The holomorphy conjecture for nondegenerate surface singularities

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Abstract.— *The holomorphy conjecture roughly states that Igusa's zeta function associated to a hypersurface and a character is holomorphic on $\mathbb{C}$ whenever the order of the character does not divide the order of any eigenvalue of the local monodromy of the hypersurface. In this article we prove the holomorphy conjecture for surface singularities that are nondegenerate over $\mathbb{C}$ with respect to their Newton polyhedron. In order to provide relevant eigenvalues of monodromy, we first show a relation between the normalized volumes (which appear in the formula of Varchenko for the zeta function of monodromy) of the faces in a simplex in arbitrary dimension. We then study some specific character sums that show up when dealing with false poles. In contrast with the context of the trivial character, we here need to show fakeness of certain candidate poles other than those contributed by B_1-facets.*

1 Introduction

Let K be a finite extension of the field of p-adic numbers $\mathbb{Q}_p$. Let R be the valuation ring of K, and let P be its maximal ideal. Suppose that the residue field R/P has cardinality q. For $z \in K$, let $\operatorname{ord}(z) \in \mathbb{Z} \cup \{\infty\}$ denote its valuation, $|z| = q^{-\operatorname{ord}(z)}$ its absolute value, and let $\operatorname{ac}(z) = z\pi^{-\operatorname{ord}(z)}$ be its angular component, where π is a fixed uniformizing parameter for R.

Let $f(\underline{x})$, $\underline{x} := (x_1, \ldots, x_n)$, be a non-constant polynomial over K, and let $\chi : R^\times \to \mathbb{C}^\times$ be a multiplicative character of $R^\times$,"i.e.," a homomorphism with finite image. We formally put $\chi(0) = 0$. Let $Z_{f,0}(\chi, K, s)$, "resp." $Z_f(\chi, K, s)$, be the corresponding *local Igusa zeta function*, "resp." *global Igusa zeta function*, "i.e.," the meromorphic continuation to $\mathbb{C}$ of the integral function

$$Z_0(s) = \int_{P^n} \chi\left(\operatorname{ac}(f(\underline{x}))\right)|f(\underline{x})|^s|d(\underline{x})|, \quad \text{"resp." } Z(s) = \int_{R^n} \chi\left(\operatorname{ac}(f(\underline{x}))\right)|f(\underline{x})|^s|d(\underline{x})|,$$

for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$, where $|d(\underline{x})| = |dx_1 \wedge \ldots \wedge dx_n|$ denotes the Haar measure on K^n normalized such that the measure of R^n is 1.

For f a polynomial over R, the local and global Igusa zeta function can be described in terms of solutions of congruences. For $i \in \mathbb{N}_{>0}$ and $u \in R/P^i$, let $M_{0,i}(u)$ and $M_i(u)$ be the number of solutions of $f(\underline{x}) \equiv u \bmod P^i$ in $(P/P^i)^n$ and $(R/P^i)^n$, respectively.

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Let c be the conductor of χ, "i.e.," the smallest $a \in \mathbb{N}_{>0}$ such that χ is trivial on $1+P^a$. Then

$$Z_0(s) = \sum_{i=0}^{\infty} \sum_{u \in (R/P^c)^\times} \chi(u) M_{0,i+c}(\pi^i u) q^{-n(i+c)} q^{-is}, \qquad \text{and}$$

$$Z(s) = \sum_{i=0}^{\infty} \sum_{u \in (R/P^c)^\times} \chi(u) M_{i+c}(\pi^i u) q^{-n(i+c)} q^{-is}.$$

Igusa showed that these functions are rational functions in q^{-s} and he gave a formula for $Z_{f,0}(\chi, K, s)$ and $Z_f(\chi, K, s)$ in terms of an embedded resolution (Y, h) of $f^{-1}\{0\}$ over K (see [I]). Let E_j, $j \in T$, be the (reduced) irreducible components of $h^{-1}(f^{-1}\{0\})$, and let N_j, "resp." $\nu_j - 1$, be the multiplicity of E_j in the divisor of $f \circ h$, "resp." $h^*(dx_1 \wedge \ldots \wedge dx_n)$ on Y. Then the poles of $Z_{f,0}(\chi, K, s)$ and $Z_f(\chi, K, s)$ are among the values

$$s = \frac{-\nu_j}{N_j} + \frac{2k\pi i}{N_j \log(q)}, \quad k \in \mathbb{Z}, j \in T, \tag{1}$$

for which the order of χ divides N_j.

Let now $f \in F[\underline{x}]$, with $F \subset \mathbb{C}$ a number field, and let K be a non-archimedean completion of F, "i.e.," a completion with respect to a finite prime. Let R be its valuation ring and let $\chi : R^\times \to \mathbb{C}^\times$ be a multiplicative character. Then the poles of $Z_{f,0}(\chi, K, s)$ and $Z_f(\chi, K, s)$ seem to be related to various invariants in singularity theory, such as the eigenvalues of monodromy and the roots of the Bernstein-Sato polynomial (see for example [D2]) and such as the jumping numbers (see for example [ST]). In this article we explore another connection conjectured by Denef, called the holomorphy conjecture. It follows from (1) that when the order of χ divides no N_j at all, then the zeta functions $Z_{f,0}(\chi, K, s)$ and $Z_f(\chi, K, s)$ are holomorphic on $\mathbb{C}$. Now, the N_j are not intrinsically associated to $f^{-1}\{0\}$; however the order (as root of unity) of any eigenvalue of the local monodromy on $f^{-1}\{0\}$ divides some N_j, and those eigenvalues are intrinsic invariants of $f^{-1}\{0\}$. This observation inspired Denef to propose the following ([D2, Conjecture 4.4.2]):

Conjecture 1 (**Holomorphy conjecture**). *For almost all non-archimedean completions K of F ("i.e.," for all except a finite number) and all characters χ, the local ("resp." global) Igusa zeta function $Z_{f,0}(\chi, K, s)$ ("resp." $Z_f(\chi, K, s)$) is holomorphic, unless the order of χ divides the order of some eigenvalue of the local monodromy of f at some complex point of $f^{-1}\{0\}$.*

This conjecture has been proven by Veys in [Ve, Theorem 3.1] for plane curves, and in [DV] Denef and Veys obtained a Thom-Sebastiani type result. In [RV] Rodrigues and Veys make several progresses on the holomorphy conjecture for homogeneous polynomials. Veys and Lemahieu confirmed the conjecture for surfaces that are general for a toric idealistic cluster (see [LV, Theorem 24]). In [LVP1] the holomorphy conjecture has been introduced for ideals and was proven for ideals in dimension two.

In this article we prove the holomorphy conjecture for surface singularities that are nondegenerate over $\mathbb{C}$ with respect to their Newton polyhedron at the origin. In Section 2 we recall this notion, along with explicit formulas for the zeta functions in this context. By a formula of Varchenko the normalized volume of a face gets a key role in the search for eigenvalues of monodromy for nondegenerate singularities. In Section 3 we prove some properties on the normalized volume of faces in a simplex of arbitrary dimension. These properties might be of independent interest. We can use them in Subsection 5.1 to obtain a set of eigenvalues that is relevant for the holomorphy conjecture. Furthermore, we prove that some candidate poles of $Z_{f,0}(\chi, K, s)$ "resp." $Z_f(\chi, K, s)$ are no actual poles. This mainly concerns candidate poles contributed by so-called B_1-facets, which were formally introduced in [LVP2] in the context of the topological zeta function. In our context of the Igusa zeta function associated to a non-trivial character, some configurations of B_1-facets that give rise to false poles have been treated in [BV, §9]. It actually turns out that almost all configurations of B_1-facets give rise to fake poles (see Subsection 5.2.3 for the exact statement). We also find a configuration without B_1-facets where we need to show that the candidate pole is a false pole. Our computations rely on the study of some specific character sums (see Section 4). We can then complete our proof using a nondegeneracy argument (see Lemma 2) which was used for the first time in [LVP2].

As a preliminary remark, we note that for the purpose of proving the holomorphy conjecture one can assume that

- f has coefficients in the ring of integers $\mathcal{O}_F$ of F; indeed, multiplying f by a constant $a \in F$ affects $Z_{f,0}(\chi, K, s)$ and $Z_f(\chi, K, s)$ only for the completions K in which $\operatorname{ord}(a) \neq 0$, of which there are finitely many,
- χ is a non-trivial character with conductor equal to 1; indeed, Denef proved that for almost all non-archimedean completions K of F, if $\chi : R^\times \to \mathbb{C}^\times$ is a multiplicative character which is non-trivial on $1 + P$, then $Z_{f,0}(\chi, K, s)$ and $Z_f(\chi, K, s)$ are constant on $\mathbb{C}$ (see [D2, Theorem 3.3]).

From now on we just write $Z_{f,0}(\chi, s)$ ("resp." $Z_f(\chi, s)$) for $Z_{f,0}(\chi, K, s)$ ("resp." $Z_f(\chi, K, s)$).

2 Nondegenerate singularities and their zeta functions

2.1 Nondegenerate singularities

Assume that $f(\underline{x}) \in \mathcal{O}_F[\underline{x}]$ is a non-constant polynomial satisfying $f(\underline{0}) = 0$. Write

$$f(\underline{x}) = \sum_{\underline{k} \in \mathbb{Z}^n_{\geq 0}} c_{\underline{k}} \underline{x}^{\underline{k}},$$

where $\underline{k} = (k_1, \ldots, k_n)$ and $\underline{x}^{\underline{k}} = x_1^{k_1} \cdot \ldots \cdot x_n^{k_n}$. The *support of* f is $\operatorname{supp} f = \{\underline{k} \in \mathbb{Z}^n_{\geq 0} \mid c_{\underline{k}} \neq 0\}$. The *Newton polyhedron* Γ_0 *of* f *at the origin* is the convex hull in $\mathbb{R}^n_{\geq 0}$ of

$$\bigcup_{\underline{k} \in \operatorname{supp} f} \underline{k} + \mathbb{R}^n_{\geq 0}.$$

A *facet* of the Newton polyhedron is a face of dimension $n - 1$. For a face τ of Γ_0, one defines the polynomial $f_\tau(\underline{x}) := \sum_{\underline{k} \in \mathbb{Z}^n \cap \tau} c_{\underline{k}} \underline{x}^{\underline{k}}$.

We say that the polynomial f is *nondegenerate over $\mathbb{C}$ with respect to the compact faces of* Γ_0 ("resp." *nondegenerate over $\mathbb{C}$ with respect to the faces of* Γ_0), if for every compact face τ ("resp." for every face τ) of Γ_0, the zero locus of f_τ has no singularities in $(\mathbb{C}^\times)^n$. For a fixed Newton polyhedron Γ, almost all polynomials having Γ as their Newton polyhedron are nondegenerate with respect to the faces of Γ (see [AVG, p.157]).

Let K be a non-archimedean completion of F with valuation ring R and maximal ideal P, whose residue field we denote by $\mathbb{F}_q$. Note that $\mathcal{O}_F \subset R$, so it makes sense to consider $\bar{f}$, the polynomial over $\mathbb{F}_q$ obtained from f by reducing each of its coefficients modulo P. We say that $\bar{f}$ is *nondegenerate over $\mathbb{F}_q$ with respect to the compact faces of* Γ_0 ("resp." *nondegenerate over $\mathbb{F}_q$ with respect to the faces of* Γ_0) if for every compact face τ ("resp." for every face τ) of Γ_0, the zero locus of $\bar{f}_\tau$ has no singularities in $(\mathbb{F}_q^\times)^n$. If f is nondegenerate over $\mathbb{C}$ with respect to the compact faces ("resp." the faces) of its Newton polyhedron Γ_0, then recall that $\bar{f}$ is nondegenerate over $\mathbb{F}_q$ with respect to the compact faces ("resp." the faces) of Γ_0 for almost all choices of K. Thus in order to prove the holomorphy conjecture for polynomials that are nondegenerate over $\mathbb{C}$, it suffices to restrict to completions K for which moreover $\bar{f}$ is nondegenerate over the residue field $\mathbb{F}_q$.

Further down we use the following property of nondegeneracy ([LVP2, Lemma 9]):

Lemma 2. *If a complex polynomial $f(x,y,z)$ is nondegenerate with respect to the compact faces of its Newton polyhedron at the origin, then for almost all $k \in \mathbb{C}$ the polynomial $f(x,y,z-k)$ is nondegenerate with respect to the compact faces of its Newton polyhedron at the origin. (Analogously for the variables x and y.)*

2.2 Some combinatorial data associated to the Newton polyhedron

Let Γ_0 be as above. For $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n_{\geq 0}$ we put

$$N(\underline{a}) := \inf_{\underline{x}\in\Gamma_0} \underline{a}\cdot\underline{x}, \qquad \nu(\underline{a}) := \sum_{i=1}^{n} a_i, \qquad F(\underline{a}) := \{\underline{x}\in\Gamma_0 \mid \underline{a}\cdot\underline{x} = N(\underline{a})\}.$$

All $F(\underline{a}), \underline{a} \neq \underline{0}$, are faces of Γ_0. To a face τ of Γ_0 we associate its dual cone $\Delta_\tau = \{\underline{a} \in \mathbb{R}^n_{\geq 0} \mid F(\underline{a}) = \tau\}$. It is a rational polyhedral cone of dimension $n - \dim \tau$. In particular if τ is a facet then Δ_τ is a ray, say $\Delta_\tau = \underline{a}\mathbb{R}_{>0}$ for some non-zero $\underline{a} \in \mathbb{Z}^n_{\geq 0}$, and then the equation of the hyperplane through τ is $\underline{a}\cdot\underline{x} = N(\underline{a})$. If we demand that $\underline{a}$ is primitive, "i.e.," that $\gcd(a_1, \ldots, a_n) = 1$, then this $\underline{a}$ is uniquely defined. For a facet τ we also use the notation $N(\tau)$ - called the lattice distance of τ - and $\nu(\tau)$, meaning respectively $N(\underline{a})$ and $\nu(\underline{a})$ for this associated $\underline{a} \in \mathbb{Z}^n_{\geq 0}$. For a general proper face τ the dual cone Δ_τ is strictly positively spanned by the dual cones of the facets containing τ.

For a set of linearly independent vectors $\underline{a_1}, \ldots, \underline{a_r} \in \mathbb{Z}^n$ we define the *multiplicity* $\operatorname{mult}(\underline{a_1}, \ldots, \underline{a_r})$ as the index of the lattice $\mathbb{Z}\underline{a_1} + \ldots + \mathbb{Z}\underline{a_r}$ in the group of the points with integral coordinates of the subspace of $\mathbb{R}^n$ generated by $\underline{a_1}, \ldots, \underline{a_r}$. Alternatively, $\operatorname{mult}(\underline{a_1}, \ldots, \underline{a_r})$ is equal to the greatest common divisor of the determinants of the $(r \times r)$-matrices obtained by omitting columns from the matrix with rows $\underline{a_1}, \ldots, \underline{a_r}$. If Δ_τ is a simplicial cone then by $\operatorname{mult}(\Delta_\tau)$ we mean the multiplicity of its set of primitive generators. For a simplicial face τ we write $\operatorname{mult}(\tau)$ for the multiplicity of its set of vertices.

2.3 The Igusa zeta function with character for nondegenerate singularities

In the case where $f \in R[\underline{x}]$ is nondegenerate over $\mathbb{F}_q$ with respect to the compact faces ("resp." the faces) of its Newton polyhedron at the origin Γ_0, Hoornaert gave a formula ([H, Theorem 3.4]) for the local ("resp." global) Igusa zeta function associated to f and χ in terms of Γ_0, which we recall. Hoornaert states the formula for $R = \mathbb{Z}_p$ only, but her proof generalizes word by word to our more general setting.

Recall that we assume $\chi : R^\times \to \mathbb{C}^\times$ to be non-trivial of conductor 1. Let $pr : R^\times \to \mathbb{F}_q^\times \cong R^\times/(1+P)$ be the natural surjective homomorphism. As χ is trivial on $1+P$, there exists a unique homomorphism $\bar{\chi} : \mathbb{F}_q^\times \to \mathbb{C}^\times$ such that $\chi = \bar{\chi} \circ pr$. One formally puts $\bar{\chi}(0) = 0$. Note that the order of χ divides the order of $\bar{\chi}$. Let f be a non-zero polynomial over R satisfying $f(\underline{0}) = 0$ and let $\bar{f}$ be nondegenerate over $\mathbb{F}_q$ with respect to all the compact faces ("resp." all the faces) of its Newton polyhedron Γ_0. Let

$$L_\tau := q^{-n} \sum_{\underline{x} \in (\mathbb{F}_q^\times)^n} \bar{\chi}(\bar{f}_\tau(\underline{x})) \qquad \text{and} \qquad S(\Delta_\tau)(s) := \sum_{\underline{a} \in \mathbb{Z}^n \cap \Delta_\tau} q^{-\nu(\underline{a}) - N(\underline{a})s}.$$

Then Hoornaert proved that the local, "resp." global, Igusa zeta function associated to f and the non-trivial character χ can be computed as

$$Z_{f,0}(\chi, s) = \sum_{\substack{\tau \text{ compact} \\ \text{face of } \Gamma_0}} L_\tau S(\Delta_\tau)(s), \qquad \text{"resp."} \qquad Z_f(\chi, s) = \sum_{\substack{\tau \text{ face} \\ \text{of } \Gamma_0}} L_\tau S(\Delta_\tau)(s).$$

In the last summation also the face $\tau = \Gamma_0$ is included and $S(\Delta_{\Gamma_0}) = 1$.

If Δ_τ is simplicial, say (strictly positively) spanned by primitive linearly independent vectors $\underline{a}_1, \ldots, \underline{a}_r \in \mathbb{Z}^n_{\geq 0}$, then

$$S(\Delta_\tau)(s) = \frac{\sum_{\underline{h}} q^{\nu(\underline{h}) + N(\underline{h})s}}{\prod_i (q^{\nu(\underline{a}_i) + N(\underline{a}_i)s} - 1)}$$

where the sum runs over $\mathbb{Z}^n \cap \{\lambda_1 \underline{a}_1 + \cdots + \lambda_r \underline{a}_r \mid 0 \leq \lambda_i < 1\}$. In particular if $\mathrm{mult}(\Delta_\tau) = 1$ then the numerator is 1. In the non-simplicial case $S(\Delta_\tau)(s)$ is a sum of such expressions (obtained by subdividing Δ_τ into simplicial cones).

We clearly see that the real parts of a set of candidate poles (containing all poles) of the local and global Igusa zeta function are given by the rational numbers $-\nu(\underline{a})/N(\underline{a})$ for $\underline{a}$ orthogonal to a facet of the Newton polyhedron at the origin. Moreover we can restrict to the facets τ for which the order of $\bar{\chi}$ divides $N(\underline{a})$, because otherwise $L_\tau = 0$: this follows from Lemma 7 below. A fortiori we can restrict to those for which the order of χ divides $N(\underline{a})$. We say that such a facet *contributes* a candidate pole to $Z_{f,0}(\chi, s)$ "resp." $Z_f(\chi, s)$.

We finally remark that if f is nondegenerate over $\mathbb{C}$ with respect to the compact faces of Γ_0, then the couples $(\nu(\underline{a}), N(\underline{a}))$ are part of the numerical data (ν_j, N_j) associated to a very explicit (namely, toric) embedded resolution of $f^{-1}\{0\}$ over F, that was first described by Varchenko in [Va]. Thus the fact that we can restrict to the case where the order of χ divides $N(\underline{a})$ also follows from Igusa's seminal work.

2.4 The formula of Varchenko for the zeta function of monodromy of f in the origin

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function. Let $\mathcal{F}$ be the Milnor fibre of the Milnor fibration at the origin associated with f and write $h^i_* : H^i(\mathcal{F}, \mathbb{C}) \to H^i(\mathcal{F}, \mathbb{C})$, $i \geq 0$, for the monodromy transformations.

The *zeta function of monodromy* at the origin associated to f is

$$\zeta_{f,0}(t) := \prod_{i\geq 0} (\det(\mathrm{id}^i - t h^i_*; H^i(\mathcal{F}, \mathbb{C})))^{(-1)^{(i+1)}},$$

where id^i is the identical transformation on $H^i(\mathcal{F}, \mathbb{C})$. One calls α an *eigenvalue of monodromy* of f at the origin if α is an eigenvalue for some $h^i_* : H^i(\mathcal{F}, \mathbb{C}) \to H^i(\mathcal{F}, \mathbb{C})$. Denef proved that every eigenvalue of monodromy of f is a zero or a pole of the zeta function of monodromy at some point of $\{f = 0\}$ ([D3]). Varchenko gave in [Va] a formula for $\zeta_{f,0}$ in terms of Γ_0 if f is nondegenerate with respect to the compact faces of its Newton polyhedron at the origin Γ_0. He defines a function $\zeta_\tau(t)$ for every compact face τ of Γ_0 for which there exists a subset $I \subset \{1, \ldots, n\}$ with $\#I = \dim(\tau) + 1$ such that $\tau \subset L_I := \{x \in \mathbb{R}^n \,\big|\, \forall i \notin I : x_i = 0\}$. We call such faces *V-faces* and we denote the index set ("resp." linear space) corresponding to a V-face τ by I_τ ("resp." L_{I_τ}). If a V-face is a simplex, then we call it a *V-simplex*.

For a face τ of dimension 0, we put $\mathrm{Vol}(\tau) = 1$. For every other compact face τ, $\mathrm{Vol}(\tau)$ is defined as the *volume* of τ for the volume form ω_τ. This is a volume form on $\mathrm{Aff}(\tau)$, the affine space spanned by τ, such that the parallelepiped spanned by a lattice-basis of $\mathbb{Z}^n \cap \mathrm{Aff}(\tau)$ has volume 1. The product $(\dim \tau)!\mathrm{Vol}(\tau)$ is also called the *normalized volume* of the face τ and is denoted by $\mathrm{NV}(\tau)$.

For a V-face τ, let $\sum_{i\in I_\tau} a_i x_i = N(\tau)$ be the equation of $\mathrm{Aff}(\tau)$ in L_{I_τ}, where $N(\tau)$ and all a_i (for $i \in I_\tau$) are positive integers and their greatest common divisor is equal to 1. We put

$$\zeta_\tau(t) := \left(1 - t^{N(\tau)}\right)^{\mathrm{NV}(\tau)}.$$

In [Va] Varchenko showed that the zeta function of monodromy of f in the origin is equal to

$$\zeta_{f,0}(t) = \prod \zeta_\tau(t)^{(-1)^{\dim(\tau)}},$$

where the product runs over all V-faces τ of Γ_0.

For a fixed facet τ of Γ_0, we say that a V-face σ in Γ_0 contributes with respect to τ if $e^{-2\pi i \nu(\tau)/N(\tau)}$ is a zero of $\zeta_\sigma(t)$.

If $n = 3$, the formula of Varchenko for the zeta function of monodromy at the origin has a specific form that we describe below. We first partition every compact facet in simplices. For each such a simplex τ, we define the factor F_τ as in [LVP2]:

$$F_\tau := \zeta_\tau \prod_\sigma \zeta_\sigma^{-1} \prod_p \zeta_p, \tag{2}$$

where the first product runs over the 1-dimensional V-faces σ in τ and the second product runs over the 0-dimensional V-faces p of τ that are intersection points of two

1-dimensional V-faces in τ. In [LVP2, Proposition 8] it is shown that F_τ is a polynomial. Following the formula of Varchenko, the zeta function of monodromy in the origin can be written as

$$\zeta_{f,0}(t) = \prod_{\tau} F_\tau \prod_{\sigma} \zeta_\sigma^{-1} \prod_{p} \zeta_p, \tag{3}$$

where the first product runs over all 2-dimensional simplices τ obtained after subdividing the compact facets and the other products run over 1-dimensional V-faces σ and 0-dimensional V-faces p for which ζ_σ, respectively ζ_p, was not used in any F_τ.

3 Preliminary results on the normalized volume

When searching for eigenvalues of monodromy using the formula of Varchenko, one has to compare normalized volumes of compact faces in a facet. This is the motivation for this section. For two faces σ and σ' in a simplicial facet τ, we denote the smallest face containing σ and σ' by $\sigma + \sigma'$.

Lemma 3. *Let σ and σ' be two non-disjoint V-faces in a simplicial facet τ. Then $\sigma \cap \sigma'$ and $\sigma + \sigma'$ are also V-faces.*

Proof. Let σ be a d_1-dimensional V-simplex, and let σ' be a d_2-dimensional V-simplex, having k vertices in common. Suppose that the vertices of $\sigma + \sigma'$ have exactly s zero entries in common. Then one has

$$s \leq n - \#(\sigma + \sigma') = n - (d_1 + 1 + d_2 + 1 - k)$$

where (abusing notation) $\#(\sigma + \sigma')$ denotes the number of vertices of $\sigma + \sigma'$. On the other hand, the vertices of $\sigma \cap \sigma'$ have at most $n - k$ zero entries in common, and so

$$n - k \geq (n - d_1 - 1) + (n - d_2 - 1) - s$$

Combining these two inequalities, one finds that they are actually equalities and so $\sigma \cap \sigma'$ and $\sigma + \sigma'$ are V-simplices. ■

Recall that for a V-simplex τ, the normalized volume $\mathrm{NV}(\tau)$ is equal to its multiplicity $\mathrm{mult}(\tau)$ divided by its lattice distance $N(\tau)$. Let $B^j = (B^j_1, \ldots, B^j_n), 1 \leq j \leq n$, be the vertices of τ, and let σ be a V-face in τ with vertices $B^1, \ldots, B^k$ and $I_\sigma = \{1, \ldots, k\}$. Then $\mathrm{mult}(\tau)$ is the absolute value of the determinant of the matrix

$$\begin{pmatrix} B^1_1 & \ldots & B^1_k & 0 & \ldots & 0 \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ B^k_1 & \ldots & B^k_k & 0 & \ldots & 0 \\ * & \ldots & * & B^{k+1}_{k+1} & \ldots & B^{k+1}_n \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ * & \ldots & * & B^n_{k+1} & \ldots & B^n_n \end{pmatrix}.$$

We denote the matrix

$$M_{\tau,\sigma} := \begin{pmatrix} B^{k+1}_{k+1} & \ldots & B^{k+1}_n \\ \vdots & \ldots & \vdots \\ B^n_{k+1} & \ldots & B^n_n \end{pmatrix}.$$

Then we have that $\mathrm{mult}(\tau) = \mathrm{mult}(\sigma) \left|\det(M_{\tau,\sigma})\right|$.

Proposition 4. *Let τ be a simplicial facet of a Newton polyhedron in $\mathbb{R}^n$. If σ is a V-face in τ, then $\mathrm{NV}(\sigma) \mid \mathrm{NV}(\tau)$.*

Proof. Let us denote the equation of the affine space through τ "resp." through σ by

$$\mathrm{Aff}(\tau) \leftrightarrow a_1x_1 + \ldots + a_nx_n = N(\tau), \qquad \mathrm{Aff}(\sigma) \leftrightarrow \frac{a_1x_1 + \ldots + a_kx_k}{\gcd(a_1,\ldots,a_k)} = N(\sigma),$$

with $\gcd(a_1,\ldots,a_n) = 1$ and $N(\sigma) = N(\tau)/\gcd(a_1,\ldots,a_k)$. Let $B^j = (B^j_1,\ldots,B^j_n), 1 \leq j \leq n$, be the vertices of τ and $B^{k+1},\ldots,B^n$ the vertices of τ that are not contained in σ. Then we find that

$$\mathrm{NV}(\tau) = \frac{\mathrm{NV}(\sigma) \left|\det(M_{\tau,\sigma})\right|}{\gcd(a_1,\ldots,a_k)}.$$

Let $v_j = (B^{k+1}_j,\ldots,B^n_j)^T, k+1 \leq j \leq n$, be the jth column of the matrix $M_{\tau,\sigma}$, and let $\tilde{M_{\tau,\sigma}}$ be the matrix obtained from $M_{\tau,\sigma}$ by replacing the first column by $a_{k+1}v_{k+1}+\ldots+a_nv_n$. For every vertex B^j of τ we have that $\gcd(a_1,\ldots,a_k) \mid a_{k+1}B^j_{k+1} + \ldots + a_nB^j_n$ and hence we find that $\gcd(a_1,\ldots,a_k) \mid \det(\tilde{M_{\tau,\sigma}}) = a_{k+1}\det(M_{\tau,\sigma})$. Analogously, we obtain that $\gcd(a_1,\ldots,a_k) \mid a_j\det(M_{\tau,\sigma})$, for $k+1 \leq j \leq n$. As we supposed that $\gcd(a_1,\ldots,a_n) = 1$, we get that $\gcd(a_1,\ldots,a_k) \mid \det(M_{\tau,\sigma})$, which implies that $\mathrm{NV}(\sigma) \mid \mathrm{NV}(\tau)$. ■

Proposition 5. *Let τ be a simplicial facet of a Newton polyhedron in $\mathbb{R}^n$. If σ and σ' are V-faces in τ such that $\sigma \cap \sigma' \neq \emptyset$, then*

$$\mathrm{NV}(\tau)\,\mathrm{NV}(\sigma\cap\sigma') = \mathrm{NV}(\sigma)\,\mathrm{NV}(\sigma')M, \quad \textit{for some } M \in \mathbb{N}. \tag{4}$$

Moreover, if $\sigma + \sigma' = \tau$, then $M = 1$ if and only if $N(\sigma\cap\sigma') = \gcd(N(\sigma), N(\sigma'))$.

Proof. As $\sigma + \sigma'$ is also a V-face (see Lemma 3), it follows by Proposition 4 that it is sufficient to prove that

$$\mathrm{NV}(\sigma+\sigma')\,\mathrm{NV}(\sigma\cap\sigma') = \mathrm{NV}(\sigma)\,\mathrm{NV}(\sigma')M, \quad \text{for some } M \in \mathbb{N}.$$

Let $B^1,\ldots,B^k,B^{k+1},\ldots,B^r$ be the vertices of σ, and let $B^1,\ldots,B^k,B^{r+1},\ldots,B^s$ be the vertices of σ'. Then $\mathrm{mult}(\sigma+\sigma')$ is the absolute value of the determinant of the matrix

$$\begin{pmatrix} B^1_1 & \ldots & B^1_k & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ B^k_1 & \ldots & B^k_k & 0 & \ldots & 0 & 0 & \ldots & 0 \\ * & \ldots & * & B^{k+1}_{k+1} & \ldots & B^{k+1}_r & 0 & \ldots & 0 \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ * & \ldots & * & B^r_{k+1} & \ldots & B^r_r & 0 & \ldots & 0 \\ * & \ldots & * & 0 & \ldots & 0 & B^{r+1}_{r+1} & \ldots & B^{r+1}_s \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ * & \ldots & * & 0 & \ldots & 0 & B^s_{r+1} & \ldots & B^s_s \end{pmatrix}.$$

We write

$$\mathrm{Aff}(\sigma+\sigma') \leftrightarrow a_1x_1+\ldots+a_sx_s = N(\sigma+\sigma'), \quad \text{with } \gcd(a_1,\ldots,a_s)=1,$$
$$\alpha := \gcd(a_1,\ldots,a_k), \beta := \gcd(a_{k+1},\ldots,a_r) \text{ and } \gamma := \gcd(a_{r+1},\ldots,a_s).$$

Then we have

$$\begin{aligned}
\mathrm{Aff}(\sigma) &\leftrightarrow \frac{a_1x_1+\ldots+a_kx_k+a_{k+1}x_{k+1}+\ldots+a_rx_r}{\gcd(\alpha,\beta)} = \frac{N(\sigma+\sigma')}{\gcd(\alpha,\beta)} = N(\sigma),\\
\mathrm{Aff}(\sigma') &\leftrightarrow \frac{a_1x_1+\ldots+a_kx_k+a_{r+1}x_{r+1}+\ldots+a_sx_s}{\gcd(\alpha,\gamma)} = \frac{N(\sigma+\sigma')}{\gcd(\alpha,\gamma)} = N(\sigma'),\\
\mathrm{Aff}(\sigma\cap\sigma') &\leftrightarrow \frac{a_1x_1+\ldots+a_kx_k}{\alpha} = \frac{N(\sigma+\sigma')}{\alpha} = N(\sigma\cap\sigma').
\end{aligned}$$

By using Proposition 4 we get

$$\mathrm{NV}(\sigma+\sigma')\,\mathrm{NV}(\sigma\cap\sigma') = \mathrm{NV}(\sigma)\,\mathrm{NV}(\sigma')\frac{\alpha}{\gcd(\alpha,\beta)\gcd(\alpha,\gamma)}.$$

As $\gcd(\alpha,\beta,\gamma)=1$, the quotient $\alpha/(\gcd(\alpha,\beta)\gcd(\alpha,\gamma))$ is an integer.

To prove the second statement, let σ and σ' be two V-faces in a simplicial facet τ such that $\sigma+\sigma'=\tau$. Then one easily shows that $N(\tau)=\mathrm{lcm}(N(\sigma),N(\sigma'))$ and one can then write

$$M = \frac{\alpha}{\gcd(\alpha,\beta)\gcd(\alpha,\gamma)} = \frac{\gcd(N(\sigma),N(\sigma'))}{N(\sigma\cap\sigma')}.$$

■

Corollary 6. *Let σ and σ' be two V-faces in a simplicial facet τ. If σ and σ' contribute with respect to τ and if $\sigma\cap\sigma'$ does not, then $M\geq 2$ in Equation (4).*

4 Some character sums

In order to prove the holomorphy conjecture, we have to show that some candidate poles of $Z_{f,0}(\chi,s)$ ("resp." $Z_f(\chi,s)$) are false poles. These proofs rely on the computation of certain character sums. We first recall some well-known properties of character sums over finite fields which we need when treating B_1-facets. We then study a specific character sum (see Proposition 10) which shows up when proving fakeness of some other candidate pole.

Lemma 7. *Let $a_1,\ldots,a_n,N\in\mathbb{Z}$, and let χ be a multiplicative character of $\mathbb{F}_q^\times$ whose order is not a divisor of N. Let $f\in\mathbb{F}_q[x_1,\ldots,x_n]$ be such that each exponent $(k_1,\ldots,k_n)$ appearing in f satisfies $a_1k_1+\cdots+a_nk_n=N$. Then*

$$\sum_{(x_1,\ldots,x_n)\in(\mathbb{F}_q^\times)^n}\chi(f(x_1,\ldots,x_n))=0.$$

Proof. Pick $u\in\mathbb{F}_q^\times$ such that $\chi(u^N)\neq 1$. Then the left hand side equals

$$\sum_{(x_1,\ldots,x_n)\in(\mathbb{F}_q^\times)^n}\chi(f(u^{a_1}x_1,\ldots,u^{a_n}x_n)) = \chi\left(u^N\right)\sum_{(x_1,\ldots,x_n)\in(\mathbb{F}_q^\times)^n}\chi(f(x_1,\ldots,x_n))$$

from which the property follows. ■

Lemma 8. *Let $a \in \mathbb{N}$, and let χ be a multiplicative character of $\mathbb{F}_q^\times$ whose order is not a divisor of a, then $\sum_{x\in\mathbb{F}_q^\times} \chi(x^a) = 0$.*

Proof. Take $f(x) = x^a$ in the previous lemma. ■

Lemma 9. *Let f be a polynomial, and let g be a monomial (possibly equipped with a non-zero coefficient) over $\mathbb{F}_q$ in the variables $x_2, \ldots, x_n$, and let χ be a non-trivial multiplicative character of $\mathbb{F}_q^\times$. Then*

$$\sum_{(x_1,\ldots,x_n)\in(\mathbb{F}_q^\times)^n} \chi(f(x_2,\ldots,x_n) + x_1 g(x_2,\ldots,x_n)) = -\sum_{(x_2,\ldots,x_n)\in(\mathbb{F}_q^\times)^{n-1}} \chi(f(x_2,\ldots,x_n)).$$

Proof. One can write

$$\begin{aligned}
&\sum_{(x_1,\ldots,x_n)\in(\mathbb{F}_q^\times)^n} \chi(f(x_2,\ldots,x_n) + x_1 g(x_2,\ldots,x_n)) \\
=& \sum_{(x_2,\ldots,x_n)\in(\mathbb{F}_q^\times)^{n-1}} \sum_{x_1\in\mathbb{F}_q^\times} \chi(f(x_2,\ldots,x_n) + x_1 g(x_2,\ldots,x_n)) \\
=& \sum_{(x_2,\ldots,x_n)\in(\mathbb{F}_q^\times)^{n-1}} \left(\sum_{u\in\mathbb{F}_q} \chi(u) - \chi(f(x_2,\ldots,x_n)) \right) \\
=& -\sum_{(x_2,\ldots,x_n)\in(\mathbb{F}_q^\times)^{n-1}} \chi(f(x_2,\ldots,x_n)),
\end{aligned}$$

where we used Lemma 8 in the last step. ■

Proposition 10. *Let χ be a multiplicative character of $\mathbb{F}_q^\times$ such that its order does not divide $a \in \mathbb{N}$. Let $\alpha, \gamma \in \mathbb{F}_q$, and let $\beta, \delta \in \mathbb{F}_q^\times$ be such that $\gamma^2 - 4\beta\delta \neq 0$. Then*

$$\begin{aligned}
&\sum_{(x,y,z)\in(\mathbb{F}_q^\times)^3} \chi(\alpha x^a + \beta x^{x_2} y^2 + \gamma x^{(x_1+x_2)/2} yz + \delta x^{x_1} z^2) = \\
&\qquad - \sum_{(x,y)\in(\mathbb{F}_q^\times)^2} \chi(\alpha x^a + \beta x^{x_2} y^2) - \sum_{(x,z)\in(\mathbb{F}_q^\times)^2} \chi(\alpha x^a + \delta x^{x_1} z^2),
\end{aligned}$$

with $x_1, x_2 \in \mathbb{N}$ such that $x_1 \equiv x_2 \bmod 2$.

Proof. First assume that q is odd, and write $\Delta = \gamma^2 - 4\beta\delta$. Let

$$\varepsilon = \begin{cases} 2 & \text{if } \Delta \text{ is a square,} \\ 0 & \text{if } \Delta \text{ is not a square.} \end{cases}$$

For each $c \in \mathbb{F}_q^\times$ define

$$L_c := \#\{ (x,y,z) \in (\mathbb{F}_q^\times)^3 \mid \alpha x^a + \beta x^{x_2} y^2 + \gamma x^{(x_1+x_2)/2} yz + \delta x^{x_1} z^2 = c \},$$

$$N_{y,c} := \#\{ (x,y) \in (\mathbb{F}_q^\times)^2 \mid \alpha x^a + \beta x^{x_2} y^2 = c \},$$

$$N_{z,c} := \#\{\, (x,z) \in (\mathbb{F}_q^\times)^2 \,\big|\, \alpha x^a + \delta x^{x_1} z^2 = c \,\},$$
$$M_c := \#\{\, x \in \mathbb{F}_q^\times \,\big|\, \alpha x^a = c \,\}.$$

We rewrite the first equation as

$$\beta x^{x_2} y^2 + \gamma x^{(x_1+x_2)/2} yz + \delta x^{x_1} z^2 = c - \alpha x^a. \tag{5}$$

For each value of $x \in \mathbb{F}_q^\times$ this defines a conic in the variables y and z. The discriminant of the quadratic part equals $\Delta \cdot x^{x_1+x_2} \neq 0$. As we supposed $x_1 \equiv x_2 \bmod 2$, we have that $\Delta \cdot x^{x_1+x_2}$ is a square if and only if Δ is a square. In the M_c cases where $c - \alpha x^a = 0$ the conic degenerates either into two lines over $\mathbb{F}_q$ (if Δ is a square), or into two conjugate lines over $\mathbb{F}_{q^2}$ (if Δ is a non-square). Thus in this case it carries $\varepsilon(q-1)+1$ points $(y,z) \in \mathbb{F}_q^2$. If $c - \alpha x^a \neq 0$ then one verifies using $\Delta \neq 0$ that Equation (5) defines an absolutely irreducible conic. It has ε rational points at infinity, so we conclude that the conic carries $q+1-\varepsilon$ points in $\mathbb{F}_q^2$, because every projective nonsingular curve of genus 0 over a finite field $\mathbb{F}_q$ has $q+1$ rational points (see [FJ]). Overall we count

$$(\varepsilon(q-1)+1)M_c + (q+1-\varepsilon)(q-1-M_c)$$

solutions $(x,y,z) \in \mathbb{F}_q^\times \times \mathbb{F}_q^2$ to Equation (5). This includes M_c points of the form $(x,0,0)$, $N_{y,c}$ points of the form $(x,y,0)$ with $y \neq 0$, and $N_{z,c}$ points of the form $(x,0,z)$ with $z \neq 0$. Therefore

$$L_c = (\varepsilon(q-1)+1)M_c + (q+1-\varepsilon)(q-1-M_c) - M_c - N_{y,c} - N_{z,c}. \tag{6}$$

Summing up, for some constants λ and μ that do not depend on c, it holds that $L_c = -N_{y,c} - N_{z,c} + \lambda M_c + \mu$. Now note that

$$S_1 := \sum_{(x,y,z)\in(\mathbb{F}_q^\times)^3} \chi(\alpha x^a + \beta x^{x_2} y^2 + \gamma x^{(x_1+x_2)/2} yz + \delta x^{x_1} z^2) = \sum_{c\in\mathbb{F}_q^\times} L_c \chi(c),$$

$$S_y := \sum_{(x,y)\in(\mathbb{F}_q^\times)^2} \chi(\alpha x^a + \beta x^{x_2} y^2) = \sum_{c\in\mathbb{F}_q^\times} N_{y,c}\chi(c),$$

$$S_z := \sum_{(x,z)\in(\mathbb{F}_q^\times)^2} \chi(\alpha x^a + \delta x^{x_1} z^2) = \sum_{c\in\mathbb{F}_q^\times} N_{z,c}\chi(c),$$

$$0 = \chi(\alpha) \sum_{x\in\mathbb{F}_q^\times} \chi(x^a) = \sum_{x\in\mathbb{F}_q^\times} \chi(\alpha x^a) = \sum_{c\in\mathbb{F}_q^\times} M_c \chi(c).$$

As for the last line, the first equality follows by Lemma 8. Plugging in the expression for L_c in S_1 we find

$$S_1 = -\sum_{c\in\mathbb{F}_q^\times} N_{y,c}\chi(c) - \sum_{c\in\mathbb{F}_q^\times} N_{z,c}\chi(c) + \lambda \sum_{c\in\mathbb{F}_q^\times} M_c\chi(c) + \mu \sum_{c\in\mathbb{F}_q^\times} \chi(c) = -S_y - S_z,$$

as wanted.

If q is even then our condition $\gamma^2 - 4\beta\delta \neq 0$ amounts to $\gamma \neq 0$. The above proof still applies, except that one should now work with $\Delta = \beta\delta/\gamma^2$ and the definition of ε should be modified to

$$\varepsilon = \begin{cases} 2 & \text{if } \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\Delta) = 0, \\ 0 & \text{if } \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\Delta) = 1. \end{cases}$$

For this definition of ε one verifies that Equation (6) still holds (see [BRS, Theorem 1]), and the remainder of the proof is exactly the same. ■

Note that the exponents $(a, 0, 0)$, $(x_1, 0, 2)$, $((x_1 + x_2)/2, 1, 1)$, $(x_2, 2, 0)$ are all contained in the hyperplane $2k_1 + (a - x_2)k_2 + (a - x_1)k_3 = 2a$, so under the stronger assumption that the order of χ does not divide $2a$, or under the additional assumption that $a - x_1$ even (which holds if and only if $a - x_2$ is even), we see from Lemma 7 that all sums in the statement of the proposition are actually zero.

5 A proof of the holomorphy conjecture for nondegenerate surface singularities

Let $f(\underline{x})$ be as in Subsection 2.1 and assume that it is nondegenerate over $\mathbb{C}$ with respect to the compact faces ("resp." the faces) of its Newton polyhedron at the origin Γ_0. Let K be a non-archimedean completion with valuation ring R and residue field $\mathbb{F}_q$, such that $\bar{f}$ is nondegenerate over $\mathbb{F}_q$ with respect to the compact faces ("resp." the faces) of Γ_0. Let $\chi : R^\times \to \mathbb{C}^\times$ be a non-trivial character of conductor 1. If $Z_{f,0}(\chi, s)$ ("resp." $Z_f(\chi, s)$) is not holomorphic on $\mathbb{C}$, then by the material from Subsection 2.3 it has a pole with real part equal to $-\nu(\tau)/N(\tau)$ for some facet τ of Γ_0 for which the order of $\bar{\chi}$ divides $N(\tau)$. Here as before $\bar{\chi}$ denotes the unique character of $\mathbb{F}_q^\times$ associated to χ.

For some facets τ, in particular the B_1-facets and the X_2-facets which we introduce below, we typically have to prove that $-\nu(\tau)/N(\tau)$ can *not* be the real part of a pole of $Z_{f,0}(\chi, s)$ ("resp." $Z_f(\chi, s)$). For the other facets, we prove that $e^{-2\pi i/N(\tau)}$ is an eigenvalue of monodromy of f at some point of $f^{-1}\{0\}$ and we thus obtain that the order of χ (which as we recall divides the order of $\bar{\chi}$) divides the order of some eigenvalue of monodromy at some point of $f^{-1}\{0\}$.

Let us first recall the notion of B_1-facets, introduced in [LVP2]. A simplicial facet of an n-dimensional Newton polyhedron ($n \geq 2$) is a *B_1-simplex with respect to the variable x_i* if it is a simplex with $n - 1$ vertices in the coordinate hyperplane $x_i = 0$ and one vertex at distance one of this hyperplane. We say that a facet τ of an n-dimensional Newton polyhedron is *non-compact for the variable x_j* ($1 \leq j \leq n$) if for every point $p \in \tau$ the point $p + (0, \ldots, 0, 1, 0, \ldots, 0) \in \tau$, where $(0, \ldots, 0, 1, 0, \ldots, 0)$ is an n-tuple with 1 at place j and 0 everywhere else. We define the maps $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1} : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, \widehat{x_j}, \ldots, x_n)$ for $j = 1, \ldots, n$. A non-compact facet τ of an n-dimensional Newton polyhedron ($n \geq 3$) is a (non-compact) *B_1-facet with respect to the variable x_i* if τ is non-compact for exactly one variable x_j and if $\pi_j(\tau)$ is a B_1-simplex in $\mathbb{R}^{n-1}$ with respect to x_i. A *B_1-facet* is a B_1-simplex or a non-compact B_1-facet with respect to some variable.

Here in addition we introduce:

Definition 11. *A* facet of type X_2 *in a* 3*-dimensional Newton polyhedron is a facet whose vertices (up to permutation of the coordinates) are of the form* $p = (a, 0, 0), q = (x_1, 0, 2)$, $r = (x_2, 2, 0)$ *with* $a - x_2$ *and* $a - x_1$ *both odd.*

Remark that X_2-facets have four lattice points : besides its three vertices we have the point $((x_1 + x_2)/2, 1, 1)$. Also notice that a simplex cannot be simultaneously B_1 and X_2, except when it is spanned by $(1, 0, 0)$, $(0, 0, 2)$, $(0, 2, 0)$ (up to permutation of the coordinates), "i.e.," it is the only compact facet of Γ_0. By Lemma 14 below, this facet does not give rise to an actual pole of $Z_{f,0}(\chi, s)$ or $Z_f(\chi, s)$.

5.1 Determination of a set of eigenvalues

As in Subsection 2.4 we subdivide the compact facets of Γ_0 into simplices τ. In [LVP2, Proposition 8] Van Proeyen and Lemahieu proved that whenever τ is not a B_1-facet, then the value $e^{-2\pi i\nu(\tau)/N(\tau)}$ is a root of F_τ. In this section we show that $e^{-2\pi i/N(\tau)}$ is also a root of F_τ, except possibly if τ is a B_1-facet or an X_2-facet. Contrary to [LVP2, Proposition 8], we here rely on Proposition 5 to get a more conceptual proof.

Proposition 12. *Let* τ *be a simplex in a subdivision of a compact facet of some three-dimensional Newton polyhedron. Suppose that* τ *is not of type* B_1 *nor of type* X_2*, then* $e^{-2\pi i/N(\tau)}$ *is a zero of* F_τ*.*

Proof. Case 1: τ does not contain a segment in a coordinate plane.
By formula (2), $F_\tau = \zeta_\tau = \left(1 - t^{N(\tau)}\right)^{\mathrm{NV}(\tau)}$ and $e^{-2\pi i/N(\tau)}$ clearly is a zero of F_τ.

Case 2: τ contains exactly one 1-dimensional V-face σ.
In this case, we have

$$F_\tau = \frac{\zeta_\tau}{\zeta_\sigma} = \frac{\left(1 - t^{N(\tau)}\right)^{\mathrm{NV}(\tau)}}{\left(1 - t^{N(\sigma)}\right)^{\mathrm{NV}(\sigma)}}.$$

Then $e^{-2\pi i/N(\tau)}$ is a zero of F_τ unless $N(\sigma) = N(\tau)$ and $\mathrm{NV}(\sigma) = \mathrm{NV}(\tau)$. One easily checks that then τ would be a B_1-facet.

Case 3: τ contains exactly two 1-dimensional V-faces σ_1 and σ_2.
In this situation,

$$F_\tau = \frac{\zeta_\tau \zeta_p}{\zeta_{\sigma_1}\zeta_{\sigma_2}} = \frac{(1 - t^l)\left(1 - t^{N(\tau)}\right)^{\mathrm{NV}(\tau)}}{\left(1 - t^{N(\sigma_1)}\right)^{\mathrm{NV}(\sigma_1)}\left(1 - t^{N(\sigma_2)}\right)^{\mathrm{NV}(\sigma_2)}},$$

where without loss of generality $\{p = (l, 0, 0)\} = \sigma_1 \cap \sigma_2$.

If $N(\sigma_1) \neq N(\tau)$ or $N(\sigma_2) \neq N(\tau)$, then see Case 1 and Case 2. If $N(\tau) = N(\sigma_1) = N(\sigma_2)$, then $F_\tau = (1 - t^l)\left(1 - t^{N(\tau)}\right)^{\mathrm{NV}(\tau) - \mathrm{NV}(\sigma_1) - \mathrm{NV}(\sigma_2)}$.

Case 3.1: If $N(p) = N(\tau)$, then by Proposition 5, $\mathrm{NV}(\tau) = \mathrm{NV}(\sigma_1)\,\mathrm{NV}(\sigma_2)$ and hence $F_\tau = \left(1 - t^{N(\tau)}\right)^{(\mathrm{NV}(\sigma_1)-1)(\mathrm{NV}(\sigma_2)-1)}$. If $\mathrm{NV}(\sigma_1)$ or $\mathrm{NV}(\sigma_2)$ would be equal to 1, then it would result that $\mathrm{NV}(\tau) = \mathrm{NV}(\sigma_i)$, for some $i \in \{1, 2\}$ and again τ would be a B_1-facet. Consequently, $e^{-2\pi i/N(\tau)}$ is a zero of F_τ.

Case 3.2: Suppose that $N(p) \neq N(\tau)$. By Proposition 5, we have $\mathrm{NV}(\tau) = M\,\mathrm{NV}(\sigma_1)\,\mathrm{NV}(\sigma_2)$, with $M \geq 2$. If not both $\mathrm{NV}(\sigma_1)$ and $\mathrm{NV}(\sigma_2)$ are equal to 1, then one easily deduces that $\mathrm{NV}(\tau) - \mathrm{NV}(\sigma_1) - \mathrm{NV}(\sigma_2) > 0$. If $\mathrm{NV}(\sigma_1) = \mathrm{NV}(\sigma_2) = 1$, then $\mathrm{NV}(\tau) - \mathrm{NV}(\sigma_1) - \mathrm{NV}(\sigma_2) > 0$ if and only if $M > 2$. It remains thus to study the case $\mathrm{NV}(\tau) = M = 2$, $\mathrm{NV}(\sigma_1) = \mathrm{NV}(\sigma_2) = 1$. As we supposed that $N(\tau) = N(\sigma_1) = N(\sigma_2)$, the vertices of τ are then $p = (N(\tau)/2, 0, 0), q = (x_1, 0, 2), r = (x_2, 2, 0)$, and

$$\mathrm{Aff}(\tau) \leftrightarrow 2x + (N(\tau)/2 - x_2)y + (N(\tau)/2 - x_1)z = N(\tau).$$

From $N(\sigma_1) = N(\sigma_2) = N(\tau)$ it follows that $N(\tau)/2 - x_2$ and $N(\tau)/2 - x_1$ are odd and hence τ is of type X_2.

Case 4: τ contains three 1-dimensional V-faces σ_1, σ_2 and σ_3.
In this situation

$$F_\tau = \frac{\zeta_\tau \zeta_p \zeta_q \zeta_r}{\zeta_{\sigma_1} \zeta_{\sigma_2} \zeta_{\sigma_3}},$$

with $p = \sigma_1 \cap \sigma_2$, $q = \sigma_1 \cap \sigma_3$ and $r = \sigma_2 \cap \sigma_3$. We suppose that $N(\tau) = N(\sigma_1) = N(\sigma_2) = N(\sigma_3)$, if not then we fall back on one of the previous cases.

Case 4.1: If $N(\tau) = N(\sigma_1 \cap \sigma_2) = N(\sigma_1 \cap \sigma_3) = N(\sigma_2 \cap \sigma_3)$, then, by Proposition 5, $\mathrm{NV}(\tau) = \mathrm{NV}(\sigma_1)\,\mathrm{NV}(\sigma_2) = \mathrm{NV}(\sigma_1)\,\mathrm{NV}(\sigma_3) = \mathrm{NV}(\sigma_2)\,\mathrm{NV}(\sigma_3)$, and thus $\mathrm{NV}(\sigma_1) = \mathrm{NV}(\sigma_2) = \mathrm{NV}(\sigma_3)$. Then F_τ becomes

$$F_\tau = \frac{\left(1 - t^{N(\tau)}\right)^{\mathrm{NV}(\sigma_1)^2 + 3}}{\left(1 - t^{N(\tau)}\right)^{3\,\mathrm{NV}(\sigma_1)}}.$$

Since $\mathrm{NV}(\sigma_1)^2 + 3 > 3\,\mathrm{NV}(\sigma_1)$, it follows that $e^{-2\pi i/N(\tau)}$ is a zero of F_τ.

Case 4.2: If $N(\tau) = N(\sigma_1 \cap \sigma_2) = N(\sigma_2 \cap \sigma_3) \neq N(\sigma_1 \cap \sigma_3)$, then Proposition 5 yields $\mathrm{NV}(\tau) = \mathrm{NV}(\sigma_1)\,\mathrm{NV}(\sigma_2) = \mathrm{NV}(\sigma_2)\,\mathrm{NV}(\sigma_3) = M\,\mathrm{NV}(\sigma_1)\,\mathrm{NV}(\sigma_3)$, with $M \geq 2$. We thus get $\mathrm{NV}(\sigma_3) = \mathrm{NV}(\sigma_1)$ and $\mathrm{NV}(\sigma_2) = M\,\mathrm{NV}(\sigma_1)$ and we find then

$$\mathrm{Aff}(\tau) \leftrightarrow x + My + z = N(\tau),$$

with $p = (N(\tau), 0, 0), q = (0, N(\tau)/M, 0)$ and $r = (0, 0, N(\tau))$. In this case $e^{-2\pi i/N(\tau)}$ is a zero of F_τ if and only if $\mathrm{NV}(\tau) + 2 > \mathrm{NV}(\sigma_1) + \mathrm{NV}(\sigma_2) + \mathrm{NV}(\sigma_3)$, or equivalently, if $(M\,\mathrm{NV}(\sigma_1) - 2)\,(\mathrm{NV}(\sigma_1) - 1) > 0$. This is always the case, as $\mathrm{NV}(\sigma_1) = N(\tau)/M = 1$ would imply that τ is a B_1-facet.

Case 4.3: If $N(\tau) = N(\sigma_1 \cap \sigma_2)$, $N(\tau) \neq N(\sigma_1 \cap \sigma_3)$ and $N(\tau) \neq N(\sigma_2 \cap \sigma_3)$, then by Proposition 5, one has $\mathrm{NV}(\tau) = \mathrm{NV}(\sigma_1)\,\mathrm{NV}(\sigma_2) = M_1\,\mathrm{NV}(\sigma_1)\,\mathrm{NV}(\sigma_3) = M_2\,\mathrm{NV}(\sigma_2)\,\mathrm{NV}(\sigma_3)$, with $M_1 \geq 2$ and $M_2 \geq 2$. In this configuration we have

$$\mathrm{Aff}(\tau) \leftrightarrow x + ky + lz = N(\tau),$$

$p = (N(\tau), 0, 0), q = (0, N(\tau)/k, 0)$ and $r = (0, 0, N(\tau)/l)$ with $\gcd(k, l) = 1$. Then we find that $M_1 = k$, $M_2 = l$ and hence $\mathrm{NV}(\sigma_2) = k\,\mathrm{NV}(\sigma_1)/l$ and $\mathrm{NV}(\sigma_3) = \mathrm{NV}(\sigma_1)/l$. In this case, $e^{-2\pi i/N(\tau)}$ would be a zero of F_τ if and only if $\mathrm{NV}(\tau) + 1 > \mathrm{NV}(\sigma_1) + \mathrm{NV}(\sigma_2) + \mathrm{NV}(\sigma_3)$, or equivalently,

$$k\,\mathrm{NV}(\sigma_1)^2 - (k + l + 1)\,\mathrm{NV}(\sigma_1) + l > 0.$$

This is true because $\mathrm{NV}(\sigma_1) \geq l$ while the largest real root of the polynomial on the left hand side is

$$\frac{k+l+1+\sqrt{(k+l+1)^2-4kl}}{2k} < l;$$

the latter inequality holds because one easily rewrites it as $kl > k+l$, which holds since $k, l \geq 2$ and $k = l = 2$ is excluded by coprimality.

Case 4.4: If $N(\tau) \neq N(\sigma_1 \cap \sigma_2)$, $N(\tau) \neq N(\sigma_1 \cap \sigma_3)$ and $N(\tau) \neq N(\sigma_2 \cap \sigma_3)$, then by Proposition 5, one has $\mathrm{NV}(\tau) = M_1 \,\mathrm{NV}(\sigma_1)\,\mathrm{NV}(\sigma_2) = M_2 \,\mathrm{NV}(\sigma_1)\,\mathrm{NV}(\sigma_3) = M_3 \,\mathrm{NV}(\sigma_2)\,\mathrm{NV}(\sigma_3)$, with $M_1 \geq 2$, $M_2 \geq 2$ and $M_3 \geq 2$. In this configuration we have

$$\mathrm{Aff}(\tau) \leftrightarrow kx + ly + mz = N(\tau),$$

$p = (N(\tau)/k, 0, 0), q = (0, N(\tau)/l, 0)$ and $r = (0, 0, N(\tau)/m)$ with k, l, m pairwise coprime. Then we find that $M_1 = k$, $M_2 = l$, $M_3 = m$ and hence $\mathrm{NV}(\sigma_2) = l\,\mathrm{NV}(\sigma_1)/m$ and $\mathrm{NV}(\sigma_3) = k\,\mathrm{NV}(\sigma_1)/m$. In this case we want to establish that $\mathrm{NV}(\tau) > \mathrm{NV}(\sigma_1) + \mathrm{NV}(\sigma_2) + \mathrm{NV}(\sigma_3)$, or equivalently that $klm\,\mathrm{NV}(\sigma_1) > k + l + m$. This follows from $\mathrm{NV}(\sigma_1) \geq 1$ and $klm \geq 4\max\{k, l, m\} > 3\max\{k, l, m\} \geq k+l+m$. ■

The above result will reduce our analysis to the study of compact facets of Γ_0, all of whose subdivisions into simplices consist purely of B_1-facets or X_2-facets. In fact, these types cannot appear within the same such facet of Γ_0, because one easily verifies that a B_1-facet and an X_2-facet can never share an edge.

For now we restrict to simplicial facets of Γ_0 that are of type B_1 or X_2. More precisely, in the next section we prove the fakeness of most candidate poles contributed by such facets. Notice that if two facets only have a vertex in common, then one can subdivide the cone dual to that vertex in such a way that the contributions of these facets to the Igusa zeta function can be analyzed separately. This reduces our analysis to 'clusters' or 'configurations' of B_1-facets or X_2-facets contributing the same candidate pole, by which we mean a collection of facets of which each member has at least one edge in common with another member.

5.2 On false poles

5.2.1 Preliminary facts

We first make the following observations, which hold up to permutation of the coordinates.

Fact 1: A vertex $P = (1, \cdot, \cdot)$ does not contribute. Indeed, as χ is not the trivial character (and so $\bar{\chi}$ neither is trivial), one immediately deduces from Lemma 8 that the contribution of P is equal to 0.

Fact 2: A vertex $P = (a, 0, 0)$ does not contribute if the order of $\bar{\chi}$ is not a divisor of a (again by Lemma 8).

Fact 3: A segment $\sigma := PQ$ with $P = (1, 1, b)$ and $Q = (0, 0, a)$ does not contribute if the order of $\bar{\chi}$ is not a divisor of a. To compute the contribution of σ, we consider

$$L_\sigma = q^{-3} \sum_{(x,y,z)\in(\mathbb{F}_q^\times)^3} \bar{\chi}(c_{0,0,a}z^a + c_{1,1,b}xyz^b).$$

By using Lemma 9, this expression simplifies to

$$-q^{-3}\bar{\chi}(c_{0,0,a}) \sum_{(y,z)\in(\mathbb{F}_q^\times)^2} \bar{\chi}(z^a).$$

If the order of $\bar{\chi}$ is not a divisor of a, then it follows from Lemma 8 that the contribution of σ is equal to 0.

Fact 4: Let $\sigma := PQ$ with $P = (\cdot,\cdot,0)$ and $Q = (\cdot,\cdot,0)$, and let $\tau := PQR$ with $R = (\cdot,\cdot,1)$ be the facet not contained in $\{z = 0\}$ that contains σ, then σ and τ cancel each other out. Indeed, by Lemma 9 with $f = f_\sigma$, it follows that $L_\sigma = (1-q)L_\tau$. As $\operatorname{mult}(\Delta_\sigma) = 1$, we find that $L_\sigma S(\Delta_\sigma) + L_\tau S(\Delta_\tau) = 0$.

Fact 5: Let $\sigma := PQ$ with $P = (\cdot,\cdot,0)$ and $Q = (\cdot,\cdot,1)$, then again by Lemma 9, one finds $L_P = (1-q)L_\sigma$. Now let τ_1 and τ_2 be the facets containing σ, and let τ_0 be the facet in $\{z = 0\}$ containing the vertex P. With δ_P the cone (strictly positively) spanned by $\Delta_{\tau_0}, \Delta_{\tau_1}$ and Δ_{τ_2}, we then find that $L_\sigma S(\Delta_\sigma) + L_P S(\delta_P) = 0$.

Fact 6: Let $\sigma := PQ$ with $P = (\cdot,\cdot,0)$ and $Q = (\cdot,\cdot,1)$, and let τ_1 be a non-compact B_1-facet containing σ. Let τ_2 be the non-compact facet containing the vertex Q and sharing a half line with τ_1. Lemma 8 implies that $\tau_1 \cap \tau_2$ does not contribute in the formula for $Z_f(\chi, s)$.

Fact 7: Let $\sigma := PQ$ with $P = (\cdot,\cdot,0)$ and $Q = (\cdot,\cdot,1)$, and let τ_1 be a non-compact B_1-facet containing σ. Let τ_0 be the non-compact facet containing the vertex P and sharing a half line σ_1 with τ_1. As $L_{\sigma_1} = (1-q)L_{\tau_1}$ and $\operatorname{mult}(\Delta_{\sigma_1}) = 1$, it follows that the contributions of τ_1 and σ_1 cancel each other out.

5.2.2 On false poles contributed by X_2-facets

Case 1: the candidate pole is contributed by an isolated X_2-facet.

Lemma 13. *Let τ be a facet with vertices $p = (N(\tau)/2, 0, 0), q = (x_1, 0, 2)$ and $r = (x_2, 2, 0)$ where $N(\tau)/2 - x_1$ and $N(\tau)/2 - x_2$ are odd. If the order of $\bar{\chi}$ does not divide $N(\tau)/2$ and is different from 2, then τ does not contribute an actual pole to $Z_{f,0}(\chi, s)$ and $Z_f(\chi, s)$.*

Proof. It follows immediately from Fact 2 that the vertices p, q and r do not contribute. Using Lemma 7 one also verifies that the edge qr does not contribute. We now show that the contributions of $\sigma_1 := pq$, $\sigma_2 := pr$ and the facet τ cancel each other. As $N(\sigma_1) = N(\sigma_2) = N(\tau)$, we have that $\operatorname{mult}(\Delta_{\sigma_1}) = \operatorname{mult}(\Delta_{\sigma_2}) = 1$, and thus

$$S(\Delta_{\sigma_i}) = \frac{1}{(q-1)(q^{N(\tau)s+\nu(\tau)} - 1)}, \quad 1 \le i \le 2.$$

One gets

$$L_{\sigma_1}S(\Delta_{\sigma_1}) + L_{\sigma_2}S(\Delta_{\sigma_2}) + L_\tau S(\Delta_\tau) = 0$$
$$\Updownarrow$$
$$(q-1)L_\tau = -L_{\sigma_1} - L_{\sigma_2}.$$

The equality between these character sums is proven in Proposition 10. There $\gamma = 0$ if the point $((x_1 + x_2)/2, 1, 1)$ is not in the support of f. Note that the condition $\gamma^2 - 4\beta\delta \neq 0$ in the statement of Proposition 10 follows from the non-degeneracy of $\overline{f}$ with respect to the edge qr. ■

If $x_1 = x_2 = 0$ (in which case the X_2-facet is the only compact facet of Γ_0) then we can prove something slightly stronger.

Lemma 14. *Let τ be a facet with vertices $p = (N(\tau)/2, 0, 0), q = (0, 0, 2)$ and $r = (0, 2, 0)$ where $N(\tau)/2$ is odd. If the order of $\bar{\chi}$ does not divide $N(\tau)/2$, then τ does not contribute an actual pole to $Z_{f,0}(\chi, s)$ and $Z_f(\chi, s)$.*

Proof. The previous proof remains valid, except for the conclusions that q, r and $\sigma_3 := qr$ do not contribute, where we used that the order of $\bar{\chi}$ is not 2. We show that the contributions cancel. Indeed, since $\text{mult}(\Delta_q) = \text{mult}(\Delta_r) = \text{mult}(\Delta_{\sigma_3}) = N(\tau)/2$ we have

$$S(\Delta_{\sigma_3}) = \frac{N}{(q-1)(q^{N(\tau)s+\nu(\tau)} - 1)}, \quad S(\Delta_q) = S(\Delta_r) = \frac{N}{(q-1)^2(q^{N(\tau)s+\nu(\tau)} - 1)}$$

for some common numerator N. One gets

$$L_q S(\Delta_q) + L_r S(\Delta_r) + L_{\sigma_3} S(\Delta_{\sigma_3}) = 0$$
$$\Updownarrow$$
$$(q-1)L_{\sigma_3} = -L_q - L_r.$$

This again follows from Proposition 10 (with $\alpha = 0$). ■

Case 2: the candidate pole is contributed by two X_2-facets sharing a 1-dimensional face.

Two different X_2-facets τ and τ' can appear in a cluster in one way only, and this determines the entire Newton polyhedron, as shown in Figure 1. As in the proof of Lemma 13 we see that q, r, qr do not contribute. Therefore the lemma also applies to this joint configuration.

5.2.3 On false poles contributed by B_1-facets

In [BV, Proposition 9.6] it is shown that if a candidate pole contributed only by B_1-facets is an actual pole of $Z_{f,0}(\chi, s)$, then it is contributed by two B_1-facets with respect to different variables having a 1-dimensional intersection. We revisit and extend this analysis and show that even in that situation the candidate pole is almost always a false pole of $Z_{f,0}(\chi, s)$. We need this strengthening here, because for the holomorphy conjecture we want to verify whether or not $1/N_j$ gives rise to an eigenvalue of monodromy, rather than the quotient ν_j/N_j (that is potentially simplifiable). We also study when

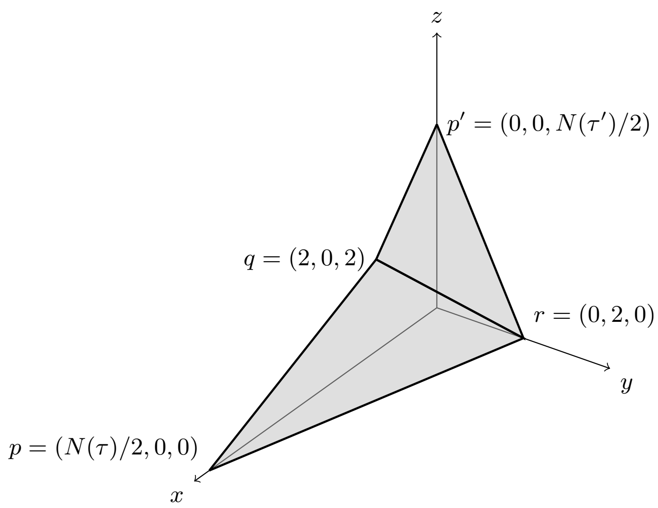

Figure 1

candidate poles of the global Igusa zeta function $Z_f(\chi, s)$ corresponding to B_1-facets are false poles.

From the preliminary work in Subsection 5.2.1 one can derive the contributions of all possible clusters of B_1-facets. We begin with the configuration studied (in the local case over $\mathbb{Q}_p$) in [BV, Proposition 9.6] that we mentioned at the beginning of this section.

Case 1: the candidate pole is contributed by a configuration of B_1-facets in which no two facets that share a 1-dimensional face are B_1 only for different variables.

For the contributions to the local Igusa zeta function, one can derive from Facts 1, 4 and 5 that the candidate pole is a false pole. For the global Igusa zeta function, one in addition uses Facts 6 and 7.

Case 2: the candidate pole is contributed by exactly two compact B_1-facets with respect to different variables, having a line segment in common.

If the common line segment is compact, then the configuration is as in Figure 2 with $A = (.,0,.), B = (1,1,b), C = (0,.,.)$ and $D = (0,0,a)$. If the order of $\bar{\chi}$ is not a divisor of a, then it follows from Facts 1 to 5 that the candidate pole is a false pole of $Z_{f,0}(\chi, s)$ and $Z_f(\chi, s)$.

Case 3: the candidate pole is contributed by two non-compact B_1-facets with respect to different variables, having a line segment in common.

If the common line segment is non-compact, then the configuration is as in Figure 3, with $A = (.,0,.), B = (0,.,.)$ and $C = (1,1,.)$. For the contributions to the local

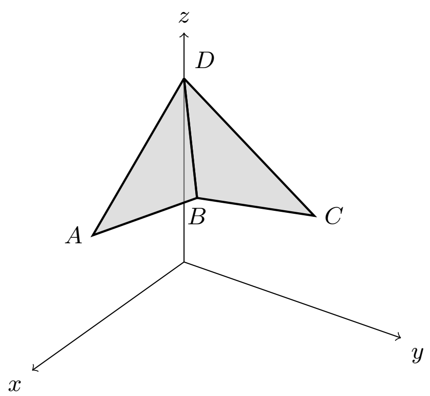

Figure 2

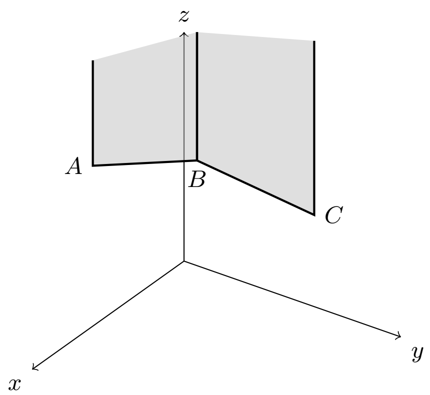

Figure 3

Igusa zeta function, one deduces from Facts 1, 4 and 5 that the candidate pole is a false pole. For the global Igusa zeta function, one also has to use Facts 6 and 7.

If the common line segment is compact, then its vertices are given by $A = (0, 0, a)$ and $B = (1, 1, b)$. If the order of $\bar{\chi}$ is not a divisor of a, then by Facts 1 to 7, it follows again that the candidate pole is not an actual pole of $Z_{f,0}(\chi, s)$ and $Z_f(\chi, s)$.

Case 4: the candidate pole is contributed by one compact B_1-facet and one non-compact B_1-facet with respect to different variables, having a line segment in common.

Again using Fact 1 to Fact 7, one finds that the candidate pole is a false pole of $Z_{f,0}(\chi, s)$ and $Z_f(\chi, s)$ when the order of $\bar{\chi}$ is not a divisor of a,.

Case 5: the candidate pole is contributed by at least two B_1-facets with respect to different variables, having a line segment in common.

As in Case 2 the contributions of $\tau_1 := ABD$, $\tau_2 := BCD$ and $\tau_1 \cap \tau_2 := BD$ are all equal to 0, one can deduce the fakeness of the candidate pole also when there are other B_1-facets having a 1-dimensional intersection with τ_1 or τ_2. ■

5.3 Holomorphy conjecture for nondegenerate surface singularities

We are now ready to prove the main result of this article.

Theorem 15. *Let F be a number field, and let $f(x,y,z) \in \mathcal{O}_F[x,y,z]$ be a polynomial which is nondegenerate over $\mathbb{C}$ with respect to the compact faces ("resp." the faces) of its Newton polyhedron at the origin Γ_0. Let K be a non-archimedean completion of F with valuation ring R (with maximal ideal P) and residue field $\mathbb{F}_q$, and suppose that $\bar{f} := f \bmod P$ is nondegenerate over $\mathbb{F}_q$ with respect to the compact faces ("resp." the faces) of Γ_0. Let χ be a non-trivial character of $R^\times$ which is trivial on $1+P$. Let τ be a facet of Γ_0. If $-\nu(\tau)/N(\tau)$ is the real part of a pole of $Z_{f,0}(\chi,s)$ ("resp." $Z_f(\chi,s)$), then the order of χ divides the order of an eigenvalue of monodromy at some point of $f^{-1}\{0\}$.*

Proof. We first suppose that τ is a compact facet. If every 1-dimensional V-face of Γ_0 is contained in a compact facet, then we know from Formula (3) that the zeta function of monodromy at the origin is a product of polynomials. If τ is not a union of simplices of type B_1 or X_2, then Proposition 12 implies that the order of χ divides the order of an eigenvalue of monodromy of $f^{-1}\{0\}$ at the origin.

If τ is of type B_1, then we found in Subsection 5.2.3 that there is a point $p = (0,0,a)$ in the configuration that is not the intersection of two 1-dimensional V-faces in a same compact facet, and secondly that the order of $\bar{\chi}$ divides this a. This means that the factor $1-t^a$ appears in $\zeta_{f,0}(t)$ and so one finds that the order of χ divides the order of some eigenvalue of monodromy of $f^{-1}\{0\}$ at the origin.

If τ is not a simplicial facet but a union of simplices of type B_1, then – up to permutation of the coordinates – the facet τ should have as vertices $A=(a,0,0), B=(c,d,0), C=(b,1,1)$ and $D=(e,0,f)$ for $a,\ldots,f \in \mathbb{Z}_{\geq 0}$ and so one has $\tau = \tau_1 \cup \tau_2$ with $\tau_1 := ABD$ and $\tau_2 := ACD$ two B_1-simplices. Notice that the factor $1-t^a$ appears in $\zeta_{f,0}(t)$. If the order of χ divides a, then one finds indeed that the order of χ divides the order of some eigenvalue of monodromy of $f^{-1}\{0\}$ at the origin.

So suppose now that the order of χ does not divide a. The facet τ is also the union of the simplices ABC and BCD. The simplex BCD never is of type X_2. If one of these simplices is not of type B_1, then it follows by Proposition 12 that the order of χ divides the order of some eigenvalue of monodromy of $f^{-1}\{0\}$ at the origin. If ABC and BCD are both of type B_1, then one then easily checks that d or f should be equal to 1, hence $\gcd(d,f)=1$. We have

$$\mathrm{Aff}(\tau_1) \leftrightarrow dx + y(a-c) + z\,(c-a-d(b-a)) = ad, \quad \text{and}$$

$$\mathrm{Aff}(\tau_2) \leftrightarrow fx + y\,(f(a-b)+e-a) + z\,(a-e) = af.$$

As $\mathrm{Aff}(\tau_1) = \mathrm{Aff}(\tau_2)$, we have that $N(\tau)$ divides $a \cdot \gcd(d,f) = a$. As we suppose that τ contributes a pole, it follows that the order of χ divides $N(\tau)$ and so again the order of χ divides the order of some eigenvalue of monodromy of $f^{-1}\{0\}$ at the origin.

If τ is of type X_2, say with vertices $p = (N(\tau)/2,0,0), q=(x_1,0,2)$ and $r = (x_2,2,0)$, then $F_\tau = 1 - t^{N(\tau)/2}$ and hence $e^{-2\pi i/(N(\tau)/2)}$ is an eigenvalue of monodromy of $f^{-1}\{0\}$ at the origin. Thus if the order of $\bar{\chi}$ divides $N(\tau)/2$ then we are done. If the order of $\bar{\chi}$ does not divide $N(\tau)/2$, then by Lemma 13, the order of $\bar{\chi}$ should be equal to 2. In this situation $N(\tau)/2$ is odd and x_1 and x_2 are even, while by Lemma 14, we can assume that $0 \neq x_1 \geq x_2$. Let τ' be the other facet which contains the segment qr. Notice that τ' is not of type B_1 and that $N(\tau')$ is even.

We first suppose that τ' is compact. If τ' is not of type X_2, then it follows from Proposition 12 that $e^{-2\pi i/N(\tau')}$ is a zero of $F_{\tau'}$ and so the order of χ divides the order of some eigenvalue of monodromy of $f^{-1}\{0\}$ at the origin. If τ' is of type X_2, then the configuration is as in Figure 1. In this situation, we get

$$\zeta_{f,0}(t) = (1 - t^{N(\tau)/2})(1 - t^{N(\tau')/2})(1 - t^2),$$

and so again the order of χ divides the order of an eigenvalue of monodromy of $f^{-1}\{0\}$ at the origin.

Suppose now that τ' is not compact, then necessarily $x_1 > x_2$ and

$$\mathrm{Aff}(\tau') \leftrightarrow x + \frac{x_1 - x_2}{2} y = x_1.$$

At a generic point $(0, 0, c)$ of the hypersurface, the polynomial $g(x, y, z) := f(x, y, z - c)$ is still nondegenerate with respect to the compact faces of its Newton polyhedron at the origin (see Lemma 2) and its Newton polyhedron is the projection onto $\{z = 0\}$ of the Newton polyhedron of f times $\mathbb{R}_+$. From Varchenko's formula one sees that this projected polyhedron fully determines $\zeta_{g,0}(t)$. Using [LVP2, Proposition 5] it follows that $\zeta_{g,0}(t)$ contains the factor $1/(1 - t^{x_1})$. We thus have that the order of χ divides the order of an eigenvalue of monodromy at a point of the hypersurface in the neighbourhood of the origin.

It is easy to check that a non-simplicial facet cannot decompose into a union of X_2-facets.

Suppose now that there is a 1-dimensional V-face σ, say in the coordinate plane $z = 0$, which is not contained in a compact facet. If $e^{-2\pi i/N(\tau)}$ is a zero of F_σ (we use the notation F_σ as if σ was a facet of a two-dimensional Newton polyhedron in the plane $z = 0$), then we choose $c \in \mathbb{C}$ close to zero such that $g(x, y, z) := f(x, y, z - c)$ is still nondegenerate with respect to its Newton polyhedron at the origin (see Lemma 2). Then we have

$$\zeta_{g,0}(t) = \prod_{\sigma \text{ compact facet}} F_\sigma,$$

with $F_\sigma = 1/\text{polynomial}$ (except the case where σ contains two vertices on coordinate axes, but in this case the same conclusion holds) and so we find that $e^{-2\pi i/N(\tau)}$ is an eigenvalue of monodromy of f at $(0, 0, c)$.

Finally let τ be non-compact. Again by the nondegeneracy argument (Lemma 2), we can reduce the dimension and conclude that $e^{-2\pi i/N(\tau)}$ is an eigenvalue of monodromy of f at a point in the neighbourhood the origin. ∎

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