on $\mathcal{H}$.

## 2.1 Operators and Adjoints (♡)

**Definition 2.1.** *A* (linear) operator $T$ *on* $\mathcal{H}$ *is a linear map from a subspace* $\mathcal{D}(T)$ *to* $\mathcal{H}$*. The set* $\mathcal{D}(T)$ *is then called the* domain *of* $T$*.*
*A linear operator is called* bounded *if*

$$\sup_{v\in\mathcal{D}(T), v\neq 0} \frac{\|Tv\|}{\|v\|} < \infty.$$

*If* $T$ *is a bounded linear operator with* $\mathcal{D}(T) = \mathcal{H}$*, we say that* $T$ *is a* continuous *operator. The space of continuous operators is denoted by* $\mathcal{L}(\mathcal{H})$*. It is a Banach space, equipped with the norm*

$$\|T\|_{\mathcal{H}\to\mathcal{H}} := \sup_{v\in\mathcal{D}(T), v\neq 0} \frac{\|Tv\|}{\|v\|}$$

Most of the time, we will consider operators $T$ such that $\mathcal{D}(T)$ is dense in $\mathcal{H}$. Note that a bounded linear operator such that $\mathcal{D}(T)$ is dense in $\mathcal{H}$ can be uniquely extended to a continuous operator.

### 2.1.1 Closable operators (♡)

Let $T$ be a linear operator in $\mathcal{H}$. We denote by

$$\mathrm{Gr}(T) := \{(v, Tv); v \in \mathcal{D}(T)\} \subset \mathcal{H} \times \mathcal{H}$$

its graph.

If $T_1, T_2$ are operators on $\mathcal{H}$ we will write

$$T_1 \subset T_2 \quad \text{if} \quad \mathrm{Gr}(T_1) \subset \mathrm{Gr}(T_2).$$

We then say that $T_2$ is an *extension* of $T_1$.

**Definition 2.2.** *Let* $T$ *be a linear operator in* $\mathcal{H}$*. We say that* $T$ *is*

- closed *if* $\mathrm{Gr}(T)$ *is a closed subspace of* $\mathcal{H} \times \mathcal{H}$*.*
- closable *if the closure of its graph,* $\overline{\mathrm{Gr}(T)}$ *is still the graph of an operator. We then write* $\overline{T}$ *for the operator such that* $\mathrm{Gr}(\overline{T}) = \overline{\mathrm{Gr}(T)}$*, and* $\overline{T}$ *is then called the* closure *of* $T$*.*

The following lemma, whose proof is trivial, gives a criterion for an operator to be closed.

**Lemma 2.1.** *Let $T$ be a linear operator in $\mathcal{H}$. $T$ is closed if and only if, for any sequence $x_n \in \mathcal{D}(T)$ such that*

$$x_n \longrightarrow x \in \mathcal{H}$$
$$Tx_n \longrightarrow y \in \mathcal{H},$$

*we have $x \in \mathcal{D}(T)$ and $Tx = y$.*

**Example 2.1.** *By the closed graph theorem, an operator $T$ with $\mathcal{D}(T) = \mathcal{H}$ is closed if and only if it is bounded.*

**Example 2.2** (Multiplication operators)**.** *Let $\mathcal{H} = L^2(\mathbb{R}^d)$, and take $f \in L^\infty_{loc}(\mathbb{R}^d)$. We define $M_f$ by*

$$\mathcal{D}(M_f) := \{g \in L^2(\mathbb{R}^d) | (fg) \in L^2(\mathbb{R}^d)\}, \quad M_f g = fg.$$

*Then $M_f$ is a closed operator. Indeed, let $g_n \in \mathcal{D}(M_f)$ with $g_n \longrightarrow g$ and $fg_n \longrightarrow h$. Up to extracting a subsequence, we may suppose that $g_n(x) \longrightarrow g(x)$ almost everywhere. Therefore, $f(x)g_n(x) \longrightarrow f(g)g(x)$ almost everywhere. We must therefore have $fg = h \in L^2(\mathbb{R}^d)$, so that $g \in \mathcal{D}(M_f)$.*

**Example 2.3.** *Let $\mathcal{H} = L^2(\mathbb{R})$, $T_1 f = f'$, with $\mathcal{D}(T_1) = C^\infty_c(\mathbb{R})$, and $T_2(f) = f'$, with $\mathcal{D}(T_2) = H^1(\mathbb{R})$. Let us show that $\overline{T_1} = T_2$.*

*Let us show that $T_2$ is closed. Let $f_n \in H^1(\mathbb{R})$ be such that $f_n \longrightarrow f$ in $L^2(\mathbb{R})$, and $f_n' \longrightarrow g$ in $L^2(\mathbb{R})$. One then easily shows that we must have $f \in H^1(\mathbb{R})$, and $f' = g$. Therefore, $T_2$ is closed.*

*We have $T_1 \subset T_2$, which shows that $T_1$ is closable. Now, let $(f, g) \in \mathrm{Gr}(T_2)$, that is to say, $f \in H^1(\mathbb{R}), g = f'$. Since $C^\infty_c(\mathbb{R})$ is dense in $H^1(\mathbb{R})$, we may find a sequence $f_n \in \mathcal{D}(T_1)$ such that $f_n \longrightarrow f$ in $L^2(\mathbb{R})$ and $f_n' \longrightarrow g$ in $L^2(\mathbb{R})$. This shows that $\overline{T_1} = T_2$.*

**Example 2.4.** *Let $\mathcal{H} = L^2(\mathbb{R})$, and let $g(x) = e^{-x^2}$, so that $g \in \mathcal{H}$. We define an operator $T$ by $\mathcal{D}(T) = C^0(\mathbb{R}) \cap L^2(\mathbb{R})$, and $Tf = f(0)g$. Let us show that $T$ is not closable.*

*Take $f \in \mathcal{D}(T)$, $f \neq 0$. We may find two sequences $f_n, g_n \in \mathcal{D}(T)$ such that $f_n$ and $g_n$ converge to $f$, but $f_n(0) = 1$ for all $n$, while $g_n(0) = 0$ for all $n$. We therefore have $Tg_n = 0$, but $Tf_n = g$. If the operator were closable, the two limits would have to be equal.*

### 2.1.2 Adjoints (♡)

**Proposition 2.1.** *Let $T \in \mathcal{L}(\mathcal{H})$. There exists a unique operator $T^* \in \mathcal{L}(\mathcal{H})$, such that for all $v, w \in \mathcal{H}$, we have*

$$\langle Tv, w\rangle = \langle v, T^* w\rangle.$$

*Proof.* Let $w \in \mathcal{H}$. The map $v \mapsto \langle Tv, w\rangle$ is a continuous linear map from $\mathcal{H}$ to $\mathbb{C}$. Therefore, by Riesz's representation theorem, there exists a unique vector, denoted by $T^*w$, such that we have $\langle Tv, w\rangle = \langle v, T^*w\rangle$ for all $v \in \mathcal{H}$.

Let us check that $w \mapsto T^*w$ is linear. Let $w_1, w_2 \in \mathcal{H}, \lambda \in \mathbb{C}$. We haven for all $v \in \mathcal{H}$,

$$\langle v, T^*(\lambda w_1 + w_2)\rangle = \langle Tv, \lambda w_1 + w_2\rangle = \lambda\langle Tv, w_1\rangle + \langle Tv, w_2\rangle = \lambda\langle v, T^* w_1\rangle + \langle v, T^* w_2\rangle.$$

Since this is true for all $v \in \mathcal{H}$, we deduce that $T^*(\lambda w_1 + w_2) = \lambda T^* w_1 + T^* w_2$, so that $T^*$ is linear.

For any $w \in \mathcal{H}$, we have $\|T^*w\| = \sup_{\|v\|\leq 1} |\langle v, T^*w\rangle| = \sup_{\|v\|\leq 1} |\langle Tv, w\rangle| \leq \sup_{\|v\|\leq 1} \|Tv\|\|w\|$. Therefore, $T^*$ is bounded. □

If $T$ is not a bounded operator, we define its adjoint in a similar way, as follows:

**Definition 2.3.** *Let $T$ be an operator in $\mathcal{H}$ with $\overline{\mathcal{D}(T)} = \mathcal{H}$. Its adjoint is an operator $T^*$ with domain*

$$\mathcal{D}(T^*) := \{w \in \mathcal{H}; \mathcal{D}(T) \ni v \mapsto \langle Tv, w\rangle \in \mathbb{C} \text{ is continuous}\}.$$

*By Riesz representation theorem, for any $w \in \mathcal{D}(T^*)$, there exists a unique vector $w' \in \mathcal{H}$ such that for all $v \in \mathcal{D}(T)$, we have $\langle Tv, w\rangle = \langle v, w'\rangle$. We then set $T^*w := w'$, and $T^*$ is a linear operator.*

Note that the assumption that $\mathcal{D}(T)$ is dense in $\mathcal{H}$ is essential here: otherwise, $T^*w$ would not be uniquely defined, since we could add any vector in $\mathcal{D}(T)^\perp$ to $T^*w$. The two definitions of adjoints coincide when $T$ is bounded.

Define the linear map

$$J : \mathcal{H} \times \mathcal{H} \ni (x, y) \mapsto (y, -x) \in \mathcal{H} \times \mathcal{H}.$$

**Lemma 2.2.** *Let $T$ be a linear operator with a dense domain. We have*

$$\mathrm{Gr}(T^*) = (J(\mathrm{Gr}(T)))^\perp.$$

*Proof.* Let $(x, y) \in \mathrm{Gr}(T^*)$, and let $(u, v) \in \mathrm{Gr}(T)$. We have

$$\begin{aligned}\langle (x, y), J(u, v)\rangle_{\mathcal{H}\times\mathcal{H}} &= \langle x, v\rangle - \langle y, u\rangle \\ &= \langle x, Tu\rangle - \langle T^*x, u\rangle = 0\end{aligned}$$

Therefore, we have $\mathrm{Gr}(T^*) \subset (J(\mathrm{Gr}(T)))^\perp$.

Let $(x, y) \in (J(\mathrm{Gr}(T)))^\perp$. For all $u \in \mathcal{D}(T)$, we have $\langle x, Tu\rangle = \langle y, u\rangle$. Therefore, the map $u \mapsto \langle x, Tu\rangle$ is continuous, so that $x \in \mathcal{D}(T^*)$, and we have $y = T^*x$. We therefore have $\mathrm{Gr}(T^*) \supset (J(\mathrm{Gr}(T)))^\perp$. □

In particular, $T^*$ is always a closed operator.

**Lemma 2.3.** *Let $T$ be a closable operator with a dense domain. Then $\mathcal{D}(T^*)$ is dense in $\mathcal{H}$*

*Proof.* Let $u \in \mathcal{D}(T^*)^\perp$. We thus have, for all $w \in \mathcal{D}(T^*)$

$$\langle J(w, T^*w), (0, u)\rangle_{\mathcal{H}\times\mathcal{H}} = \langle T^*w, 0\rangle - \langle w, u\rangle = 0.$$

Therefore, we have $(0, u) \in (J(\mathrm{Gr}(T^*))^\perp = J((J(\mathrm{Gr}(T^*))^\perp)^\perp = \overline{\mathrm{Gr}(T^*)}$, because $J$ is an involution, which commutes with $\perp$. Since $T$ is closable, we deduce that $u = 0$. □

Note that we have also proved that $(T^*)^* = \overline{T}$.

**Proposition 2.2.** *Let $T$ be a closable operator on a Hilbert space $\mathcal{H}$, and let $z \in \mathbb{C}$. We have*

$$\begin{aligned}\ker(T^* - \overline{z}) &= \mathrm{Ran}(T - z)^\perp \\ \overline{\mathrm{Ran}(T - z)} &= \ker(T^* - \overline{z})^\perp.\end{aligned}$$

*Proof.* The second identity follows from the first one, by taking the orthogonal complement. Let us prove the first point. Let $v \in \ker(T^* - \overline{z})$. Since $\mathcal{D}(T^*)$ is dense, this is equivalent to $\langle (T^* - \overline{z})v, w\rangle = 0$ for all $w \in \mathcal{D}(T)$. This is equivalent to having $\langle v, (T - z)w\rangle = 0$ for all $w \in \mathcal{D}(T)$. Therefore, $v \in \ker(T^* - \overline{z})$ if and only if $v \in \mathrm{Ran}(T - z)^\perp$. □

**Definition 2.4.** *Let $T$ be a linear operator on $\mathcal{H}$ with domain $\mathcal{D}(T)$. $T$ is said to be*

- symmetric *if for all $v, w \in \mathcal{D}(T)$, we have*

  $$\langle Tv, w\rangle = \langle v, Tw\rangle.$$

  *In other words, we have $T \subset T^*$.*

- self-adjoint *if $\mathcal{D}(T) = \mathcal{D}(T^*)$ and $T$ is symmetric. In other words, we have $T = T^*$.*

**Example 2.5.** *Let $M_f$ be as in Example 2.2. We have (EXERCISE) $(M_f)^* = M_{\overline{f}}$. In particular, $M_f$ is self-adjoint if and only if $f$ is real-valued.*

**Example 2.6** (Free Laplacian in $\mathbb{R}^d$)**.** *Let $\mathcal{H} = L^2(\mathbb{R}^d)$, and $T$ be given by $\mathcal{D}(T) = H^2(\mathbb{R}^d)$, $Tf = -\Delta f$. By integration by parts, $T$ is symmetric.*

*By the Riesz representation theorem, $f \in \mathcal{D}(T^*)$ if and only if there exists $g \in L^2(\mathbb{R}^d)$ such that for all $h \in H^2(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^d} (-\Delta h) f = \int_{\mathbb{R}^d} hg.$$

*In particular, we have $-\Delta f = g$ in the sense of distributions, so that $f \in H^2(\mathbb{R}^d)$. We deduce that $\mathcal{D}(T^*) = \mathcal{D}(T)$, so that $T$ is self-adjoint.*

## 2.2 A short introduction to quantum mechanics (♠)

In classical mechanics, the motion of a particle is governed by Newton's equation. Namely, if $x(t) \in \mathbb{R}^3$ denotes the position of the particle at time $t$, it satisfies the equation[1]

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = f(t),$$

where $f$ is a force, which, in the absence of friction, takes the form $f(t) = -\nabla V(x(t))$.

**Abstract quantum mechanics in Hilbert spaces** In quantum mechanics, a particle is no longer considered as a point in $\mathbb{R}^3$, but as a vector in a Hilbert space. Actually, the state of any physical system (a particle, a system of particles, me, the whole universe) is described by a vector $v$ in a (separable) Hilbert space $\mathcal{H}$. More precisely, we ask that $\|v\| = 1$, and we consider that $v$ and $e^{i\theta} v$ represent the same physical state, for any $\theta \in \mathbb{R}$.

The state of a system generally depends on time, and its time dependence is given by *Schrödinger's equation*, which can be written, in well-chosen units, as

$$i \frac{\mathrm{d}}{\mathrm{d}t} v(t) = H v(t). \tag{2.1}$$

Here, $H$ is a self-adjoint operator on $\mathcal{H}$, called the *(quantum) Hamiltonian.*

Given a physical system, one wants to measure real numbers out of it. For instance, in classical mechanics, one wants to measure the speed, position, energy, angular momentum... of a system of particles.

In quantum mechanics, the quantities one can measure, called observables, are given by self-adjoint operators. The value of an observable $A$ measured for a state $v$ is in general not deterministic. Namely, if one prepares several times the same state $v$, and measures several times the observable $A$, one will not find the same value. However, for a large number of experiments, one will *on average* find the value

$$\langle v, Av \rangle. \tag{2.2}$$

Beware that, making a measurement on a system affects it: it is no longer in the same state after the measurement.

**Quantum mechanics for a single particle** Very often, one is interested in the motion of a single particle (for instance, an electron). A natural choice of Hilbert space is then $\mathcal{H} = L^2(\mathbb{R}^3)$, or more generally, $\mathcal{H} = L^2(\mathbb{R}^d)$. A state $\psi \in L^2(\mathbb{R}^3)$ has the following physical interpretation. It does not have a definite position as a classical particle does, but only a *probabilistic* position. Namely, the probability that, when we measure the position of the particle, we find it in an open set $\Omega \subset \mathbb{R}^3$ is given by

$$\int_\Omega |\psi(x)|^2 \mathrm{d}x.$$

This corresponds to taking, in (2.2), the observable of multiplication by $\mathbf{1}_\Omega$. Note that the condition that $\|\psi\| = 1$ ensures that the total probability is one.

The quantum Hamiltonian governing the motion of an electron is often a *Schrödinger operator*, of the form

$$H = -h^2 \Delta + V,$$

[1] Here, we choose a system of units such that the mass of the particle is equal to 1.

for some measurable function $V$, called the potential, which has the same expression as in classical mechanics.

The Schrödinger equation then takes the expression of a partial differential equation

$$i\frac{\partial\psi(t,x)}{\partial t} = -\Delta\psi(t,x) + V(x)\psi(t,x).$$

## 2.3 The resolvent and the spectrum (♡)

**Definition 2.5.** *Let $T$ be a linear operator on $\mathcal{H}$. The* resolvent set *of $T$ is the subset of $\mathbb{C}$*

$$\rho(T) := \{z \in \mathbb{C}; (T - z\mathrm{Id}) : \mathcal{D}(T) \to \mathcal{H} \text{ is invertible, with a bounded inverse}\}.$$

*The spectrum of $T$ is defined as*

$$\sigma(T) = \mathbb{C}\backslash\rho(T).$$

*When $z \in \sigma(T)$ is such that $\ker(T - z) \neq \{0\}$, we say that $z$ is an* eigenvalue *for $T$. The* multiplicity *of $z$ is then $\dim\ker(T - z)$. The set of eigenvalues is denoted by $\sigma_p(T)$.*

In the sequel, we will often write $T - z$ instead of $T - z\mathrm{Id}$.

**Lemma 2.4.** *Let $T$ be a closed operator. We then have the following equivalence.*

$$z \in \rho(T) \iff \begin{cases} \ker(T - z) = \{0\} \\ \mathrm{Im}(T - z) = \mathcal{H}. \end{cases}$$

*Proof.* The $\Longrightarrow$ inclusion is obvious. Suppose that $\ker(T - z) = \{0\}$ and $\mathrm{Im}(T - z) = \mathcal{H}$. The inverse of $T - z$, $(T - z)^{-1}$ is thus defined everywhere. Since the graph of $T - z$ is closed, the graph of $(T - z)^{-1}$ is also closed. By the closed graph theorem, $(T - z)^{-1}$ is a continuous operator, so that $z \in \rho(T)$. □

**Definition 2.6.** *Let $\Omega \subset \mathbb{C}$ be an open set, and let $T(z)$ be a family of operators in $\mathcal{L}(\mathcal{H})$ depending on $z \in \Omega$. We say that the map $\Omega \ni z \mapsto T(z) \in \mathcal{L}(\mathcal{H})$ is holomorphic if, for any $v_1, v_2 \in \mathcal{H}$, the map $\Omega \ni z \mapsto \langle T(z)v_1, v_2\rangle$ is holomorphic.*

**Proposition 2.3.** *Let $T$ be a linear operator. The set $\rho(T)$ is open, the set $\sigma(T)$ is closed. The map $\rho(T) \ni z \mapsto R_T(z) := (T - z)^{-1} \in \mathcal{L}(\mathcal{H})$, called the* resolvent *is holomorphic. It satisfies the identities*

$$R_T(z_1) - R_T(z_2) = (z_1 - z_2)R_T(z_1)R_T(z_2) \tag{2.3}$$

$$R_T(z_1)R_T(z_2) = R_T(z_2)R_T(z_1) \tag{2.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}R_T(z) = R_T(z)^2. \tag{2.5}$$

*for all $z, z_1, z_2 \in \rho(T)$.*

*Proof.* If $z, z_0 \in \rho(T)$, we have

$$T - z = (T - z_0)(\mathrm{Id} - (z - z_0)R_T(z_0)). \tag{2.6}$$

If $|z - z_0| < R_T(z_0)$, then the right-hand side is invertible, with inverse $(T - z_0)\sum_{k=0}^{\infty}(z - z_0)^k R_T^k(z_0))$, which is a bounded operator.

Therefore, $\rho(T)$ is an open set, and we have

$$(T - z)^{-1} = (T - z_0)\sum_{k=0}^{\infty}(z - z_0)^k R_T^k(z_0)).$$

This shows that $\rho(T) \ni z \mapsto R_T(z) \in \mathcal{L}(\mathcal{H})$ is holomorphic. Differentiating with respect to $z$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}(T-z)^{-1} = (T-z_0)\sum_{k=0}^{\infty}(k+1)(z-z_0)^k R_T^{k+1}(z_0)).$$

Taking the value at $z = z_0$, we obtain (2.5).

From (2.6), we obtain that $(T-z_0)^{-1} = \big(\mathrm{Id} - (z-z_0)R_T(z_0)\big)R_T(z)$, from which (2.3) follows. Finally, (2.4) follows from the symmetries in (2.3). □

**Proposition 2.4** (The spectrum of a bounded operator)**.** *Let $T \in \mathcal{L}(\mathcal{H})$. Then $\sigma(T) \neq \emptyset$, and*

$$\sigma(T) \subset \{z \in \mathbb{C}; |z| \leq \|T\|\}.$$

*Proof.* Let $z \in \mathbb{C}$ with $|z| > \|T\|$. We may write $(T-z) = -z\big(\mathrm{Id} - \frac{1}{z}T\big)$. Since $\|T/z\| < 1$, this operator is invertible, with inverse

$$(T-z)^{-1} = -\sum_{k=0}^{\infty}\frac{T^k}{z^{k+1}}. \tag{2.7}$$

Let us show that the spectrum is non-empty. Suppose for contradiction that $\sigma(T) = \emptyset$. Then, by Proposition 2.3, for any $v, w \in \mathcal{H}$, $\mathbb{C} \ni z \mapsto \langle v, (T-z)^{-1}w\rangle \in \mathbb{C}$ is holomorphic. Furthermore, by (2.7), it goes to zero at infinity. Therefore, by Liouville's theorem, it must be constant equal to zero. We therefore have $R_T(z) = 0$, which is absurd, since $R_T(z) : \mathcal{H} \to \mathcal{D}(T)$ is invertible. □

### 2.3.1 The spectrum of multiplication operators (♡)

Let $\mathcal{H} = L^2(\mathbb{R}^d)$, and $f \in L^\infty_{loc}(\mathbb{R}^d)$, and $M_f$ be defined as in Example 2.2. The essential range of $f$ is defined as

$$\text{ess ran} f := \big\{\lambda \in \mathbb{C} \mid \forall \varepsilon > 0, \mathrm{Leb}\{x; |f(x)-\lambda| < \varepsilon\} > 0\big\}.$$

**Proposition 2.5.** *We have*

$$\sigma(M_f) = \text{ess ran} f$$
$$\sigma_p(M_f) = \big\{\lambda \in \mathbb{C} \mid \mathrm{Leb}\{x; f(x) = \lambda\} > 0\big\}.$$

*Proof.* Let $\lambda \notin \text{ess ran} f$. Then $M_{1/(f-\lambda)}$ is a bounded operator, and it is the inverse of $M_f - \lambda$, so that $\lambda \notin \sigma(M_f)$.

Conversely, let $\lambda \in \text{ess ran} f$. Let $\Omega_n := \{x \in \mathbb{R}^d; |f(x)-\lambda| < n^{-1}\}$, which has positive measure. We let $v_n \in L^2(\mathbb{R}^d)$ be the indicator function of $\Omega_n$. We have $v_n \in \mathcal{D}(M_f)$, and

$$\|(M_f-\lambda)v_n\|^2 = \int_{\Omega_n}|f(x)-\lambda|^2\mathrm{d}x \leq \frac{1}{n^2}\|v_n\|^2,$$

so that $(M_f - \lambda)$ cannot have a bounded inverse. Therefore, $\lambda \in \sigma(M_f)$.

Finally, note that $\lambda \in \sigma_p(M_f)$ if and only if there exists $g \in L^2(\mathbb{R}^d)$, $g \neq 0$ such that $f(x)g(x) = \lambda g(x)$ for almost every $x \in \mathbb{R}^d$. Therefore, $g$ must vanish for every $x$ with $f(x) \neq \lambda$, so that $\{x; f(x) = \lambda\}$ must have positive measure. Conversely, if $\{x; f(x) = \lambda\}$ has positive measure, any function supported on $\{x; f(x) = \lambda\}$ is an eigenfunction. □

### 2.3.2 The spectrum of a self-adjoint operator (♡)

**Proposition 2.6.** *Let $T$ be a self-adjoint operator. Then $\sigma(T) \subset \mathbb{R}$, and we have for all $z \in \mathbb{C}\backslash\mathbb{R}$*

$$\|(T-z)^{-1}\| \leq \frac{1}{\Im z}. \tag{2.8}$$

*Proof.* Let $z \in \mathbb{C}\backslash\mathbb{R}$, and $v \in \mathcal{H}$. We have

$$\langle v, (T-z)v\rangle = \langle v, Tv\rangle - \Re z\langle v, z\rangle - i\Im z\langle v, z\rangle.$$

Since $T$ is self-adjoint, $\langle v, Tv\rangle$ is real. Therefore, we have

$$|\Im z|\|v\|^2 \leq |\langle v, (T-z)v\rangle| \leq \|v\| \cdot \|(T-z)v\|,$$

so that

$$\|(T-z)v\| \geq |\Im z|\|v\|. \tag{2.9}$$

Therefore, $\ker(T-z) = \{0\}$, and $\overline{\mathrm{Ran}(T-z)} = \mathcal{H}$ by Proposition 2.2. Equation (2.9) also implies that $\mathrm{Ran}(T-z)$ is closed. Indeed, let $y_n$ be a sequence in $\mathrm{Ran}(T-z)$, converging to some $y \in \mathcal{H}$. There exists $x_n \in \mathcal{H}$ such that $y_n = (T-z)(x_n)$. We have $\|x_n - x_m\| \leq \frac{1}{\Im z}\|y_n - y_m\|$, so that $(x_n)$ is a Cauchy sequence. It is thus convergent, and since $T$ is self-adjoint, $T-z$ is closed, so that $y \in \mathrm{Ran}(T)$.

By Lemma 2.4, we have $z \in \rho(T)$, and (2.8) follows from (2.9). □

**Proposition 2.7.** *Let $T \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator. Write*

$$m := \inf_{v \in \mathcal{H}\backslash\{0\}} \frac{\langle v, Tv\rangle}{\|v\|^2}$$
$$M := \sup_{v \in \mathcal{H}\backslash\{0\}} \frac{\langle v, Tv\rangle}{\|v\|^2}.$$

*Then $\sigma(T) \subset [m, M]$, and $m, M \subset \sigma(T)$.*

*Proof.* We have already proved that $\sigma(T) \subset \mathbb{R}$. Let $\lambda > M$. We have $\langle v, (\lambda - T)v\rangle \geq (\lambda - M)\|v\|^2$, so that $\lambda - T$ is invertible with a bounded inverse, by the same argument as in the proof of the previous Proposition. We therefore have $\sigma(T) \subset (-\infty, M]$. We show similarly that $\sigma(T) \subset [m, +\infty)$.

Let us show that $M \in \sigma(T)$. We will then deduce that $m \in \sigma(T)$, by replacing $T$ by $-T$.

Let $u, v \in \mathcal{H}$. The map bilinear map $(u, v) \mapsto \langle u, (M-T)v\rangle$ is positive, so that, by the Cauchy-Schwarz inequality, we have

$$|\langle u, (M-T)v\rangle|^2 \leq \langle u, (M-T)u\rangle\langle v, (M-T)v\rangle$$

Taking the supremum over $u \in \mathcal{H}$ with $\|u\| = 1$, we obtain

$$\|(M-T)v\| \leq \|M-T\|\langle v, (M-T)v\rangle.$$

Now, by definition of $M$, there exists a sequence $v_n \in \mathcal{H}$ with $\|v_n\| = 1$ such that $\langle v_n, Tv_n\rangle \longrightarrow M$, so that $\|(M-T)v_n\| \longrightarrow 0$. Therefore, $M - T$ cannot be invertible with a bounded inverse. We deduce that $M \in \sigma(T)$. □

**Remark 2.1.** *If $T \in \mathcal{L}(\mathcal{H})$ and $m, M$ are as in Proposition 2.7, then*

$$\|T\| = \max(|m|, |M|).$$

*Indeed, we have $|m|, |M| \leq \|T\|$. On the other hand, by Cauchy-Schwarz inequality, we have*

$$|\langle u, Tv\rangle|^2 \leq |\langle u, Tu\rangle||\langle v, Tv\rangle|.$$

*Taking the supremum over $u, v \in \mathcal{H}$ with $\|u\|, \|v\| = 1$, we deduce that $\|T\| \leq \max(|m|, |M|)$.*

## 2.4 Criteria for self-adjointness (♣)

### 2.4.1 The Kato-Rellich theorem (♣)

**Theorem 2.1.** *Let $T$ be a closed symmetric operator on $\mathcal{H}$. The following assertions are equivalent:*

1. *$T$ is self-adjoint.*
2. $\ker(T^* + i) = \ker(T^* - i) = \{0\}$.
3. $\mathrm{Ran}(T + i) = \mathrm{Ran}(T - i) = \mathcal{H}$.

*Proof.* The implication 1. $\Longrightarrow$ 2. is easy: a self-adjoint operator cannot have non-real eigenvalues.

Let us prove that 2. $\Longrightarrow$ 3. Recall that by Proposition 2.2, we have $\ker(T^* \pm i) = \mathrm{Ran}(T \mp i)^\perp$. It is therefore sufficient to show that $\mathrm{Ran}(T \pm i)$ is closed. Let $v \in \mathcal{H}$. We have

$$\|(T \pm i)v\|^2 = \|Tv\|^2 + \|v\|^2 \pm i(\langle Tv, v\rangle - \langle v, Tv\rangle) = \|Tv\|^2 + \|v\|^2,$$

since $T$ is symmetric. Now, let $w_n \in \mathrm{Ran}(T \mp i)$ converge to some $w \in \mathcal{H}$. By assumption, there exists $v_n \in \mathcal{D}(T)$ such that $w_n = (T \mp i)v_n$. By the previous equality, we deduce that $v_n$ and $Tv_n$ are Cauchy sequences. $T$ being closed, $v_n$ converges to $v \in \mathcal{D}(T)$, $Tv_n$ converges to $Tv$, and we have $w = (T \pm i)v$, which proves 3.

Let us prove that 3. $\Longrightarrow$ 1. Let $v \in \mathcal{D}(T^*)$. We want to show that $v \in \mathcal{D}(T)$. Since $T \pm i$ is surjective, we may find $w \in \mathcal{D}(T)$ such that $(T - i)w = (T^* - i)v$. We have $T \subset T^*$, so that $(T^* - i)(v - w) = (T - i)w - (T^* - i)w = 0$.

We have $\mathrm{Ran}(T + i) = \mathcal{H}$, so that $\ker(T^* - i) = \{0\}$. Therefore, $v = w$ belongs to $\mathcal{D}(T)$. □

**Definition 2.7.** *Let $A$ be a self-adjoint operator on $\mathcal{H}$, and let $B$ be an operator on $\mathcal{H}$ with $\mathcal{D}(A) \subset \mathcal{D}(B)$. We say that $B$ is* relatively bounded *with respect to $A$ if there exists $a, b > 0$ such that for any $v \in \mathcal{D}(A)$, we have*

$$\|Bv\| \leq a\|Av\| + b\|v\|. \tag{2.10}$$

*The infimum of all $a > 0$ such that (2.10) holds is called the* relative bound *of $B$ with respect to $A$.*

**Theorem 2.2** (Kato-Rellich)**.** *Let $A$ be a self-adjoint operator, and let $B$ be a symmetric operator, relatively bounded with respect to $A$, with a relative bound $< 1$. The operator $A + B$ with domain $\mathcal{D}(A + B) = \mathcal{D}(A)$ is self-adjoint.*

*Proof.* Let $a \in (0, 1)$, $b > 0$ be such that

$$\|Bv\| \leq a\|Av\| + b\|v\| \quad \forall v \in \mathcal{D}(T).$$

Let us fix $\lambda > b$. We have for any $v \in \mathcal{D}(A)$

$$\|(A + B \pm i\lambda)v\|^2 = \|(A + B)v\|^2 + \lambda^2\|v\|^2 \pm i\lambda(\langle (A + B)v, v\rangle - \langle v, (A + B)v\rangle) = \|(A + B)v\|^2 + \lambda^2\|v\|^2,$$

Therefore, using the fact that $(a + b)^2 \leq 2a^2 + 2b^2$, we have

$$\begin{aligned} \|(A + B \pm i\lambda)v\| &\geq \frac{1}{2}\|(A + B)v\| + \frac{\lambda}{2}\|v\| \\ &\geq \frac{1}{2}\big(\|Av| - \|Bv\|\big) + \frac{\lambda}{2}\|v\| \\ &\geq \frac{1 - a}{2}\|Av\| + \frac{\lambda - b}{2}\|v\|. \end{aligned} \tag{2.11}$$

**Step 1** Let us show that $(A + B)$, with domain $\mathcal{D}(A)$, is a closed operator. Let $(v_n) \in \mathcal{D}(A)$ be such that $(v_n)$ and $w_n := (A + B)v_n$ converge in $\mathcal{H}$. By (2.11), $(Av_n)$ is a Cauchy sequence, so it it is convergent. Since $A$ is closed, $v_n$ converges to some $v \in \mathcal{D}(A)$, and $Av_n$ converges to $Av$. Since $B$ is relatively bounded with respect to $A$, $(Bv_n)$ is a Cauchy sequence, so it converges to $w \in \mathcal{H}$. We have to show that $w = Bv$. This is not trivial, since we did not assume that $B$ is closable. Let $h \in \mathcal{D}(A)$. We have

$$\langle w, h\rangle = \lim_{n\to\infty} \langle Bv_n, h\rangle = \lim_{n\to\infty} \langle v_n, Bh\rangle = \langle v, Bh\rangle = \langle Bv, h\rangle.$$

Since this is true for every $h \in \mathcal{D}(A)$, and $D(A)$ is dense in $\mathcal{H}$, we deduce that $w = Bv$.

**Step 2** Let us show that, for $\lambda$ large enough, $A + B \pm i\lambda$ is bijective. We have, for any $v \in \mathcal{D}(A)$, $\|(A \pm i\lambda)v\|^2 = \|Av\|^2 + \lambda^2\|v\|^2$, so that

$$\|Bv\| \le a\|Av\| + b\|v\| \le a\|(A \pm i\lambda)v\| + \frac{b}{\lambda}\|(A \pm i\lambda)v\| = \left(a + \frac{b}{\lambda}\right)\|(A \pm i\lambda)v\|.$$

Let us choose $\lambda$ large enough so that $a + \frac{b}{\lambda} < 1$. We then have $\|B(A \pm i\lambda)^{-1}\| < 1$. Therefore, we may write

$$A + B \pm i\lambda = \big(1 + B(A \pm i\lambda)^{-1}\big)(A \pm i\lambda).$$

$(A \pm i\lambda)$ is bijective, and $B(A \pm i\lambda)^{-1}$ is also bijective. Therefore, $A + B \pm i\lambda$ is bijective, which concludes the proof. □

### 2.4.2 Self-adjointness of Schrödinger operators (♣)

**Theorem 2.3.** *Let $d \le 3$. Let $V \in L^2(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ be real valued. Then the operator $T = -\Delta + V$ with $\mathcal{D}(T) = H^2(\mathbb{R}^d)$ is self-adjoint.*

*Proof.* Let $f \in \mathcal{S}(\mathbb{R}^d)$, and let $\lambda \in \mathbb{C}\backslash\mathbb{R}$. We have

$$\begin{aligned} f(x) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (\mathcal{F}f)(\xi) e^{ix\cdot\xi} \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (|\xi|^2 + \lambda)(\mathcal{F}f)(\xi) \frac{1}{|\xi|^2 + \lambda} e^{ix\cdot\xi} \mathrm{d}\xi. \end{aligned}$$

Noting that, since $d \le 3$, the map $\xi \mapsto \frac{1}{|\xi|^2+\lambda}$ belongs to $L^2(\mathbb{R}^d)$, we have

$$\begin{aligned} |f(x)| &\le \frac{1}{(2\pi)^{d/2}} \big\|(|\xi|^2 + \lambda)\hat{f}(\xi)\big\|_{L^2(\mathbb{R}^d)} \left\|\frac{1}{|\xi|^2 + \lambda}\right\|_{L^2(\mathbb{R}^d)} \\ &= \frac{1}{(2\pi)^{d/2}} \left\|\frac{1}{|\xi|^2 + \lambda}\right\|_{L^2(\mathbb{R}^d)} \|-\Delta f\|_{L^2(\mathbb{R}^d)} + \frac{\lambda}{(2\pi)^{d/2}} \left\|\frac{1}{|\xi|^2 + \lambda}\right\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

By density, we obtain that for any $f \in H^2(\mathbb{R}^d)$ and any $\lambda \in \mathbb{R}^d$, we have

$$\|f\|_{L^\infty} \le a_\lambda\|-\Delta f\| + b_\lambda\|f\|,$$

where $a_\lambda := \frac{1}{(2\pi)^{d/2}}\left\|\frac{1}{|\xi|^2+\lambda}\right\|$ and $b_\lambda := \frac{\lambda}{(2\pi)^{d/2}}\left\|\frac{1}{|\xi|^2+\lambda}\right\|$.

Now, by assumption, we can write $V = V_1 + V_2$, with $V_1 \in L^\infty(\mathbb{R}^d)$, and $V_2 \in L^2(\mathbb{R}^d)$. We have, for any $f \in H^2(\mathbb{R}^d)$,

$$\begin{aligned} \|Vf\| \le \|V_1 f\| + \|V_2 f\| &\le \|V_1\|_\infty \|f\| + \|V_2\|_2 \|f\|_\infty \\ &\le \|V_2\|_2 a_\lambda \|-\Delta f\| + \big(b_\lambda + \|V_1\|_\infty\big)\|f\|. \end{aligned}$$

By taking $\lambda$ large enough, we may assume that $a_\lambda < 1$. We may then apply the Kato-Rellich theorem to conclude. □

**Example 2.7** (The Coulomb potential). *Let $q \in \mathbb{R}$, and let $d = 2$ or $d = 3$. Consider the potential $V(x) = \frac{q}{|x|}$. For any bounded domain $\Omega \subset \mathbb{R}^d$ containing the origin, we have $\mathbf{1}_\Omega V \in L^2(\mathbb{R}^d)$, while $(1 - \mathbf{1}_\Omega)V \in L^\infty(\mathbb{R}^d)$. Therefore, $-\Delta + V$ is self-adjoint on $H^2(\mathbb{R}^d)$.*

# Chapter 3

# The spectral theorem

## 3.1 Three versions of the spectral theorem (♡)

Let $T$ be a bounded self-adjoint operator on $\mathcal{H}$. If $P(X) = \sum_{j=0}^{n} a_j X^j$ is a polynomial in one variable, we define

$$P(T) = \sum_{j=0}^{n} a_j T^j,$$

where $T^2 = T \circ T$, $T^3 = T \circ T \circ T$,...

**Lemma 3.1.** *Let $T \in \mathcal{L}(\mathcal{H})$, and let $P \in \mathbb{C}[X]$. We have*

$$\sigma(P(T)) = P(\sigma(T)).$$

*Proof.* Let $\lambda \in \mathbb{C}$. We may write $P(X) - \lambda = \alpha \prod_{i=1}^{n}(X - \lambda_i)$, where $\alpha, \lambda_1, ..., \lambda_n \in \mathbb{C}$ and the $\lambda_i$ are solutions of $P(\lambda_i) - \lambda = 0$.

We have $P(T) - \lambda = \alpha \prod_{i=1}^{n}(T - \lambda_i)$, which is invertible if and only if each of the operators $T - \lambda_i$ is invertible. Therefore, $\lambda \in \sigma(P(T))$ if and only if there exists $\lambda_i \in \sigma(T)$ such that $P(\lambda_i) = \lambda$. This shows the result. □

**Lemma 3.2.** *Let $T \in \mathcal{L}(\mathcal{H})$, and $P \in \mathbb{C}[X]$. We have*

$$\|P(T)\|_{\mathcal{H} \to \mathcal{H}} = \sup_{\lambda \in \sigma(T)} |P(\lambda)|.$$

*Proof.* By Lemma 3.1, we have $\sup_{\lambda \in \sigma(T)} |P(\lambda)| = \sup_{\mu \in \sigma(P(T))} |\mu|$.

By Proposition 2.7 and Remark 2.1, this quantity is equal to $\|P(T)\|$. □

#### Continuous functional calculus

Let $T \in \mathcal{L}(\mathcal{H})$, and let $C(\sigma(T))$ denote the space of complex-valued continuous functions on the compact set $\sigma(T)$. By the Stone-Weierstrass theorem, for any function $f \in C(\sigma(T))$, there exists a sequence $P_n \in \mathbb{C}[X]$ such that $\|P_n - f\|_{C^0(\sigma(T))} \longrightarrow 0$. We define the operator

$$f(A) = \lim_{n \to \infty} P_N(A).$$

Thanks to Lemma 3.2, the limit exists and is independent of the choice of $P_n$.

**Proposition 3.1.** *Let $T \in \mathcal{L}(\mathcal{H})$. The map $C(\sigma(T)) \ni f \mapsto f(T) \in \mathcal{L}(\mathcal{H})$ has the following properties:*

1. $(\alpha f + \beta g)(T) = \alpha f(T) + \beta g(T)$.
2. $(f \cdot g)(T) = f(T)g(T)$.
3. $\overline{f}(T) = f(T)^*$.
4. *If $f \geq 0$, then for all $v \in \mathcal{H}$, we have $\langle v, f(T)v\rangle$.*
5. $\|f(T)\| = \|f\|_\infty$.
6. *If $f(z) = \frac{1}{z-z_0}$ for some $z_0 \in \rho(T)$, then $f(T) = R_T(z_0)$.*

*Proof.* The first five points follow from Lemma 3.2, while the last point follows from the second one. □

**Continuous functional calculus for semi-bounded operators** It is actually possible, for any self-adjoint operator $T$, not necessarily bounded, to build a map $C_0(\sigma(T)) \ni f \mapsto f(T) \in \mathcal{L}(\mathcal{H})$ having the same properties as in Proposition 3.1, where $C_0(\sigma(T))$ is the space of functions in $C(\sigma(T))$ which vanish at infinity.

For simplicity, we will not do the construction for general operators, but only for a wide class of self-adjoint operators, which includes the Laplacian with any reasonable boundary conditions.

**Definition 3.1.** *Let $T$ be a self-adjoint operator. We say that $T$ is* semi-bounded from below *if there exists $c \in \mathbb{R}$ such that, for all $v \in \mathcal{D}(T)$, we have*

$$\langle v, Tv\rangle \geq c\|v\|^2. \tag{3.1}$$

If $T$ is semi-bounded from below and $c$ is as in (3.1), then for any $\lambda_0 < c$, we have $\lambda_0 \in \rho(T)$, by the same argument as in the first part of the proof of Proposition 2.7. The operator $(T - \lambda_0)^{-1}$ is then self-adjoint (cf the exercises), and bounded by definition.

From now on, all the operators we will consider will be semi-bounded from below, even though this assumption is never necessary.

**Lemma 3.3.** *The map $r_{\lambda_0} : \sigma(T) \ni \lambda \mapsto (\lambda - \lambda_0)^{-1}$ is a bijection, and we have $\sigma(R_{\lambda_0}(T)) = r_{\lambda_0}(\sigma(T))$.*

*Proof.* It is clear that $r_{\lambda_0}$ is a bijection. Now, if $\lambda \neq 0$, we have

$$(T - \lambda_0)^{-1} - \lambda : -\lambda(T + \lambda_0)^{-1}\Big[T - \big(\lambda_0 - \frac{1}{\lambda}\big)\mathrm{Id}\Big].$$

Therefore, $\lambda \neq 0$ is in $\sigma((T - \lambda_0)^{-1})$ if and only if $\lambda = \frac{1}{\lambda_0 - x}$ for some $x \in \sigma(T)$. Since $0 \notin \sigma((T - \lambda_0)^{-1})$ by definition, the result follows. □

If $f \in C_0(\sigma(T))$, then $f \circ r_{\lambda_0}^{-1} \in C(\sigma(T - \lambda_0)^{-1})$, and we simply define

$$f(T) := \big(f \circ r_{\lambda_0}^{-1}\big)\big((T - \lambda_0)^{-1}\big). \tag{3.2}$$

Thus defined, the map $C_0(\sigma(T)) \ni f \mapsto f(T) \in \mathcal{L}(\mathcal{H})$ has the same properties as in Proposition 3.1.

**Remark 3.1.** *If $f \in C_0(\mathbb{R})$, then we can define $f(T) := f_{|\sigma(T)}(T)$. In particular, $f(T) = 0$ if and only if $f$ vanishes on $\sigma(T)$.*

### Borelian functional calculus

Let $v \in \mathcal{H}$. The map $C_0(\mathbb{R}) \ni f \mapsto \langle v, f(T)v \rangle$ is continuous, and is positive when $f$ is positive. Therefore, by the Riesz representation theorem, there exists a finite positive measure $\mu_v$, supported on $\sigma(T)$ such that for all $f \in C_0(\mathbb{R})$,

$$\langle v, f(T)v \rangle = \int_{\mathbb{R}} f(\lambda) \mathrm{d}\mu_v(\lambda). \tag{3.3}$$

The measure $\mu_v$ is called the *spectral measure at* $v$. Note that we have

$$\mu(\mathbb{R}) = \|v\|^2. \tag{3.4}$$

By the polarization identities, we can find, for any $v, w \in \mathcal{H}$, a complex-valued measure $\mu_{v,w}$ such that for all $f \in C_0(\mathbb{R})$, we have

$$\langle v, f(T)w \rangle = \int_{\mathbb{R}} f(\lambda) \mathrm{d}\mu_{v,w}(\lambda). \tag{3.5}$$

Thanks to equation (3.5), we can define $\langle v, f(T)w \rangle$ for any $f \in L^\infty(\mu_{v,w})$. In particular, if $M \subset \mathbb{R}$ is a Borelian set, and if $\chi_M$ is its characteristic function, then $\chi_M(T) \in L^\infty(\mu_{v,w})$ for all $v, w \in \mathcal{H}$, so that $\chi_M(T)$ is well-defined.

**Proposition 3.2.** *Let $T$ be a self-adjoint operator. The operators $\chi_M(T)$ satisfy the following properties*

1. *$\chi_M(T)$ is a projection.*
2. *$\chi_\emptyset(T) = 0$, $\chi_{\sigma(T)}(T) = \mathrm{Id}$.*
3. *$\chi_{M\cap N}(T) = \chi_M(T)\chi_N(T)$*
4. *If $(M_n)_{n\in\mathbb{N}}$ are mutually disjoint Borelian sets, and if $v \in \mathcal{H}$, then*

$$\chi_{\cup_{n\in\mathbb{N}} M_n}(T)v = \sum_{n\in\mathbb{N}} \chi_{M_n}(T)v.$$

5. *Let $a < b \in \mathbb{R}$ $\chi_{(a,b)}(T) = 0$ if and only if $\sigma(T) \cap (a, b) = \emptyset$.*
6. *For any $\lambda \in \mathbb{R}$, we have $\mathrm{Ran}\chi_{\{\lambda\}}(T) = \ker(T - \lambda)$.*

The family of operators $\chi_M(T)$ is called the *projection-valued measure associated to* $T$, or the *projection-valued spectral measure of* $T$.

*Proof.* The first four points follow by noting that $\chi_M^2 = \chi_M$, $\chi_\emptyset = 0$, $\chi_{\sigma(T)} = 1$ on $\sigma(T)$, $\chi_{M\cap N} = \chi_M\chi_N$ and, if $(M_n)_{n\in\mathbb{N}}$ are mutually disjoint Borelian sets, then $\chi_{\cup_{n\in\mathbb{N}} M_n} = \sum_{n\in\mathbb{N}} \chi_{M_n}$. We then simply apply the definition (3.5).

For the fifth point, it is clear from the definition of $\chi_M(T)$ that if $\sigma(T)\cap(a, b) = \emptyset$, then $\chi_{(a,b)}(T) = 0$. The converse follows from point 5 in Proposition 3.1.

For the last point, note that $v \in \ker(T - \lambda)$ if and only if $(T - \zeta)^{-1}v = (\lambda - \zeta)^{-1}v$ for all $\zeta \in \rho(T)$. Using the continuous functional calculus for $(T - \zeta)^{-1}$, this is equivalent to having $\langle v, f(T)v \rangle = f(\lambda)$, that is to say, to having $\chi_{\{\lambda\}}v = v$. □

### $L^2$ functional calculus

**Definition 3.2.** *Let $T$be a self-adjoint operator on $\mathcal{H}$. Let $L \subset \mathcal{H}$. We say that $L$ is*

- *an* invariant *subspace for $T$ if, for any $z \in \mathbb{C}\backslash\mathbb{R}$, we have $(T - z)^{-1}(L) \subset L$.*
- *a* cyclic *subspace for $T$ with cyclic vector $v \in \mathcal{H}$ if we have*

$$L = \overline{\mathrm{Span}\{(T - z)^{-1}v; z \in \mathbb{C}\backslash\mathbb{R}\}}.$$

**Remark 3.2.** *1. If $L$ is invariant for $T$, then $L^\perp$ is also invariant for $T$. Indeed, if $v \in L$ and $w \in L^\perp$, we have*

$$\langle v, (T-z)^{-1}w\rangle = \langle (T-\overline{z})^{-1}v, w\rangle = 0,$$

*since $(T-\overline{z})^{-1}v \in L$.*

2. *If $L$ is invariant for $T$, and $v \in \mathcal{D}(T)\cap L$, then $Tv \in L$. Indeed, let $z \in \mathbb{C}\backslash\mathbb{R}$. There exists $w \in \mathcal{H}$ such that $v = (T-z)^{-1}w$. We may write $w = w_0 + w_1$, with $w_0 \in L$, and $w_1 \in L^\perp$. We therefore have $v = (T-z)^{-1}w_0 + (T-z)^{-1}w_1$, and the first term belongs to $L$ while the second belongs to $L^\perp$. Therefore, we must have $w \in L$. Now, we have*

$$Tv = zv + (T-z)v = zv + w \in L.$$

3. *If $L$ is a cyclic subspace for $T$ with cyclic vector $v$, then $L$ is the smallest $T$-invariant subspace containing $T$. Note that the fact that $L$ is a vector space follows from (2.3).*

4. *If $L$ is an invariant subspace for $T$, then the restriction of $T$ to $L$ defines a self-adjoint operator. (The proof is left as an easy exercise).*

**Lemma 3.4.** *Let $T$ be a self-adjoint operator on $\mathcal{H}$, and suppose that $\mathcal{H}$ is cyclic for $T$ with cyclic vector $v$. Then there exists a unitary isomorphim $U : \mathcal{H} \to L^2(\sigma(T), \mathrm{d}\mu_v)$ with the following properties*

- *Let $h : \sigma(T) \longrightarrow \mathbb{R}$ be given by $h(s) = s$. A vector $w \in \mathcal{H}$ belongs to $\mathcal{D}(T)$ if and only if $hUw$ belongs to $L^2(\sigma(T), \mathrm{d}\mu_v)$.*
- *For any $\psi \in U\mathcal{D}(T)$, we have $UTU^{-1}\psi = h\psi$.*

*Proof.* Consider the map $\Theta : C_0(\mathbb{R}) \longrightarrow L^2(\mathbb{R}, \mathrm{d}\mu_v)$ given by $\Theta(f) = f$. We have

$$\begin{aligned}\langle \Theta f, \Theta g\rangle &= \int_{\sigma(T)} \overline{f} g \mathrm{d}\mu_v \\ &= \langle v, f(T)^* g(T) v\rangle = \langle f(T)v, g(T)v\rangle.\end{aligned}$$

Let us write $\mathcal{M} := \{f(T)v; f \in C_0(\mathbb{R})\} \subset \mathcal{H}$. We have just shown that the map

$$U : \mathcal{H} \supset \mathcal{M} \longrightarrow C_0(\mathbb{R}) \subset L^2(\mathbb{R}, \mathrm{d}\mu_v) \qquad U(f(T)v) = f$$

is one-to-one and isometric.

Now, since $\mathcal{H}$ is a cyclic subspace, $\mathcal{M}$ is dense in $\mathcal{H}$. $C_0(\mathbb{R})$ is dense in $L^2(\mathbb{R}, \mathrm{d}\mu_v)$. Therefore, $U$ can be extended in a unique way to a one-to-one isometric map from $\mathcal{H}$ to $L^2(\mathbb{R}, \mathrm{d}\mu_v)$.

Let $f, f_1, f_2 \in C_0(\mathbb{R})$, and write $w_1 = f_1(T)v$, $w_2 = f_2(T)v$. We have

$$\begin{aligned}\langle w_1, f(T)w_2\rangle &= \langle f_1(T)v, f(T)f_2(T)v\rangle \\ &= \langle v, \big(\overline{f_1} f f_2\big)(T)v\rangle \\ &= \int_{\mathbb{R}} f\overline{f}_1 f_2 \mathrm{d}\mu_v \\ &= \langle Uw_1, M_f U w_2\rangle,\end{aligned}$$

where $M_f$ denotes the multiplication by $f$ in $L^2(\mathbb{R}, \mathrm{d}\mu_v)$. We therefore have, for any $f \in C_0(\mathbb{R})$ and $\psi \in L^2(\mathbb{R}, \mathrm{d}\mu)$,

$$f\psi = Uf(T)U^*\psi.$$

In particular, if $\zeta \in \mathbb{C}\backslash\mathbb{R}$, consider the map $r_\zeta(z) = (z-\zeta)^{-1}$. We have

$$r_\zeta\psi = U(T-\zeta)^{-1}U^*\psi.$$

Therefore, $U$ must be a bijection from $\mathrm{Ran}\big((T-\zeta)^{-1}\big) = \mathcal{D}(T)$ to $\mathrm{Ran}(M_{r_\zeta})$. But we have

$$\mathrm{Ran}(M_{r_\zeta}) = \{\psi \in L^2(\mathbb{R}, \mathrm{d}\mu_v); h\psi \in L^2(\mathbb{R}, \mathrm{d}\mu_v)\} = \mathcal{D}(M_h).$$

The first point follows.

Now, let $\psi \in U(\mathcal{D}(T))$. By what precedes, there exists $\varphi$ such that $\psi = r_\zeta \varphi$. We have

$$Tr_\zeta(T)U^*\varphi = (T-\zeta)r_\zeta(T)U^*\varphi + \zeta r_\zeta(T)U^*\varphi = U^*\varphi + \zeta r_\zeta(T)U^*\varphi,$$

so that

$$\begin{aligned} UTU^*\psi = UTU^* r_\zeta \varphi &= UTr_\zeta(T)U^*\varphi \\ &= UU^*\varphi + \zeta U r_\zeta(T)U^*\varphi \\ &= (1 + \frac{\zeta}{x-\zeta})\varphi \qquad = x r_\zeta \varphi = h\psi \end{aligned}$$

□

**Theorem 3.1.** *Let $T$ be a self-adjoint operator on $\mathcal{H}$. There exists a set $N \subset \mathbb{N}$, a finite measure $\mu$ on $N \times \sigma(T)$, and a unitary isomorphism $U : \mathcal{H} \to L^2(N \times \sigma(T), \mathrm{d}\mu)$ with the following properties*

- *Let $h : N \times \sigma(T) \to \mathbb{R}$ be given by $h(n,s) = s$. A vector $v \in \mathcal{H}$ belongs to $\mathcal{D}(T)$ if and only if $hUv$ belongs to $L^2(N \times \sigma(T), \mathrm{d}\mu)$.*
- *For any $\psi \in U\mathcal{D}(T)$, we have $UTU^{-1}\psi = h\psi$.*

This theorem says that, any self-adjoint operator is unitarily equivalent to a multiplication operator in some $L^2$ space.

*Proof.* Using Remark 3.2, we may find a family of subspaces $\mathcal{H}_n \subset \mathcal{H}$ indexed by $N = \mathbb{N}$ or $N$ finite, such that the $H_n$ are mutually orthogonal, with

$$\mathcal{H} = \oplus_{n \in N} \mathcal{H}_n,$$

and each $\mathcal{H}_n$ is cyclic with a cyclic vector $v_n$ such that $\|v_n\| = 2^{-n}$.

Indeed, we pick a first vector $v_1 \in \mathcal{H}$, with $\|v_1\| = 1$, and build a cyclic vector space out of it. We then take a vector $v_2 \in \mathcal{H}_1^\perp$, with $\|v_2\| = 1/2$, and build a cyclic vector space out of it. Since the space $\mathcal{H}$ is separable, this procedures ends after a finite or countable number of steps.

We then consider the restriction $T_n$ of $T$ to $\mathcal{H}_n$. It is a self-adjoint operator, and we may apply Lemma 3.4 to it. We obtain a measure $\mu_n$, with $\mu_n(\mathbb{R}) = 4^{-n}$. We then simply define $\mu$ by $\mu(n \times \Omega) = \mu_n(\Omega)$, which defines a finite measure. Similarly, $U$ is defined acting component by component, and has the required properties. □

**Remark 3.3.** *This construction is not at all intrinsic.*

**Corollary 3.1.** *Let $v \in \mathcal{H}$. We have $v \in \mathcal{D}(T)$ if and only if $x \mapsto x$ is in $L^2(\mathbb{R}, \mathrm{d}\mu_v)$.*

**Definition 3.3.** *Let $T$ be a self-adjoint operator on $\mathcal{H}$, and let $N, \mu, U$ be as in Theorem 3.1. If $f \in C(\sigma(T))$, let $\tilde{f} : N \times \sigma(T) \longrightarrow \mathbb{R}$ be given by $\tilde{f}(n,s) = f(s)$. Note that $\tilde{f} \in L^\infty_{loc}(N \times \sigma(T))$. We then define the operator $f(T)$ on $\mathcal{H}$ by $\mathcal{D}(f(T)) = U^*\mathcal{D}(M_{\tilde{f}})$, and $f(T) = U^* M_{\tilde{f}} U$.*

We leave it as an exercise to show that this definition does not depend on the construction made in Theorem 3.1, and is coherent with the definition of $f(T)$ presented before when $f \in C_0(\sigma(T))$.

## 3.2 Useful formulas for the spectral projectors (♣)

**Lemma 3.5.** *Let $f_n, f$ be Borelian functions, such that $\|f_n\|_{L^\infty} \le c$ for some $c$ independent of $n$, and such that $f_n(x) \longrightarrow f(x)$ for all $x \in \mathbb{R}$.*
*Let $T$ be a self adjoint-operator on $\mathcal{H}$, and let $v \in \mathcal{H}$. Then*

$$f_n(T)v \longrightarrow f(T)v.$$

*Proof.* We have

$$\|f_n(T)v\|^2 = \langle v, |f_n(T)|^2 v\rangle = \int_{\mathbb{R}} |f_n(\lambda)|^2 \mathrm{d}\mu_v(\lambda).$$

By the dominated convergence theorem, this converges to $\int_{\mathbb{R}} |f(\lambda)|^2 \mathrm{d}\mu_v(\lambda) = \|f(T)v\|^2$.

Let $w \in \mathcal{H}$. By the dominated convergence theorem, we have

$$\langle w, f_n(T)v\rangle = \int_{\mathbb{R}} f_n(T)\mathrm{d}\mu_{v,w}(T) \longrightarrow \int_{\mathbb{R}} f(T)\mathrm{d}\mu_{v,w}(T) = \langle w, f(T)v\rangle.$$

Therefore, we have

$$\|f_n(T)v - f(T)v\|^2 = \langle f(T)v - f_n(T), f(T)v\rangle + \langle f_n(T)v, f_n(T)v\rangle - \langle f_n(T)v, f(T)v\rangle \longrightarrow 0,$$

which proves the result. □

**Proposition 3.3.** *Let $T$ be a self-adjoint operator on $\mathcal{H}$, and $\lambda \in \mathbb{R}$. For any $v \in \mathcal{H}$, we have*

$$\chi_{\{\lambda\}}(T)v = -i \lim_{\varepsilon\to 0} \varepsilon(T - \lambda - i\varepsilon)^{-1}v$$

*Proof.* Consider the function $f_\varepsilon(x) := -\frac{i\varepsilon}{x-\lambda-i\varepsilon}$. It satisfies $\|f_\varepsilon\|_{L^\infty} \leq 1$, $f_\varepsilon(\lambda) = 1$, and $f_\varepsilon(x) \longrightarrow 0$ if $x \neq \lambda$. Therefore, we may apply the previous lemma to conclude. □

**Proposition 3.4** (Stone's formula)**.** *Let $T$ be a self-adjoint operator on $\mathcal{H}$, and $a < b \in \mathbb{R}$. For any $v \in \mathcal{H}$, we have*

$$\frac{1}{2}\big(\chi_{(a,b)} + \chi_{[a,b]}\big)v = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_a^b \Im R(\lambda + i\varepsilon)v\mathrm{d}\lambda.$$

*Proof.* For any $\varepsilon > 0$, we set

$$\begin{aligned} f_\varepsilon(x) &:= \frac{1}{\pi}\int_a^b \Im \frac{1}{x - \lambda - i\varepsilon}\mathrm{d}\lambda \\ &= \frac{1}{\pi}\int_a^b \frac{\varepsilon}{(\lambda - x)^2 + \varepsilon^2}\mathrm{d}\lambda \\ &= \frac{1}{\pi}\Big(\arctan\big(\frac{b-x}{\varepsilon}\big) - \arctan\big(\frac{a-x}{\varepsilon}\big)\Big). \end{aligned}$$

Therefore, we have $\|f_\varepsilon\|_{L^\infty} \leq 1$, and $f_\varepsilon(x) \longrightarrow 0$ if $x \notin [a,b]$, $f_\varepsilon(x) \longrightarrow 1$ if $x \in (a,b)$, and $f_\varepsilon(x) \longrightarrow 1/2$ if $x \in \{a; b\}$. The result follows by applying Lemma 3.5. □

**Proposition 3.5.** *Let $T$ be a self-adjoint operator on $\mathcal{H}$, let $\lambda \in \mathbb{C}$ and let $v \in \mathcal{H}$. We have*

$$\mathrm{d}(\lambda, \sigma(T))\|v\| \leq \|(T - \lambda)v\|$$

*Proof.* If $\lambda \in \sigma(T)$, then the statement is trivial. Suppose that $\lambda \notin \sigma(T)$. The statement is then equivalent to

$$\|(T-\lambda)^{-1}\| \leq \frac{1}{\mathrm{d}(\lambda, \sigma(T))}, \tag{3.6}$$

which is a refinement over (2.8).

The map $x \mapsto \frac{1}{x-\lambda}$ belongs to $C_0(\sigma(T))$, and its supremum is given by $\frac{1}{\mathrm{d}(\lambda,\sigma(T))}$. The statement therefore follows from Proposition 3.1. □

**Corollary 3.2** (Weyl's criterion)**.** *Let $T$ be a self-adjoint operator on $\mathcal{H}$, and let $\lambda \in \mathbb{R}$. Then $\lambda \in \sigma(T)$ if and only if there exists a sequence $v_n \subset \mathcal{D}(T)$ with $\|v_n\| = 1$ such that $(T-\lambda)v_n \longrightarrow 0$.*

*Proof.* The "if" part follows from the previous proposition.

For the "only if" part, note that $\lambda \in \sigma(T)$ if and only if for any $\varepsilon > 0$, we have $\chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(T) \neq 0$, as can be seen using point 5 in Proposition 3.2.

Since $\chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(T)$ is an orthogonal projection, we may find for each $\varepsilon > 0$, a $v_\varepsilon$ such that $\|v_\varepsilon\| = 1$ and $\chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(T)v_\varepsilon = v_\varepsilon$.

We have $(T-\lambda)v_\varepsilon = (T-\lambda)\chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(T)v_\varepsilon = f_\varepsilon(T)v_\varepsilon$, where

$$f_\varepsilon(x) = \chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(x)(\lambda - x),$$

so that $\|f_\varepsilon\|_{L^\infty} = \varepsilon$. We therefore have $(T-\lambda)v_\varepsilon \longrightarrow 0$. □

## 3.3 The measure-theoretic decomposition of the spectrum (♣)

**Definition 3.4.** *Let $\mu$ be a finite positive Borelian measure on $\mathbb{R}$. We say that $\mu$ is*

- pure point *if it is supported on a countable set, i.e., there exists a countable set $A \subset \mathbb{R}$ such that $\mu(\mathbb{R}\backslash A) = 0$. Such a measure can be written as $\mu = \sum_i \alpha_i \delta_{x_i}$, where $\alpha_i > 0$. The $x_i$ are called the* atoms *of $\mu$.*
- continuous *if it has no atom, i.e., if we have $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$.*
- absolutely continuous *if there exists a function $f \in L^1(\mathbb{R})$ such that $\mathrm{d}\mu = f(x)\mathrm{d}x$. By the Radon-Nikodym theorem, this is equivalent to having $\mu(A) = 0$ for all Borelian sets with zero Lebesgue measure.*
- singular continuous *if $\mu$ is continuous, but is supported on a set of zero Lebesgue measure.*

By the Lebesgue decomposition theorem, any finite positive Borelian measure $\mu$ on $\mathbb{R}$ can be decomposed in a unique way as

$$\mu = \mu_{pp} + \mu_{ac} + \mu_{sc},$$

where $\mu_{pp}$ is pure point, $\mu_{ac}$ is absolutely continuous, and $\mu_{sc}$ is singular continuous.

Let us define the subspaces

$$\begin{aligned}
\mathcal{H}_{pp} &:= \{v \in \mathcal{H}; \mu_v \text{ is pure point}\} \\
\mathcal{H}_{ac} &:= \{v \in \mathcal{H}; \mu_v \text{ is absolutely continuous}\} \\
\mathcal{H}_{sc} &:= \{v \in \mathcal{H}; \mu_v \text{ is singular continuous}\}.
\end{aligned}$$

**Proposition 3.6.** *The spaces $\mathcal{H}_{pp}, \mathcal{H}_{ac}$ and $\mathcal{H}_{sc}$ are closed vector spaces, are mutually orthogonal, and we have*

$$\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sc}.$$

*Furthermore, if $\Pi_{pp}, \Pi_{ac}$ and $\Pi_{sc}$ denote the orthogonal projections on these spaces, we have*

$$\begin{aligned}
\Pi_{pp}\mathcal{D}(T) \subset \mathcal{D}(T), &\quad T\big(\Pi_{pp}\mathcal{D}(T)\big) \subset \mathcal{H}_{pp} \\
\Pi_{ac}\mathcal{D}(T) \subset \mathcal{D}(T), &\quad T\big(\Pi_{ac}\mathcal{D}(T)\big) \subset \mathcal{H}_{ac} \\
\Pi_{sc}\mathcal{D}(T) \subset \mathcal{D}(T), &\quad T\big(\Pi_{sc}\mathcal{D}(T)\big) \subset \mathcal{H}_{sc}.
\end{aligned} \tag{3.7}$$

Thanks to (3.7), we see that $T$ defines self-adjoint operators when restricted to the subspaces $\mathcal{H}_{pp}$, $\mathcal{H}_{ac}$, $\mathcal{H}_{sc}$. We denote them by $T_{pp}, T_{ac}$ and $T_{sc}$ respectively. We define

$$\sigma_{pp}(T) := \sigma(T_{pp}), \quad \sigma_{ac}(T) := \sigma(T_{ac}), \quad \sigma_{sc}(T) := \sigma(T_{sc}).$$

*Proof.* First of all, let us note that, if $v_n \in \mathcal{H}$ is a sequence converging to some $v \in \mathcal{H}$, and if $A \subset \mathbb{R}$ is a Borelian set, we have $\mu_{v_n}(A) = \langle v_n, \chi_A(T)v_n\rangle \longrightarrow \langle v, \chi_A(T)v\rangle = \mu_v(A)$.

We easily deduce from this that $\mathcal{H}_{pp}, \mathcal{H}_{ac}$ and $\mathcal{H}_{sc}$ are closed.

Next, suppose that $v, w$ belong to two different spaces. We have $v = \chi_{supp(\mu_v)}(T)v$, so that

$$\begin{aligned} |\langle v, w\rangle| &= |\langle \chi_{supp(\mu_v)}(T)v, w\rangle| \\ &= |\langle \chi_{supp(\mu_v)}(T)v, \chi_{supp(\mu_v)}(T)w\rangle| \\ &\le |\langle \chi_{supp(\mu_v)}(T)v, \chi_{supp(\mu_v)}(T)v\rangle|^{1/2} |\langle \chi_{supp(\mu_v)}(T)w, \chi_{supp(\mu_v)}(T)w\rangle|^{1/2} \\ &= |\langle v, \chi_{supp(\mu_v)}(T)v\rangle|^{1/2} |\langle w, \chi_{supp(\mu_v)}(T)w\rangle|^{1/2} \\ &= \mu_w(supp(\mu_v)) = 0. \end{aligned}$$

Therefore, the spaces $\mathcal{H}_{pp}, \mathcal{H}_{ac}$ and $\mathcal{H}_{sc}$ are mutually orthogonal.

Let $v \in \mathcal{H}$. We may decompose $\mu_v$ as $\mu_v = \mu_{pp,v} + \mu_{ac,v} + \mu_{sc,v}$. Let us write $\Pi_{pp,v} := \chi_{supp(\mu_{pp,v})}(T)$, $\Pi_{ac,v} := \chi_{supp(\mu_{ac,v})}(T)$, $\Pi_{sc,v} := \chi_{supp(\mu_{sc,v})}(T)$, so that $v = \Pi_{pp,v}v + \Pi_{ac,v}v + \Pi_{sc,v}v$.

Now, if $f \in C_0(\mathbb{R})$, we have

$$\langle \Pi_{pp,v}v, f(T)\Pi_{pp,v}v\rangle = \langle v, (\chi_{supp(\mu_{pp,v})}f)(T)\rangle,$$

so that $\mu_{\Pi_{pp,v}v}$ is purely punctual. The other components are dealt with similarly, and we obtain that $\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sc}$.

Finally, let $v \in \mathcal{D}(T)$. By Corollary 3.1, the measure $\mu_v$ has a finite second moment. This implies that its purely punctual, absolutely continuous and singular continuous parts do also have finite second moments, so that $\Pi_{pp}\mathcal{D}(T) \subset \mathcal{D}(T), \Pi_{ac}\mathcal{D}(T) \subset \mathcal{D}(T), \Pi_{sc}\mathcal{D}(T) \subset \mathcal{D}(T)$.

Finally, if $v \in \mathcal{D}(T) \cap \mathcal{H}_i$ for $i = pp, sc$ or $ac$, then we have

$$\begin{aligned} \langle Tv, f(T)Tv\rangle &= \langle \Pi_{i,v}Tv, f(T)T\Pi_{i,v}v\rangle \\ &= \langle v, \big(x^2 f(x)\chi_{i,v}(x)\big)(T)v\rangle, \end{aligned}$$

so that $Tv \in \mathcal{H}_i$ □

**Lemma 3.6.** *Let $(z_i)_{i\in I}$ denote the set of eigenvalues of $T$. We have*

$$\mathcal{H}_{pp} = \overline{\oplus_{i\in I} \ker(T - z_i)}.$$

*Proof.* If $v \in \ker(T - z_i)$ for some $i \in I$, then $\mu_v = \delta_{z_i}$, so that $v \in \mathcal{H}_{pp}$. Since $\mathcal{H}_{pp}$ is a closed vector space, we have $\overline{\oplus_{i\in I} \ker(T - z_i)} \subset \mathcal{H}_{pp}$.

Conversely, if $v \in \mathcal{H}_{pp}$, we have $\mu_v = \sum_{j\in J} \alpha_j \delta_j$ for some countable set $J$. Since $\mu_v(\mathbb{R}) = \|v\|$, we may find for any $\varepsilon > 0$ a finite subset $J_\varepsilon$ such that $\sum_{j\in J\setminus J_\varepsilon} |\alpha_j| < \varepsilon$. We can write

$$v = \sum_{j\in J_\varepsilon} \chi_{z_j}(T)v + w,$$

where $w = \sum_{j\in J\setminus J_\varepsilon} \chi_{z_j}(T)v$. We have $\mu_w = \sum_{j\in J\setminus J_\varepsilon} \alpha_j \delta_{z_j}$, so $\|w\| = \mu_w(\mathbb{R}) \le \varepsilon$. Therefore, $v$ can be $\varepsilon$-approximated by a finite linear combination of eigenfunctions, which proves the result. □

## 3.4 From spectral measures to long time behaviour (♠)

In this section, we work in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$, and study the family of operators $e^{-itH}$. The basic properties of this family of operators are given in the exercise sheet. The following theorems are called the RAGE theorems, after Ruelle, Amrein, Georgescu and Enss.

**Theorem 3.2.** *Let $H$ be a self-adjoint operator on $\mathcal{H}$, and let $\psi \in \mathcal{H}_{pp}$. We have*

$$\lim_{R\longrightarrow\infty} \sup_{t\ge 0} \int_{B(0,R)} |e^{-itH}\psi|^2(x)\mathrm{d}x = \|\psi\|^2$$

$$\lim_{R\longrightarrow\infty} \sup_{t\ge 0} \int_{\mathbb{R}^d\setminus B(0,R)} |e^{-itH}\psi|^2(x)\mathrm{d}x = 0.$$

*Proof.* First of all, note that $e^{-itH}$ is unitary, so we have

$$\|\psi\|^2 = \int_{B(0,R)} |e^{-itH}\psi|^2(x)\mathrm{d}x + \int_{\mathbb{R}^d\setminus B(0,R)} |e^{-itH}\psi|^2(x)\mathrm{d}x.$$

Therefore, the two equations above are equivalent, and we will only prove the first one.

When $\psi$ is an eigenfunction, we have $H\psi = \lambda\psi$ for some $\lambda \in \mathbb{R}$, so $e^{-itH}\psi = e^{-i\lambda t}\psi$, and the result follows from the fact that $\lim_{R\longrightarrow\infty} \int_{B(0,R)} |\psi|^2 = \|\psi\|^2$.

Suppose now that $\psi$ is a finite linear combination of eigenfunctions:

$$\psi = \sum_{k=1}^{n} \alpha_k \psi_k,$$

where $H\psi_k = \lambda_k\psi_k$. We have

$$\begin{aligned}\|e^{-itH}\psi\|_{L^2(\mathbb{R}^d\setminus B(0,R))} &= \Big\|e^{-itH}\sum_{k=1}^{n}\alpha_k\psi_k\Big\|_{L^2(\mathbb{R}^d\setminus B(0,R))}\\ &\leq \sum_{k=1}^{n} |\alpha_k| \big\|e^{-itH}\psi_k\big\|_{L^2(\mathbb{R}^d\setminus B(0,R))}\\ &= \sum_{k=1}^{n} |\alpha_k| \big\|\psi_k\big\|_{L^2(\mathbb{R}^d\setminus B(0,R))}.\end{aligned}$$

For any $\varepsilon > 0$, we may find $R$ large enough such that this quantity is smaller than $\varepsilon\sum_{k=1}^{n}|\alpha_k|$. This proves the result for finite linear combinations of eigenfunctions.

Now, if $\psi \in \mathcal{H}_{pp}$, by Lemma 3.6, for any $\varepsilon > 0$, we may find $\alpha_1, ..., \alpha_n \in \mathbb{R}$ and eigenfunctions $\psi_1, ..., \psi_n$ such that

$$\psi = \sum_{k=1}^{n} \alpha_k\psi_k + \varphi,$$

where $\|\varphi\| \leq \varepsilon$. We have

$$\begin{aligned}\|e^{-itH}\psi\|_{L^2(\mathbb{R}^d\setminus B(0,R))} &\leq \Big\|e^{-itH}\sum_{k=1}^{n}\psi\Big\|_{L^2(\mathbb{R}^d\setminus B(0,R))} + \|e^{-itH}\varphi\|_{L^2(\mathbb{R}^d\setminus B(0,R))}\\ &\leq \Big\|e^{-itH}\sum_{k=1}^{n}\psi\Big\|_{L^2(\mathbb{R}^d\setminus B(0,R))} + \varepsilon.\end{aligned}$$

Taking the supremum over $t$, the limit $R \longrightarrow \infty$ and then taking $\varepsilon$ to zero finishes the proof. ☐

**Theorem 3.3.** *Let $H$ be a self-adjoint operator on $\mathcal{H}$, and let $\psi \in \mathcal{H}_{ac}$. We have, for any $\varphi \in L^2(\mathbb{R}^d)$*

$$\lim_{t\longrightarrow\infty} \int_{\mathbb{R}^d} \varphi(x)\big(e^{-itH}\psi\big)(x)\mathrm{d}x = 0.$$

*In other words, $e^{-itH}\psi$ converges weakly to zero.*

*Proof.* We have

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(x)\big(e^{-itH}\psi\big)(x)\mathrm{d}x &= \langle\varphi, e^{-itH}\psi\rangle\\ &= \int_{\mathbb{R}} e^{-it\lambda}\mathrm{d}\mu_{\varphi,\psi}(\lambda).\end{aligned}$$

We claim that $\mu_{\varphi,\psi}$ is absolutely continuous. Indeed, let $A$ have zero Lebesgue measure. We have

$$\begin{aligned}|\mu_{\varphi,\psi}(A)| &= |\langle\varphi, \chi_A(H)\psi\rangle|\\ &= |\langle\chi_A(H)\varphi, \chi_A(H)\psi\rangle|\\ &\leq |\langle\chi_A(H)\varphi, \chi_A(H)\varphi\rangle|^{1/2}|\langle\chi_A(H)\psi, \chi_A(H)\psi\rangle|^{1/2}\\ &= \mu_\varphi(A)^{1/2}\mu_\psi(A)^{1/2} = 0.\end{aligned}$$

By the Riemann-Lebesgue lemma, the Fourier transform of an absolutely continuous measure goes to zero at infinity. The result follows. □

# Chapter 4

# The spectrum of Schrödinger operators

## 4.1 Discrete and essential spectrum (♡)

**Definition 4.1.** *Let $T$ be a self-adjoint operator on $\mathcal{H}$. We define its* discrete spectrum *as*

$$\sigma_{disc}(T) = \{\lambda \in \sigma(T) : \exists \varepsilon > 0 \text{ with } \dim \mathrm{Ran}\chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(T) < \infty\}.$$

*The* essential spectrum *of $T$ is then defined as*

$$\sigma_{ess}(T) = \sigma(T) \backslash \sigma_{disc}(T).$$

**Lemma 4.1.** *Let $\lambda \in \mathbb{R}$. $\lambda$ belongs to the discrete spectrum of $T$ if and only if $\lambda$ is an isolated eigenvalue of $T$ of finite multiplicity.*

*Proof.* Let $\lambda \in \sigma_{disc}(T)$. Then there exists $\varepsilon_0 > 0$ such that $\chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(T)$ does not depend on $\varepsilon$ for $0 < \varepsilon < \varepsilon_0$. This operator is not zero, since $\lambda \in \sigma(T)$. Therefore, we have

$$\chi_{\{\lambda\}}(T) = \lim_{\varepsilon \to 0} \chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(T) \neq 0.$$

Using Proposition 3.2, point 6, we obtain that $\lambda$ is an eigenvalue with finite multiplicity. Now, since $\chi_{(\lambda-\varepsilon_0,\lambda)} = 0$ and $\chi_{(\lambda,\lambda+\varepsilon_0)} = 0$, we have $\sigma(T) \cap (\lambda - \varepsilon_0, \lambda + \varepsilon_0) = \{\lambda\}$, so that $\lambda$ is isolated in the spectrum.

Conversely, suppose that $\lambda$ is an isolated eigenvalue of finite multiplicity. There exists $\varepsilon_0 > 0$ such that $\sigma(T) \cap (\lambda - \varepsilon_0, \lambda + \varepsilon_0) = \{\lambda\}$, so that $\chi_{(\lambda-\varepsilon_0,\lambda)}(T) = \chi_{(\lambda,\lambda+\varepsilon_0)}(T) = 0$. Since $\dim \mathrm{Ran}\chi_{\{\lambda\}}(T) = \dim \ker(T - \lambda)$ is finite, we have

$$\dim \mathrm{Ran}\chi_{(\lambda-\varepsilon_0,\lambda+\varepsilon)}(T) = \dim \mathrm{Ran}\chi_{(\lambda-\varepsilon_0,\lambda)}(T) + \dim \mathrm{Ran}\chi_{\{\lambda\}}(T) + \dim \mathrm{Ran}\chi_{(\lambda,\lambda+\varepsilon_0)}(T) < \infty.$$

□

**Example 4.1.** *Let $T$ be a compact operator. Then $\sigma_{ess}(T) = \{0\}$.*

**Proposition 4.1** (Weyl's criterion for essential spectrum). *Let $\lambda \in \mathbb{R}$. Then $\lambda \in \sigma_{ess}(T)$ if and only if there exists a sequence $v_n \subset \mathcal{D}(T)$ such that*

1. $\|v_n\| = 1$
2. *$v_n$ converges weakly to zero*
3. $\|(T-\lambda)v_n)\| \longrightarrow 0$.

*Such a sequence is called a* singular Weyl sequence. *Furthermore, the first two conditions can actually be replaced by 1'. $v_n$ is orthonormal sequence.*

*Proof.* Suppose that $\lambda$ is such that there exists a sequence $v_n$ satisfying the three conditions. By Weyl's criterion, we know that $\lambda \in \sigma(T)$. Suppose for contradiction that $\lambda \in \sigma_{disc}(T)$. Let $\Pi$ be the orthogonal projector on $\ker(T-\lambda)$. It is finite-dimensional, hence compact, so $\Pi v_n \longrightarrow 0$. In particular, there exists $n_0$ such that for all $n \geq n_0$, we have $\|(\mathrm{Id}-\Pi)v_n\| \geq 1/2$.

One the one hand, $\lambda$ being isolated in the spectrum, there exists $c > 0$ such that

$$\|(T-\lambda)(\mathrm{Id}-\Pi)v_n)\| \geq c\|(\mathrm{Id}-\Pi)v_n)\| \geq c/2.$$

On the other hand, we have $\|(T-\lambda)(\mathrm{Id}-\Pi)v_n)\| = \|(\mathrm{Id}-\Pi)(T-\lambda)v_n)\| \longrightarrow 0$, which gives us the desired contradiction.

Conversely, suppose that $\lambda \in \sigma_{ess}(T)$. We then have $\dim \mathrm{Ran}\chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(T) = \infty$ for all $\varepsilon > 0$. Therefore, we may find a sequence $\varepsilon_n$ going to zero such that $\dim \mathrm{Ran}\chi_{I_n \setminus I_{n+1}}(T) = \infty$ for all $n$, where $I_n = (\lambda - \varepsilon_n, \lambda + \varepsilon_n)$.

For each $n$, we choose a $v_n \in \mathrm{Ran}\chi_{I_n \setminus I_{n+1}}$ with $\|v_n\| = 1$. The vectors $v_n$ are then orthogonal to each other, and hence converge weakly to zero. We have

$$\|(T-\lambda)v_n\| = \|(T-\lambda)\chi_{I_n \setminus I_{n+1}} v_n\| \leq \varepsilon_n,$$

which goes to zero. $\square$

**Theorem 4.1** (Stability of the essential spectrum). *Let $A, B$ be self-adjoint operators on $\mathcal{H}$. Suppose that there exists $z \in \rho(A) \cap \rho(B)$ such that $K(z) := (A-z)^{-1} - (B-z)^{-1}$ is a compact operator. Then $\sigma_{ess}(A) = \sigma_{ess}(B)$.*

Note that, using the resolvent identities, if $K(z)$ is a compact operator for some $z \in \rho(A) \cap \rho(B)$, then $K(z)$ is compact for all $z \in \rho(A) \cap \rho(B)$.

*Proof.* Let $\lambda \in \sigma_{ess}(A)$, and let $(v_n)$ be an associated singular Weyl sequence. Let us write $w_n := \frac{1}{\|(B-z)^{-1}v_n\|}(B-z)^{-1}v_n$, and show that $w_n$ is a singular Weyl sequence for $B$. We have

$$\begin{aligned}
\frac{1}{z-\lambda}(B-\lambda)(B-z)^{-1}v_n &= \Big((B-z)^{-1} - \frac{1}{\lambda - z}\Big)v_n \\
&= \Big((A-z)^{-1} - \frac{1}{\lambda - z}\Big)v_n - K(z)v_n \\
&= \frac{1}{z-\lambda}(A-\lambda)(A-z)^{-1}v_n - K(z)v_n \longrightarrow 0,
\end{aligned}$$

since $K$ is compact, and $v_n$ converges weakly to zero. Now, since $K(z)v_n \longrightarrow 0$, we have

$$\|(B-z)^{-1}v_n\| \sim \|(A-z)^{-1}v_n\| \longrightarrow \frac{1}{\lambda - z},$$

by the previous computations. We therefore have $\|(B-\lambda)w_n\| \longrightarrow 0$. Finally, one easily shows that $w_n$ converges weakly to zero. $\square$

**Definition 4.2.** *Let $A$ be a self-adjoint operator, and $B$ be a closable operator. We say that $B$ is $A$-compact if there exists $z \in \rho(A)$ such that $B(A-z)^{-1}$ is compact.*

By the resolvent identities, we see that if $B(A-z)^{-1}$ is compact for some $z \in \rho(A)$, then it is compact for all $z \in \rho(A)$.

**Lemma 4.2.** *Suppose that $B$ is $A$-compact. Then $B$ is relatively bounded with respect to $A$, with relative bound $0$.*

*Proof.* The key of the proof is to show that

$$\lim_{\lambda \to \infty} \|B(A - i\lambda)^{-1}\| = 0. \tag{4.1}$$

Suppose that (4.1) holds. Then, for any $a > 0$, we can find $\lambda > 0$ such that for any $v \in \mathcal{H}$, we have

$$\|B(A - i\lambda)^{-1} v\| \le a\|v\|.$$

Since $(A - i\lambda)^{-1}$ is a bijection from $\mathcal{H}$ to $\mathcal{D}(A)$, we deduce that for any $w \in \mathcal{D}(A)$, we have

$$\|Bw\| \le a\|(A - i\lambda)w\| \le a\|A\|w + a\lambda\|w\|,$$

which shows that $B$ is relatively bounded with respect to $A$, with relative bound $\le a$ for any $a > 0$.

Let us prove (4.1). Suppose for contradiction that (4.1) does not hold. Then there exists $c > 0$ such that, for any $n \in \mathbb{N}$, we may find $v_n \in \mathcal{H}$ such that $\|B(A - in)^{-1} v_n\| \ge c\|v_n\|^2$ for every $n \in \mathbb{N}$. Write $w_n := (A - in)^{-1} v_n$. We have

$$\|v_n\|^2 = \|(A - in)w_n\|^2 = \|Aw_n\|^2 + n^2\|w_n\|^2.$$

Therefore,

$$\|Bw_n\|^2 \ge c\|Aw_n\|^2 + cn^2\|w_n\|^2.$$

We may change the normalization to assume that $\|Bw_n\|^2 = 1$. Therefore, $Aw_n$ is bounded, and $w_n$ converges to zero.

Let $z \in \rho(A)$. The sequence $(A - z)w_n$ is bounded, so we may extract a converging subsequence from $Bw_n = B(A - z)^{-1}(A - z)w_n$. The limit is then some vector $u$ with $\|u\| = 1$. But we have shown that $w_n \longrightarrow 0$. Therefore, since $B$ is closable, we must have $u = B(0_{\mathcal{H}}) = 0_{\mathcal{H}}$, which gives the desired contradiction. □

**Theorem 4.2.** *Let $A$ be a self-adjoint operator on $\mathcal{H}$, and let $B$ be a symmetric operator on $\mathcal{H}$. Suppose that $B$ is $A$-compact. Then the operator $A + B$ with domain $\mathcal{D}(A + B) = \mathcal{D}(A)$ is self-adjoint, and we have*

$$\sigma_{ess}(A + B) = \sigma_{ess}(A).$$

*Proof.* The fact that $A + B$ is self-adjoint follows from Theorem 2.2, and from Lemma 4.2.

To show that $\sigma_{ess}(A + B) = \sigma_{ess}(A)$, we use Theorem 4.1, noting that for $z \notin \mathbb{R}$, we have

$$(A - z)^{-1} - (A + B - z)^{-1} = (A + B - z)^{-1} B (A - z)^{-1}.$$

□

### 4.1.1 Essential spectrum of Schrödinger operators (♡)

**Definition 4.3.** *Let $d \le 3$, and let $V : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a measurable function. We say that $V$ belongs to the* Kato class *if for every $\varepsilon > 0$, we can find $V_1 \in L^\infty(\mathbb{R}^d)$ and $V_2 \in L^2(\mathbb{R}^d)$ such that $V = V_1 + V_2$, and $\|V_1\|_\infty \le \varepsilon$.*

**Theorem 4.3.** *Let $T = -\Delta$, with $\mathcal{D}(T) = H^2(\mathbb{R}^d)$. Let $V$ belong to the Kato class. Then $M_V$ is relatively compact with respect to $T$. In particular, $\sigma_{ess}(-\Delta + V) = [0, +\infty)$.*

*Proof.* Let $z \in \mathbb{C}\backslash\mathbb{R}$. For any $f \in L^2(\mathbb{R}^d)$, we have

$$\mathcal{F}\big((T-z)^{-1}f\big)(\xi) = (|\xi|^2 - z)^{-1}\big(\mathcal{F}f\big)(\xi).$$

We therefore have $(T-z)^{-1}f = G_z \star f$, where $G_z \in L^2(\mathbb{R}^d)$ is the function such that $\mathcal{F}G_z(\xi) = (2\pi)^{-d/2}(|\xi|^2 - z)^{-1}$, and where $\star$ denotes the convolution product.

Let $\varepsilon > 0$, $V_1 \in L^\infty(\mathbb{R}^d)$, $V_2 \in L^2(\mathbb{R}^d)$ be such that $\|V_1\|_{L^\infty} < \varepsilon$ and $V = V_1 + V_2$. The operator $V_2(T-z)^{-1}$ is an integral operator, with integral kernel $K(x,y) = V_2(x)G_z(x-y)$. We have

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|K(x,y)|^2\mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|V_2(x)|^2|G_z(x-y)|^2\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^d}|V_2(x)|^2\mathrm{d}x\int_{\mathbb{R}^d}|G_z(y)|^2\mathrm{d}y \\ &= \|V_2\|_{L^2}^2\|G_z\|_{L^2}^2 < \infty.\end{aligned}$$

Therefore, $V_2(T-z)^{-1}$ is a Hilbert-Schmidt operator, hence compact (cf the exercises)

We have $\|V_1(T-z)^{-1}\| \le \varepsilon\|(T-z)^{-1}\|$, which goes to zero with $\varepsilon$. The operator $V(T-z)^{-1}$ is the limit of a sequence of compact operators. It is therefore compact (cf the exercises), which concludes the proof. ☐

**Example 4.2** (Essential spectrum of the Coulomb operator)**.** *Let $d = 2$ or $d = 3$. The potential $V = \frac{q}{|x|}$ can be written as $V = V_{|B(0,R)} + V_{|\mathbb{R}^d\backslash B(0,R)}$. For $R$ large enough, we have $\|V_{|\mathbb{R}^d\backslash B(0,R)}\|_{L^\infty} \le \varepsilon$, while $V_{|B(0,R)} \in L^2(\mathbb{R}^d)$. Therefore, we have $\sigma_{ess}(-\Delta+V) = [0,+\infty)$.*

**Theorem 4.4.** *Let $V \in L^\infty(\mathbb{R}^d)$. Suppose that there exists $\alpha \in \mathbb{R}$ such that the set $\Omega := \{x \in \mathbb{R}^d; V(x) < \alpha\}$ has finite Lebesgue measure. Then the operator $-\Delta + V$ with domain $H^2(\mathbb{R}^d)$ is self-adjoint, and has purely discrete spectrum in $(-\infty,\alpha)$.*

*Proof.* Let us write $U := (V-\alpha)\mathbf{1}_\Omega$, and $W = V - U$. The potential $U$ belongs to $L^2$, since it is bounded with a support of finite measure. In particular, $U$ is in the Kato class, so $U(-\Delta-z)^{-1}$ is compact for any $z \in \mathbb{C}\backslash\mathbb{R}$. Now, for any $z \in \mathbb{C}\backslash\mathbb{R}$, we have

$$U(-\Delta+W-z)^{-1} = U(-\Delta-z)^{-1} + U(-\Delta-z)^{-1}W(-\Delta+W-z)^{-1},$$

so that $U(-\Delta+W-z)^{-1}$ is compact. We deduce that

$$\sigma_{ess}(-\Delta+V) = \sigma_{ess}(-\Delta+W).$$

But , since $W \ge \alpha$, we have $\sigma(-\Delta+W) \subset [\alpha,+\infty)$. Therefore, $\sigma_{ess}(-\Delta+V) \subset [\alpha,+\infty)$. ☐

### 4.1.2 Negative eigenvalues of Schrödinger operators (♣)

**Proposition 4.2.** *Let $V \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be such that*

$$\int_{\mathbb{R}^2} V(x)\mathrm{d}x < 0.$$

*Then the operator $-\Delta + V$ has at least one negative eigenvalue.*

**Remark 4.1.** *One can show that such a general result cannot hold when $d \ge 3$, because of Hardy's inequality.*

*Proof.* The potential $V$ belongs to the Kato class, so $\sigma_{ess}(-\Delta+V) = [0,\infty)$. If we can find $\psi \in \mathcal{D}(-\Delta+V) = H^2(\mathbb{R}^2)$, such that $\langle v, (-\Delta+V)v\rangle < 0$, then this will show the existence of a negative eigenvalue. Indeed, we have $\langle v, (-\Delta+V)v\rangle = \int_{\sigma(-\Delta+V)}\lambda\mathrm{d}\mu_\psi$, so this quantity can be negative only if $\sigma(-\Delta+V) \cap (-\infty,0) \neq \emptyset$.

Let $\varepsilon > 0$. Consider $\psi_\varepsilon(x) = e^{-|x|^\varepsilon/2}$. We have

$$\begin{aligned}\langle \psi_\varepsilon, -\Delta\psi_\varepsilon\rangle &= \int_{\mathbb{R}^2} |\nabla\psi_\varepsilon(x)|^2 \mathrm{d}x \\ &= \frac{\varepsilon^2}{4}\int_{\mathbb{R}^2} |x|^{2\varepsilon-2} e^{-|x|^\varepsilon}\mathrm{d}x \\ &= \pi\frac{\varepsilon^2}{2}\int_0^{+\infty} r^{2\varepsilon-1}e^{-r^\varepsilon}\mathrm{d}r \\ &= \pi\frac{\varepsilon}{2}\int_0^{+\infty} ue^{-u}\mathrm{d}u = c\varepsilon.\end{aligned}$$

On the other hand,

$$\int_{\mathbb{R}^2} |\psi_\varepsilon|^2 V(x)\mathrm{d}x \longrightarrow c' \int_{\mathbb{R}^2} V(x)\mathrm{d}x < 0$$

Therefore, for $\varepsilon > 0$ small enough, $\psi_\varepsilon$ satisfies $\langle\psi_\varepsilon, (-\Delta + V)\psi_\varepsilon\rangle < 0$. □

### Rellich's theorem

We will not prove the following theorem, which says that a Schrödinger operator cannot have *positive* eigenvalues, when the potential decays fast enough at infinity.

**Theorem 4.5** (Rellich). *Let $V \in C_c^\infty(\mathbb{R}^d)$. If $u \in H^2(\mathbb{R}^d)$ satisfies*

$$(-\Delta + V)u = \lambda u$$

*for some $\lambda \geq 0$, then we have $u \equiv 0$.*

Therefore, when $V \in C_c^\infty(\mathbb{R}^d)$, the Schrödinger operator has only finitely many eigenvalues. However, in the next section, we will define its *scattering* resonances, which often play the role of "generalized eigenvalues".

## 4.2 Scattering theory and resonances (♠)

In this section, we will always work in $d = 3$ dimensions, for simplicity.

### 4.2.1 The free resolvent in dimension 3 (♠)

For $\Im\lambda > 0$, consider the meromorphic family of operators

$$R_0(\lambda) := (-\Delta - \lambda^2)^{-1} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3).$$

As an easy application of a the spectral theorem, we have

$$\|R_0(\lambda)\|_{L^2\to L^2} \leq \frac{1}{|\lambda|\Im\lambda}.$$

**Proposition 4.3.** *For any $f \in L^2(\mathbb{R}^3)$, and any $\lambda$ with $\Im\lambda > 0$, we have*

$$\big(R_0(\lambda)f\big)(x) = \int_{\mathbb{R}^3} R_0(x, y; \lambda)f(y)\mathrm{d}y, \tag{4.2}$$

*where*

$$R_0(x, y; \lambda) = \frac{e^{i\lambda|x-y|}}{4\pi|x - y|}.$$

*Proof.* We have $f = (-\Delta - \lambda^2)R_0(\lambda)f$, so, applying the Fourier transform, we obtain

$$\mathcal{F}f(\xi) = (|\xi|^2 - \lambda^2)\mathcal{F}\big(R_0(\lambda)f\big)(\xi),$$

so that

$$\begin{aligned}\big(R_0(\lambda)f\big) &= \mathcal{F}^{-1}\Big(\frac{1}{|\xi|^2-\lambda^2}\mathcal{F}f\Big)\\ &= \frac{1}{(2\pi)^{3/2}}\mathcal{F}^{-1}\Big(\frac{1}{|\xi|^2-\lambda^2}\Big)\star f.\end{aligned}$$

Therefore, (4.2) holds, with $R_0(x,y;\lambda) = R_0(x-y,\lambda)$, where

$$\begin{aligned}R_0(x;\lambda) &= \frac{1}{(2\pi)^{3/2}}\mathcal{F}^{-1}\Big(\frac{1}{|\xi|^2-\lambda^2}\Big)\\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{ix\cdot\xi}}{|\xi|^2-\lambda^2}\mathrm{d}x\\ &= \frac{1}{(2\pi)^3}\int_0^{+\infty}\int_{\mathbb{S}^2}\frac{e^{irx\cdot\omega}}{r^2-\lambda^2}r^2\mathrm{d}r\mathrm{d}\omega.\end{aligned}$$

Now, we have

$$\int_{\mathbb{S}^2} e^{ir\omega\cdot x}\mathrm{d}\omega = \frac{2\pi}{ir|x|}\big(e^{ir|x|} - e^{-ir|x|}\big). \tag{4.3}$$

Indeed, Note that the left-hand side of (4.3) is a function of $r$ and $|x|$, but not of $x/|x|$. We may therefore assume that $x = (0,0,|x|)$ in the canonical basis of $\mathbb{R}^3$. Working in spherical coordinates, we obtain

$$\begin{aligned}\int_{\mathbb{S}^2} e^{ir\omega\cdot x}\mathrm{d}\omega &= \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} e^{ir|x|\cos\varphi}\sin\varphi\mathrm{d}\varphi\mathrm{d}\theta\\ &= 2\pi\Big[\frac{e^{ir|x|\cos\varphi}}{ir|x|}\Big]_{\varphi=0}^{\pi}\\ &= \frac{2\pi}{ir|x|}\big(e^{ir|x|} - e^{-ir|x|}\big).\end{aligned}$$

Using (4.3), we obtain

$$\begin{aligned}R_0(x,\lambda) &= \frac{1}{(2\pi)^2 i|x|}\int_0^{+\infty}\frac{r}{r^2-\lambda^2}\big(e^{ir|x|} - e^{-ir|x|}\big)\mathrm{d}r\\ &= \frac{1}{8i\pi^2|x|}\int_{\mathbb{R}}\frac{r}{r^2-\lambda^2}\big(e^{ir|x|} - e^{-ir|x|}\big)\mathrm{d}r\\ &= \frac{1}{8i\pi^2|x|}\int_{\mathbb{R}}\frac{r}{r^2-\lambda^2}e^{ir|x|}\mathrm{d}r - \frac{1}{4i\pi}\int_{\mathbb{R}}\frac{r}{r^2-\lambda^2}e^{-ir|x|}\mathrm{d}r\end{aligned}$$

Now, the map $r \mapsto \frac{r}{r^2-\lambda^2}e^{ir|x|}$ is meromorphic, with simple poles at $r = \pm\lambda$. It goes to zero when $r \to \infty$ with $\Im r \geq 0$, so we can use the residue theorem to obtain

$$\begin{aligned}\int_{\mathbb{R}}\frac{r}{r^2-\lambda^2}e^{ir|x|}\mathrm{d}r &= 2i\pi\mathrm{Res}\Big(\frac{r}{r^2-\lambda^2}e^{ir|x|}\Big)\Big|_{r=\lambda}\\ &= 2i\pi\frac{\lambda}{2\lambda}e^{i\lambda|x|} = i\pi e^{i\lambda|x|}\end{aligned}$$

Similarly, we have

$$\int_{\mathbb{R}}\frac{r}{r^2-\lambda^2}e^{-ir|x|}\mathrm{d}r = -2i\pi\frac{(-\lambda)}{-2\lambda}e^{i\lambda|x|} = -i\pi e^{i\lambda|x|},$$

so that

$$R_0(x,\lambda) = \frac{1}{8i\pi^2|x|}2i\pi e^{i\lambda|x|} = \frac{e^{i\lambda|x|}}{4\pi|x|},$$

which proves the result. □

The map $\lambda \mapsto R_0(x, y; \lambda)$ is holomorphic in all $\mathbb{C}$, smooth in $x$ and $y$, but it decays when $|x-y| \longrightarrow \infty$ only when $\Im\lambda > 0$. Therefore, for any $\rho \in C_c^\infty(\mathbb{R}^3)$, the map

$$\begin{aligned} \rho R_0(\lambda)\rho : L^2(\mathbb{R}^3) &\longrightarrow H^2(\mathbb{R}^3) \\ \big(\rho R_0(\lambda)\rho f\big)(x) &= \int_{\mathbb{R}^3} \rho(x) \frac{e^{i\lambda|x-y|}}{4\pi|x-y|} \rho(y) f(y) \mathrm{d}y \end{aligned}$$

is well defined for all $\lambda \in \mathbb{C}$. In other words, the map $R_0(\lambda) : L^2_{comp}(\mathbb{R}^3) \longrightarrow H^2_{loc}(\mathbb{R}^3)$, initially defined for $\Im\lambda > 0$, can be holomorphically continued to $\mathbb{C}$.

**Remark 4.2.** *Here, we defined $R_0(\lambda)$ for $\Im\lambda > 0$, and we extended it to $\Im\lambda \le 0$. We could also have started by defining $R_0(\lambda)$ for $\Im\lambda < 0$, and then extend it to $\Im\lambda \ge 0$. The two procedures don't give the same result!*

*The first procedure gives what is called the* outgoing *resolvent, sometimes denoted by $R_0(\lambda + i0)$, to recall that it was first defined for $\Im\lambda > 0$, while the second one is called the* incoming *resolvent, and is sometimes denoted by $R_0(\lambda - i0)$.*

### 4.2.2 Scattering resonances (♠)

From now on, we fix $V \in C_c^\infty(\mathbb{R}^3)$.

**Lemma 4.3.** *There exists $C(V) > 0$ such that, for all $\lambda \in \mathbb{C}$ with $\Im\lambda > C(V)$, $(-\Delta + V - \lambda^2)$ is invertible*

*Proof.* First of all, note that for all $\lambda \in \mathbb{C}$ with $\Im\lambda > 0$, we have

$$(-\Delta + V - \lambda^2)R_0(\lambda) = Id + VR_0(\lambda). \tag{4.4}$$

Multiplication by $V$ is a bounded operator, and $R_0$ is small if $\Im\lambda$ is large enough. Therefore, $\|VR_0(\lambda)\|_{L^2\to L^2} < 1$ for $\Im\lambda$ large enough. We may hence invert $Id + VR_0(\lambda)$ by a Neumann series for $\Im\lambda >> 1$. The result follows. □

Equation (4.4) still holds when $\Im\lambda \le 0$, as an identity between operators from $L^2_{comp}(\mathbb{R}^3)$ to $L^2_{loc}(\mathbb{R}^3)$.

**Theorem 4.6.** *Let $V \in C_c^\infty(\mathbb{R}^3)$. The map*

$$\lambda \mapsto (-\Delta + V - \lambda^2)^{-1} : L^2_{comp}(\mathbb{R}^3) \longrightarrow L^2_{loc}(\mathbb{R}^3)$$

*extends as a meromorphic family of operators to $z \in \mathbb{C}$. Its poles are called the* (scattering) resonances *of $-\Delta + V$. If $\lambda \in \mathbb{C}$ is a resonance, then* $\ker(-\Delta + V - \lambda^2)$ *is finite dimensional.*

*Proof.* We start with a few resolvent identities.

**Lemma 4.4.** *Let $\rho \in C_c^\infty(\mathbb{R}^3)$ such that $\rho = 1$ on the support of $V$. The following identities hold for all $\lambda \in \mathbb{C}$, as identities between operators from $L^2_{comp}$ to $L^2_{loc}$:*

$$\begin{aligned} \big(I - VR_0(\lambda)(1-\rho)\big)^{-1} &= Id + VR_0(\lambda)(1-\rho) \\ (-\Delta + V - \lambda^2)R_0(\lambda) &= \big(Id + VR_0(\lambda)(1-\rho)\big)\big(Id + VR_0(\lambda)\rho\big). \end{aligned}$$

*Proof.* For the first formula, we invert $\big(I - VR_0(\lambda)(1-\rho)\big)$ by a Neumann series, noticing that all the terms vanish starting from the third.

The second equality is proven as follows:

$$\begin{aligned} (-\Delta + V - \lambda^2)R_0(\lambda) &= Id + VR_0(\lambda)\rho + VR_0(\lambda)(1-\rho) \\ &= \big(Id + VR_0(\lambda)(1-\rho)\big)\big(Id + VR_0(\lambda)\rho\big) \end{aligned}$$

□

**Lemma 4.5.** *Let $\Omega \subset \mathbb{C}$ be a connected open set, and $(K(z))_{z\in\Omega}$ be a holomorphic family of compact operators. Suppose that there exists $z_0 \in \Omega$ such that $\big(\mathrm{Id} + K(z_0)\big)^{-1}$ exists. Then the family $(\big(\mathrm{Id} + K(z)\big)^{-1})_{z\in\Omega}$ is meromorphic and has poles of finite rank. In other words, $(\big(\mathrm{Id} + K(z)\big)$ is invertible on $\mathbb{C}\backslash S$, where $S$ is a discrete set, and at each $z \in S$,* $\ker\big(\mathrm{Id} + K(z)\big)$ *has finite rank.*

*Sketch of proof.* We first work in a neighbourhood of a point $z_0$ such that $\big(\mathrm{Id} + K(z_0)\big)$ is invertible. We may then use a connectedness argument to show that the conclusions are valid on all of $\mathbb{C}$.

We may find $r > 0$ such that for all $z \in B(z_0, r)$, we have $\|K(z) - K(z_0)\| < 1/4$. Since $K(z_0)$ is compact, we may find a finite rank operator $F$ such that $\|K(z_0) - F\| < 1/4$. Therefore, $\|K(z) - F\| < 1/2$, so that $\mathrm{Id} + K(z) - F$ is invertible, with an analytic inverse. We then write

$$\mathrm{Id} + K(z) = \big(\mathrm{Id} + K(z) - F\big)\Big(\mathrm{Id} + \big(\mathrm{Id} + K(z) - F\big)^{-1}F\Big).$$

Therefore, $\mathrm{Id} + K(z)$ is invertible if and only if $g(z) := \Big(\mathrm{Id} + \big(\mathrm{Id} + K(z) - F\big)^{-1}F\Big)$ is invertible. Using the fact that $F$ has finite rank, we may then show that $g(z)$ is invertible if and only if the determinant of some finite matrix depending analytically on $z$ does not vanish. This proves the result. $\square$

By the first equation in Lemma 4.4, $Id + VR_0(\lambda)(1-\rho)$ is always invertible. Therefore, by the second identity in Lemma 4.4, $(-\Delta + V - \lambda^2)$ is invertible if and only if $\big(Id + VR_0(\lambda)\rho\big)$ is invertible.

$z \mapsto \big(Id + VR_0(\lambda)\rho\big)$ is a holomorphic family of operators. Furthermore, since $VR_0(\lambda)\rho$ maps $L^2$ into $H^2$, and since $H^2$ embeds compactly in $L^2$, $VR_0(\lambda)\rho : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is compact. We saw in the first question that $\big(Id + VR_0(\lambda)\rho\big)$ is invertible for $\Im\lambda >> 1$. Therefore, we may apply Lemma 4.5 to conclude that $\rho(-\Delta + V - \lambda^2)^{-1}\rho$ extends to $\lambda \in \mathbb{C}$ as a meromorphic family of operators.

Let us check that its poles do not depend on $\rho$. $\big(Id + VR_0(\lambda)\rho\big)$ is not invertible at $\lambda$ if and only if there exists $u \in L^2(\mathbb{R}^3)$ such that $u = -VR_0(\lambda)\rho u$. Since $u$ must be supported on the support of $V$, the precise choice of $\rho$ we make does not matter. $\square$

**Remark 4.3.** *It is not hard to see that the scattering resonances of $-\Delta + V$ with positive imaginary part correspond precisely to the square roots of the negative eigenvalues of $-\Delta + V$.*

*Furthermore, Rellich's theorem can be improved to show that $-\Delta + V$ has no resonances on the real axis.*

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