KATHOLIEKE UNIVERSITEIT LEUVEN

Faculteit Wetenschappen

Departement Wiskunde

# Poincaré series and zeta functions

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Proefschrift ingediend tot het
behalen van de graad van
Doctor in de Wetenschappen

maart 2007

ISBN 978-90-8649-086-8

# Dankwoord

Na deze jaren waarin ik vol plezier en enthousiasme heb mogen werken aan dit doctoraat, wil ik mijn promotoren Prof. dr. Antonio Campillo en Prof. dr. Willem Veys van harte bedanken. De dubbele wetenschappelijke begeleiding vanuit verschillende invalshoeken maakten het een privilege dat jullie beiden mijn promotoren waren.

In het bijzonder, Wim, bedankt om me te begeleiden, voor het piekfijn nalezen van manuscripten, om me de kans te geven congressen en scholen bij te wonen (of te organiseren), voor de vrijheid die je me gaf, en voor het voorstel om samen op de monodromieconjectuur te werken! Antonio....pienso en los innumerables momentos de ilusión en los que me contaste de series de Poincaré y de tantas otras cosas que hacen que mi cultura general ha crecido mucho. Gracias también por querer conversar sobre cualquier problema, por responder a mis preguntas en cualquier momento, por estimularme y animarme siempre, por tu alegría. Eres un poeta de ciencias, y también de la vida. Muchísimas gracias por todo el tiempo que has dedicado a mi investigación. Nunca voy a olvidar la estancia en Valladolid en la primavera de 2006.

Tevens wil ik mijn dank betuigen aan de andere juryleden. Bedankt Prof. dr. Wolfgang Ebeling voor het beoordelen van mijn thesis, een extra dankjewel Joost voor je vele nuttige opmerkingen en aan Jan en Karel voor jullie optimisme en om me te steunen in het halen van een postdoc.

Ook de collega's uit onze 'afdeling' zorgen voor onvergetelijke herinneringen. Jullie zorgden voor heel variërende gesprekken, van botsende mieren tot opblazingen, van verre reisbestemmingen tot onze gezamelijke tocht door Hartshorne. Dankjewel allemaal!

También quiero agradecer a los becarios de Valladolid, gracias por recibirme siempre tan cordialmente, por organizar juntos el YMIS y por las noches de tapas. Más agradecimientos van a Félix Delgado por haberme permitido hacer una estancia en Valladolid y por las conversaciones.

Vervolgens wil ik ook graag mijn vrienden in de bloemetjes zetten, bedankt voor jullie interesse en jullie eeuwige aanmoedigingen, voor het plezier en de babbels, voor de heerlijke reizen, voor jullie vriendschap. Een speciale klaproos gaat hierbij naar Hanne.

Mama en papa, ik bedank jullie vanuit het diepste van mijn hart. Jullie en Inge zorgen voor het meest liefdevol denkbare gezinnetje; jullie zijn er altijd voor mij, zowel voor ontspannende momenten als voor het helpen met praktische zaken. Ik bedank jullie ontroerd voor alle kansen en vrijheid die jullie me geven.

Ook wil ik de andere leden van mijn familie bedanken die nieuwsgierig zijn naar het onderzoek dat ik verricht en voor de leuke feesten.

Andreas, c'est un cadeau divin de t'avoir rencontré...

Ann

# Contents

# Introduction

Poincaré series and zeta functions appear in many fields in mathematics. Poincaré series are typically series for which the coefficients represent discrete information about the class of the objects one is studying. Another kind of functions in which one puts relevant information are zeta functions. The goal is then to deduce new properties of the object out of these functions.

In this thesis we study a specific Poincaré series and some zeta functions that are classical in the domain of singularity theory. The Poincaré series that we investigate is defined for an algebraic variety with respect to a set of discrete valuations. This Poincaré series became of high interest when Campillo, Delgado and Gusein-Zade proved that for irreducible plane curve singularities this series coincides with the zeta function of monodromy. Since then, the Poincaré series has been studied e.g. for reducible plane curve singularities, for rational surface singularities and for quasi-homogenous singularities. A lot of interesting facts were discovered. It has been shown that the Poincaré series can be written as an integral with respect to the topological Euler-Poincaré characteristic; also for quasi-homogeneous singularities a relation was found between the Poincaré series and the zeta function of monodromy; for reducible plane curve singularities, the Poincaré series equals the Alexander polynomial of the link of the singularity; for a curve in a surface with a rational singularity, the Poincaré series of both varieties were shown to be related when using a resolution for the surface that gives an embedded resolution for the curve, etc.

A second object of study in this thesis is about the topological zeta function and about the zeta function of monodromy. These are functions that are associated to a function and they are rational. It is striking that for a fixed function, the possible poles of its topological zeta function are often not really poles. Segers and Veys determined the complete set of numbers that occur as a pole of some topological zeta function in dimension 2 and 3. Veys also showed that it can be seen on the dual resolution graph of a plane curve what the poles are.

A second reason why these poles are very interesting is because they are one of the protagonists in the intriguing monodromy conjecture. This conjecture states that the poles of the topological zeta function associated to a germ of a function $f$ induce eigenvalues of the local monodromy of $f$. This makes it very interesting to do research on the topological zeta function. In dimension 2, the conjecture was completely proven by Loeser. In higher dimensions Loeser proved the statement for particular cases. One of the conditions is that the polynomial should be nondegenerate with respect to its Newton polyhedron. Other contributions were made by Artal-Bartolo, Cassou-Noguès, Luengo and Melle-Hernández, Rodrigues and Veys.

By means of the first three chapters, we want the reader to feel comfortable while going through this thesis. We introduce toric varieties, clusters and the Poincaré series that we investigate. We provide examples, sometimes complemented with some observations or questions.

In Chapter 4 the Poincaré series for an affine toric variety is studied. Where for curves and rational surface singularities it is obvious which valuations to take to obtain a possible interesting series (namely the ones of the minimal resolution), for toric varieties this is a priori not clear. That is why we study the Poincaré series of a toric variety with respect to an arbitrary set of valuations. Chapter 4 is an extension of the results in [Le].

First we compute the series. Since the coordinate ring for a toric variety is graded and since the ideals involved in the Poincaré series are then monomial, we can obtain a formula for the Poincaré series in an easy way. In Chapter 1 it is shown that the Poincaré series can always be written as the generating function of the topological Euler-Poincaré characteristics of the projectivisations of the fibres of the extended semigroup and as an integral with respect to the topological Euler-Poincaré characteristic. For curves and rational surface singularities, Campillo, Delgado and Gusein-

Zade also obtained a description for the Poincaré series at the level of the modification space. It was a nice challenge to try to understand this in higher dimensions. I thank professor Gusein-Zade for the stimulating conversation about this. I generalised the description and proved it for affine toric varieties. As a consequence we get a formula for the Poincaré series when the variety is $\mathbb{C}^d$ and when the valuations are induced by a toric constellation. This means that we blow up in 0-dimensional orbits of a smooth variety. The formula that we obtain is similar to the one for curves. Finally we show that the Poincaré series for a toric complete intersection is cyclotomic.

In Chapter 5 we introduce the zeta functions that we study in this thesis. These are the zeta function of monodromy and the topological zeta function. We end this chapter by stating the monodromy conjecture for the topological zeta function.

Chapter 6 and 7 contain our investigation about the topological zeta function. The inspiration of Chapter 6 comes from a result of Segers and Veys. Let us consider the set

$$\mathcal{P}_d := \{s_0 \mid \exists f \in \mathbb{C}[x_1, \ldots, x_d] \,:\, Z_{top,f}(s) \textit{ has a pole in } s_0\}.$$

Segers and Veys showed that $\mathcal{P}_2 \cap (-\infty, -\frac{1}{2}) = \{-\frac{1}{2} - \frac{1}{i} \mid i \in \mathbb{Z}_{>1}\}$ and that $\mathcal{P}_3 \cap (-\infty, -1) = \{-1 - \frac{1}{i} \mid i \in \mathbb{Z}_{>1}\}$. They expected that this could be generalised to

$$\mathcal{P}_d \cap (-\infty, -\frac{d-1}{2}) = \{-\frac{d-1}{2} - \frac{1}{i} \mid i \in \mathbb{Z}_{>1}\}, \quad \text{for all } d \in \mathbb{Z}_{>1}.$$

We will show for all $d \geq 4$ that $\{-(d-1)/2 - 1/i \mid i \in \mathbb{Z}_{>1}\} \subset \mathcal{P}_d$ and we prove that any rational number in the remaining interval $[-(d-1)/2, 0)$ is a pole of some topological zeta function. We will obtain these results by picking out functions for which their topological zeta functions gives this list of poles. This chapter corresponds to [Le,Se,Ve].

An interesting remark is that the hypersurfaces determined by these functions have an embedded resolution given by a sequence of blowing-ups at centres which are concrete orbits for the toric action.

Chapter 7 is taking place in the setting of toric idealistic clusters in dimension 3. This means that one blows up in the 0-dimensional orbits of the constellation to which some weights are assigned. The cluster is called idealistic if these weights satisfy the proximity inequalities. They guarantee that there exist hypersurfaces which pass through the infinitely near points of the constellation exactly with the multiplicity given by the weight. Campillo, Gonzalez-Sprinberg and Lejeune-Jalabert provided whole sets of hypersurfaces for which the blowing-up of a cluster gives an embedded resolution for them.

In concrete, we take a finitely supported ideal in $\mathbb{C}[x, y, z]$ and we consider the constellation of base points associated to the ideal. Our objective is to write down the topological zeta function for general surfaces in this ideal. We show that the topological zeta function can explicitly be written down by using the data in the cluster. This formula can be implemented and so provides a quick way to compute the topological zeta function. Moreover, it offers maybe a good way to handle the monodromy conjecture in this specific context.

In Chapter 8 we continue working in the same context as in Chapter 7. For these surfaces, satisfying also a generic condition, we study a phenomenon related to the monodromy conjecture. In particular, for an irreducible exceptional component $E_j$ created by blowing up the constellation, we study when $\chi(\overset{\circ}{E_j}) < 0$. We then show that, if $\chi(\overset{\circ}{E_j}) > 0$, then $e^{-2\pi i \frac{\nu_j}{N_j}}$ is an eigenvalue of monodromy of such a generic surface for the cluster.

In the appendix we consider some ways to define Poincaré series for hypersurfaces and we formulate questions about them. We situate some known results in this outline and we provide a new result for toric hypersurfaces. In particular, we compute a Poincaré series for them in terms of the Newton polyhedron.

# Chapter 1

# Poincaré series

## 1.1 Introduction

As two series can only be equal if all their coefficients coincide, a series is a good tool to encode discrete information. For example if $f(x)$ is a polynomial in $\mathbb{Z}[x_1 \cdots, x_d]$, a classical series associated to $f$ is

$$P(t) := \sum_{i=0}^{\infty} N_i (p^{-d} t)^i.$$

In this formula $p$ is a prime number and $N_i$ denotes the number of solutions in $\mathbb{Z}/p^i\mathbb{Z}$ of $f(x) \equiv 0 \bmod p^i$. Another example concerns semigroups $S \subset \mathbb{N}^d$. Algebraists study the series

$$Q(u_1, \cdots, u_d) := \sum_{\underline{s} \in S} u_1^{s_1} \cdots u_d^{s_d}.$$

Such series are called *Poincaré series.*

A typical kind of Poincaré series are those that are induced by a filtration. For an algebraic variety $X$, a *one-index filtration* on the ring of germs of functions $\mathcal{O}_{X,o}$ on $(X, o)$ is a function that associates to each nonnegative integer $n \in \mathbb{Z}_{\geq o}$ an ideal $I_n$ in $\mathcal{O}_{X,o}$ such that

$$\mathcal{O}_{X,o} =: I_0 \supseteq I_1 \supseteq \cdots I_n \supseteq \cdots$$

If the dimensions $\dim \frac{I(n)}{I(n+1)}$ are finite for all $n \in \mathbb{Z}_{\geq 0}$, then the Poincaré series of this filtration is

$$P(t) := \sum_{n=0}^{\infty} \dim \frac{I(n)}{I(n+1)} t^n.$$

## 1.2 Complete ideals and valuations

The Poincaré series that we study is induced by a filtration coming from valuations. The ideals that are then appearing in the Poincaré series are exactly the complete ideals. In this section we recall some basic and interesting properties about valuations and complete ideals. For more details, we refer to [Ha] and [Z,Sa].

Let $K$ be a field and let $G$ be a totally ordered Abelian group. A *valuation* $\nu$ on $K$ is a map $\nu : K^* \to G$ such that for all $x, y \in K$:

1. $\nu(xy) = \nu(x) + \nu(y)$;
2. $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$.

The ring $R_\nu := \{x \in K^* \mid \nu(x) \geq 0\} \cup \{0\}$ is called the *valuation ring* of $\nu$. Often one also writes '$\nu : R_\nu \to G$'. If for all $x$ in a subfield $F$ of $K$, $x \neq 0$, holds that $\nu(x) = 0$, then one calls $\nu$ a *valuation of* $K/F$. If $A$ and $B$ are local rings contained in a field $K$ with maximal ideal respectively $\mathsf{m}_A, \mathsf{m}_B$, then we say that $B$ *dominates* $A$ if $A \subset B$ and $\mathsf{m}_B \cap A = \mathsf{m}_A$.

Let $(X, \mathcal{O}_X)$ be an integral quasi-projective scheme over a field $k$. We denote its function field by $k(X)$. A valuation $\nu$ of $k(X)/k$ is said to have *centre* $\xi$ in $X$ if its valuation ring dominates the local ring $\mathcal{O}_{X,\xi}$.

Observe that centres for valuations do not always exist. Let us consider the example where $X = \mathbb{C}$ and $\nu : \mathbb{C}(x) \to \mathbb{Z} : f/g \mapsto \deg(g) - \deg(f)$, $\deg(\cdot)$ denoting the degree of a polynomial. Then $\nu$ does not have a centre in $X$. However, if $X$ is *proper* over $k$, then every valuation has a unique non-empty centre in $X$. For an affine scheme $X := \operatorname{Spec} R$ and a valuation $\nu : R \to G$ with centre in $R$, the centre of $\nu$ is the irreducible variety corresponding to the prime ideal $\{x \in R \mid \nu(x) > 0\} \cup \{0\}$. Often this ideal is also called the centre of the valuation $\nu$.

In singularity theory one is often using valuations that are induced by divisors that are created during a resolution process of singularities. If $\pi : X' \to X$ is a resolution of singularities and if $E$ is an irreducible exceptional divisor on $X'$ with generic point $\xi_E$, then $\mathcal{O}_{X',\xi_E}$ is a 1-dimensional Noetherian local domain. Since $X'$ is smooth, the Weil divisor $E$ is also Cartier and the maximal ideal of $\mathcal{O}_{X',\xi_E}$ is principal. This means that $\mathcal{O}_{X',\xi_E}$ is a discrete valuation ring. If $\nu$ is the induced valuation on $k(X)$, then $\nu(f)$ is the order of $f \circ \pi$ along $E$.

The above situation not only holds for resolutions of singularities. It is also true for proper birational morphisms where $X'$ is normal. Note that the projective birational morphisms correspond to the blowing-ups in coherent sheaves of ideals. For a proper birational map $X' \to X$ and a valuation $\nu$ with centre in $X$, holds that $\nu$ has also a centre in $X'$.

Fix an integral quasi-projective scheme $(X, \mathcal{O}_X)$ over a field $k$. We look at all projective, birational morphisms $(X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$.

For two schemes $(X', \mathcal{O}'_X)$ and $(X'', \mathcal{O}''_X)$, we write $X'' \succeq X'$ if there exists a projective birational map $X'' \to X'$ such that the diagram

$$\begin{tikzcd} & X' \arrow[dl] \\ X & \\ & X'' \arrow[ul] \arrow[uu] \end{tikzcd}$$

is commutative. The order '$\succeq$' determines a projective system. In general its projective limit $\varprojlim_{X' \to X} X'$ is not a scheme. Only when $X$ is a curve it is; it is then equal to the normalisation of the curve. This projective limit is a locally ringed space and we will denote by $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$. It is called the *Zariski-Riemann space*. Zariski showed that the points of $\mathcal{Z}$ are in one-to-one correspondence with the valuations on $k(X)$ with centre at $X$. For a point $P \in \mathcal{Z}$ and its corresponding valuation $\nu$ on $k(X)$, it holds that $\mathcal{O}_{\mathcal{Z},P} = R_\nu$.

Let $I \subset \mathcal{O}_{X,o}$ be an ideal and let $\mathcal{Z}$ be the Zariski-Riemann space associated to $X$. The *completion* $\overline{I}$ of $I$ is the biggest ideal in $\mathcal{O}_{X,o}$ such that $\overline{I}\mathcal{O}_{\mathcal{Z}} = I\mathcal{O}_{\mathcal{Z}}$. The ideal $I$ is called *complete* if $\overline{I} = I$.

Let $I$ be an ideal in a integral domain $R$. Denote the quotient field of $R$ by $K$. An element $x$ of $K$ is called *integral over* $I$ if there exists an integer $n \in \mathbb{Z}_{>0}$ and elements $a_i \in I^i$, $1 \leq i \leq n$, such that

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0.$$

The set of elements in $K$ that are integral over $I$ is called the *integral closure* of $I$. The *integral dependence criterium* says that the completion of an ideal is equal to the integral closure of the ideal.

Let $\mathcal{V}(R)$ be the set of valuations $\nu$ on $K$ such that $R_\nu \supseteq R$. An ideal $I$ in $R$ is called a *valuation ideal* if there exists a valuation $\nu \in \mathcal{V}(R)$ and an ideal $I_\nu$ of $R_\nu$ such that $I_\nu \cap R = I$. There is a very nice result that states that an ideal of an integral domain $R$ is complete if and only if it is the intersection of valuation ideals of $R$ ([Ki,V, p. 440]). In particular, one has

$$\overline{I} = \bigcap_{\nu \in \mathcal{V}(R)} IR_\nu = \bigcap_{\nu \in \mathcal{V}(R)} (IR_\nu \cap R).$$

Denote the blowing-up of the ideal $I$ by $Bl_I R$ and its normalisation by $\overline{Bl_I R}$. Then $\overline{I}$ is also the biggest ideal such that $\overline{I}\mathcal{O}_{\overline{Bl_I R}} = I\mathcal{O}_{\overline{Bl_I R}}$. For $j \in J := \{1, \cdots, r\}$, let $E_j$ be an irreducible component of the exceptional divisor on $\overline{Bl_I R}$ and let $\zeta_{E_j}$ be its generic element. The valuations $\nu_1, \cdots, \nu_r$ induced by the discrete valuation rings $\mathcal{O}_{\overline{Bl_I R}, \zeta_{E_j}}$ are the so called *Rees valuations* of $I$. One has

$$\overline{I} = \bigcap_{j=1}^{r} IR_{\nu_j}$$

and if we denote $\nu_j(I) := \min\{\nu_j(x) \mid x \in I\}, j \in J$, then it follows that

$$x \in \overline{I} \Leftrightarrow \forall j \in J : \nu_j(x) \geq \nu_j(I).$$

## 1.3 Poincaré series

The Poincaré series that we investigate appeared for the first time in [C,D,K] for studying plane curve singularities over arbitrary fields. In this section we introduce this Poincaré series and we show when it is well defined.

Let $X$ be an algebraic variety over a field $k$. Consider the ideals

$$I_n := \{g \in \mathcal{O}_{X,o} \mid \nu(g) \geq n\}$$

with $\nu : \mathcal{O}_{X,o} \to \mathbb{Z} \cup \{\infty\}$ a discrete valuation. These ideals give rise to a filtration on $\mathcal{O}_{X,o}$. One can also consider several valuations and one then gets a *multi-index filtration*: for discrete valuations $\nu_1, \cdots, \nu_r : \mathcal{O}_{X,o} \to \mathbb{Z} \cup \{\infty\}$ and for $\underline{v} \in \mathbb{Z}^r$, consider the ideals

$$I(\underline{v}) := \{g \in \mathcal{O}_{X,o} \mid \nu_j(g) \geq v_j, 1 \leq j \leq r\}.$$

If the centre of each valuation $\nu_j$ $(j \in J)$ is the maximal ideal $\mathsf{m}$ of $\mathcal{O}_{X,o}$, then $\mathsf{m}I(\underline{v}) \subset I(\underline{v}+\underline{1})$. As $I(\underline{v})/\mathsf{m}I(\underline{v})$ is a finite dimensional $k$-vector space, it follows that also the $k$-vector spaces $I(\underline{v})/I(\underline{v}+\underline{1})$ have finite dimension.

In 1994 Campillo, Delgado and Kiyek put the dimensions $\dim \frac{I(\underline{v})}{I(\underline{v}+\underline{1})}$ in a Poincaré series. In fact, setting $d(\underline{v}) := \dim(I(\underline{v})/I(\underline{v}+\underline{1}))$ and writing $\underline{t} := (t_1, \cdots, t_r)$ and $\underline{v} := (v_1, \cdots, v_r)$, they define the Laurent series $L(\underline{t}) := \sum_{\underline{v}\in\mathbb{Z}^r} d(\underline{v})\underline{t}^{\underline{v}}$.

Let $i \in J$. Note that for $\underline{v}$ and $\underline{v}'$ vectors in $\mathbb{Z}^r$ such that $v_j = v'_j$ for all $j \in J \setminus i$ and such that $v_i \le 0$ and $v'_i \le 0$, one has that $I(\underline{v}) = I(\underline{v}')$ and hence $\prod_{j=1}^r (t_j - 1)L(t_1, \cdots, t_r) \in \mathbb{Z}[[t_1, \cdots, t_r]]$. They define the *Poincaré series of $X$ with respect to the valuations* $\{\nu_1, \cdots, \nu_r\}$ as

$$P(t_1, \cdots, t_r) := \frac{\prod_{j=1}^r (t_j - 1)L(t_1, \cdots, t_r)}{(t_1 \cdots t_r - 1)}.$$

Note that $\mathbb{Z}[[t_1, \cdots, t_r, t_1^{-1}, \cdots, t_r^{-1}]]$ is a $\mathbb{Z}[t_1, \cdots, t_r, t_1^{-1}, \cdots, t_r^{-1}]$-module but not a ring. As the element $\prod_{j=1}^r (t_j - 1)$ is not invertible in the ring $\mathbb{Z}[t_1, \cdots, t_r, t_1^{-1}, \cdots, t_r^{-1}]$, it follows that in general $P$ contains less information than $L$.

In Section 1.4, we will see that involving all $\underline{v} \in \mathbb{Z}^r$ makes that the Poincaré series can be described in a nice geometrical way.

**1.1 Remark** If one of the valuations $\nu_1, \cdots, \nu_r$ does not have its centre at $\mathsf{m}$, then one can not define the series $L$. Indeed, suppose $\nu_j$ $(j \in J)$ is a valuation with centre at the prime ideal $\mathsf{p}_j$ which is different from $\mathsf{m}$. If $g \in \mathcal{O}_{X,o}$ and $\underline{\nu}(g) = \underline{v}$, then choose a function $h \in \mathsf{m} \setminus \mathsf{p}_j$. Now for each $n \in \mathbb{Z}_{\ge 0}$ one has that $gh^n \in I(\underline{v})$ and $gh^n \notin I(\underline{v}+\underline{1})$. As all the $gh^n (n \in \mathbb{Z}_{\ge 0})$ are linearly independent over $k$, it follows that $d(\underline{v})$ is infinite. $\square$

In Section 1.6 we will have a closer look at the Poincaré series for curves and rational surface singularities. In particular, we will comment which valuations Campillo, Delgado and Gusein-Zade use to define the Poincaré series for these varieties.

The Poincaré series for curves has a very deep meaning. For example for irreducible plane curve singularities, the Poincaré series is equal to the monodromy zeta function ([C,D,G-Z2]). It is a highly interesting question what kind of valuations one can incorporate in general in the Poincaré series such that the series would contain a lot of information. Good candidates could maybe be essential valuations and arc space valuations. Some pioneering work in this direction is done by Ebeling and Gusein-Zade in [Eb,G-Z3] and in [Eb,G-Z4].

Whatever the valuations used to define the Poincaré series are, the Poincaré series can always be written as the generating function of the topological Euler-Poincaré characteristics of the projectivisations of the fibres of the extended semigroup and even as an integral with respect to the topological Euler-Poincaré characteristic. Campillo, Delgado and Gusein-Zade introduced these equivalent descriptions for the Poincaré series in [C,D,G-Z3], [C,D,G-Z7] and in [C,D,G-Z5]. We discuss them in Section 1.4 and in Section 1.5.

## 1.4 Geometric view on Poincaré series

We will study the Poincaré series for algebraic varieties over $\mathbb{C}$. Let $\mathcal{V} := \{\nu_1, \cdots, \nu_r\}$ be a set of discrete valuations on $\mathbb{C}(X)$. We denote the vector $(\nu_1, \cdots, \nu_r)$ by $\underline{\nu}$. The *semigroup of values of $X$ with respect to the set of valuations $\mathcal{V}$* is $S_{\mathcal{V}} := \{\underline{v} \in \mathbb{Z}^r_{\geq 0} \mid \underline{v} = \underline{\nu}(g) \text{ for some } g \in \mathcal{O}_{X,o}\}$.

For $j \in J$, denote by $D_j(\underline{v})$ the complex vector space $I(\underline{v})/I(\underline{v} + \underline{e_j})$ where $e_j$ is the $r$-tuple with $j$-th component equal to 1 and the other components equal to 0. Let us consider the map

$$\begin{array}{rcl} j_{\underline{v}} : I(\underline{v}) & \longrightarrow & D_1(\underline{v}) \times \cdots \times D_r(\underline{v}) \\ g & \longmapsto & (a_1(g), \cdots, a_r(g)) =: \underline{a}(g), \end{array}$$

where $a_j(g)$ is the projection of $g$ on $D_j(\underline{v})$. The set $\hat{S}_{\mathcal{V}} := \{(\underline{\nu}(g), \underline{a}(g)) \mid g \in \mathcal{O}_{X,o}\}$ is a semigroup with respect to the summation of the components $\underline{\nu}$ and multiplication of the parts $\underline{a}$ and is called the *extended semigroup*. This notion showed up for the first time in [C,D,G-Z1] where it was introduced for plane curves. Let $D(\underline{v})$ be the image of the map $j_{\underline{v}}$, then $D(\underline{v}) \simeq I(\underline{v})/I(\underline{v} + \underline{1})$. We define $F_{\underline{v}}$ as $D(\underline{v}) \cap (D_1^*(\underline{v}) \times \cdots \times D_r^*(\underline{v}))$ where $D_j^*(\underline{v})$ denotes $D_j(\underline{v}) \setminus \{\underline{0}\}$, $j \in J$. Having the map

$$\begin{array}{rcl} \rho : \quad \hat{S}_{\mathcal{V}} & \longrightarrow & S_{\mathcal{V}} \\ (\underline{\nu}(g), \underline{a}(g)) & \longmapsto & \underline{\nu}(g), \end{array}$$

$F_{\underline{v}}$ can also be expressed as $\rho^{-1}(\underline{v})$ and therefore one also calls the spaces $F_{\underline{v}}$ the *fibres of the extended semigroup* $\hat{S}_{\mathcal{V}}$. From its definition it follows that, for $\underline{v} \in S$, the space $F_{\underline{v}}$ is the complement to an arrangement of vector subspaces in the vector space $D(\underline{v})$. Moreover, $F_{\underline{v}}$ is invariant with respect to multiplication by nonzero constants. Let $\mathbb{P}F_{\underline{v}} := F_{\underline{v}}/\mathbb{C}^*$ be the projectivisation of $F_{\underline{v}}$. Then $\mathbb{P}F_{\underline{v}}$ is the complement to an arrangement of projective subspaces in the projective space $\mathbb{P}D(\underline{v})$.

**1.2 Theorem** *Let $X$ be an algebraic variety and let $\mathcal{V} := \{\nu_1, \cdots, \nu_r\}$ be a set of discrete valuations on $\mathbb{C}(X)$ with centre in the maximal ideal of $\mathcal{O}_{X,o}$. The Poincaré series with respect to $X$ and $\mathcal{V}$ is then*

$$P(\underline{t}) = \sum_{\underline{v}\in\mathbb{Z}^r} \chi(\mathbb{P}F_{\underline{v}})\underline{t}^{\underline{v}}.$$

*Proof.* Choose an element $\underline{w} \geq \underline{v}+\underline{1}$ in $\mathbb{Z}^r_{\geq 0}$, let $b(\underline{v}) :=\dim(I(\underline{v})/I(\underline{w}))$. For $A \subset J$, let $\underline{1_A}$ be the element of $\mathbb{Z}^r_{\geq 0}$ for which the $j$-th component is equal to 1, respectively 0 if $j \in A$, respectively $j \notin A$. Let $L_A$ be the subspace $\{(a_1, \cdots, a_r) \in D_1(\underline{v}) \times \cdots \times D_r(\underline{v}) \mid a_j = 0 \text{ for } j \in A\}$. One has

$$\begin{aligned}
\chi(\mathbb{P}F_{\underline{v}}) &= \chi(\mathbb{P}D(\underline{v})) - \chi(\bigcup_{j=1}^{r} \mathbb{P}(D(\underline{v}) \cap L_j)) \\
&= \chi(\mathbb{P}D(\underline{v})) - \sum_{A\subset J, A\neq\emptyset} (-1)^{\#A-1}\chi(\mathbb{P}(D(\underline{v}) \cap L_A)) \\
&= \sum_{A\subset J} (-1)^{\#A}\chi(\mathbb{P}(D(\underline{v}) \cap L_A)) \\
&= \sum_{A\subset J} (-1)^{\#A}\dim(D(\underline{v}) \cap L_A).
\end{aligned}$$

Since $\underline{v} \leq \underline{w}-\underline{1}$, one has $\dim(D(\underline{v})\cap L_A) = b(\underline{v}+\underline{1_A})-b(\underline{v}+\underline{1})$. This implies that the coefficient at $\underline{t}^{\underline{v}}$ in the series $(\sum_{\underline{v}\in\mathbb{Z}^r} \chi(\mathbb{P}F_{\underline{v}})\underline{t}^{\underline{v}})(t_1 \cdots t_r - 1)$ is equal to

$$\sum_{A\subset J} (-1)^{\#A}(b(\underline{v} + \underline{1_A} - \underline{1}) - b(\underline{v})) - \sum_{A\subset J} (-1)^{\#A}(b(\underline{v} + \underline{1_A}) - b(\underline{v} + \underline{1})).$$

Since $\sum_{A\subset J}(-1)^{\#A} = 0$, this coefficient is also equal to

$$\sum_{A\subset J} (-1)^{\#A}(b(\underline{v} + \underline{1_A} - \underline{1}) - b(\underline{v} + \underline{1_A})) = \sum_{A\subset J} (-1)^{\#A}(d(\underline{v} + \underline{1_A} - \underline{1}).$$

The coefficient at $\underline{t}^{\underline{v}}$ in $L(\underline{t}) \prod_{j=1}^{r}(t_j-1)$ is also equal to $\sum_{A\subset J}(-1)^{\#A}(d(\underline{v}+\underline{1_A} - \underline{1})$. ∎

## 1.5 Poincaré series as an integral

The Poincaré series has also the shape of an integral with respect to the topological Euler-Poincaré characteristic over the projectivisation $\mathbb{P}\mathcal{O}_{X,o}$. This notion has been introduced in [C,D,G-Z5] and is inspired by the notion of motivic integration (see for example [De,L4]). It was developed to integrate over $\mathbb{P}\mathcal{O}_{\mathbb{C}^d,o}$, what is not allowed by the usual Viro construction where one integrates with respect to the topological Euler-Poincaré characteristic over finite dimensional spaces (see [Vi]). It can be extended to integrals over $\mathbb{P}\mathcal{O}_{X,o}$ for an arbitrary variety $X$.

Let $\mathsf{m}$ be the maximal ideal in the ring $\mathcal{O}_{X,o}$ and for $k \in \mathbb{Z}_{\geq 0}$, let $\mathcal{J}^k_{X,o} := \mathcal{O}_{X,o}/\mathsf{m}^{k+1}$ be the space of *k-jets of functions on the variety* $X$. For a complex vector space $L$ (finite of infinite dimensional) let $\mathbb{P}L := (L \setminus \{0\})/\mathbb{C}^*$ be its projectivisation and let $\mathbb{P}^*L$ be the disjoint union of $\mathbb{P}L$ with a point. One has a natural map $\pi_k : \mathbb{P}\mathcal{O}_{X,o} \mapsto \mathbb{P}^*\mathcal{J}^k_{X,o}$.

A subset $A \subset \mathbb{P}\mathcal{O}_{X,o}$ is said to be a *cylindrical subset* if $A = \pi_k^{-1}(B)$ for a constructible subset (i.e. a finite union of locally closed subsets) $B \subset \mathbb{P}\mathcal{J}^k_{X,o} \subset \mathbb{P}^*\mathcal{J}^k_{X,o}$. For a cylindrical subset $A \subset \mathbb{P}\mathcal{O}_{X,o}$ ($A = \pi_k^{-1}(B)$, $B \subset \mathbb{P}\mathcal{J}^k_{X,o}$), its *Euler characteristic* $\chi(A)$ is defined as the Euler characteristic $\chi(B)$ of the set $B$.

Let $\psi : \mathbb{P}\mathcal{O}_{X,o} \to G$ be a function which takes values in an Abelian group $G$. We say that the function $\psi$ *is cylindrical* if, for each $g \in G, g \neq 0$, the set $\psi^{-1}(g) \subset \mathbb{P}\mathcal{O}_{X,o}$ is cylindrical. The *integral of a cylindrical function* $\psi$ *over the space* $\mathbb{P}\mathcal{O}_{X,o}$ *with respect to the Euler characteristic* is

$$\int_{\mathbb{P}\mathcal{O}_{X,o}} \psi d\chi := \sum_{g \in G, g \neq 0} \chi(\psi^{-1}(g)) \cdot g$$

if this sum makes sense in $G$. If the integral exists, the function $\psi$ is said to be *integrable*.

We have the map $\underline{\nu} := (\nu_1, \cdots, \nu_r) : \mathbb{P}\mathcal{O}_{X,o} \longrightarrow \mathbb{Z}^r$. Let $\underline{t}^{\underline{\nu}}$ be the corresponding function with values in $\mathbb{Z}[[\underline{t}]] := \mathbb{Z}[[t_1, \cdots, t_r]]$.

**1.3 Proposition**

$$P(\underline{t}) = \int_{\mathbb{P}\mathcal{O}_{X,o}} \underline{t}^{\underline{\nu}} d\chi.$$

*Proof.* For $\underline{v} \in \mathbb{Z}^r_{\geq 0}$, let $N := 1+ \max\{v_j \mid 1 \leq j \leq r\}$ and let $Y_{\underline{v}} := \{j^N g \in \mathbb{P}\mathcal{J}^N_{X,o} \mid g \in \mathbb{P}F_{\underline{v}}\} \subset \mathbb{P}\mathcal{J}^N_{X,o}$. Then $\{g \in \mathbb{P}\mathcal{O}_{X,o} \mid \underline{\nu}(g) = \underline{v}\} = \pi_N^{-1}(Y_{\underline{v}})$. Consider the map

$$\begin{aligned} \alpha : Y_{\underline{v}} &\longrightarrow \mathbb{P}F_{\underline{v}} \\ j^N g &\longmapsto \underline{a}(g). \end{aligned}$$

As $\alpha$ is a locally trivial fibration whose fibre is a complex affine space, we obtain

$$\int_{\mathbb{P}\mathcal{O}_{X,o}} \underline{t}^{\underline{\nu}} d\chi = \sum_{\underline{v}\in\mathbb{Z}^r} \chi(Y_{\underline{v}})\underline{t}^{\underline{v}} = \sum_{\underline{v}\in\mathbb{Z}^r} \chi(\mathbb{P}F_{\underline{v}})\underline{t}^{\underline{v}} = P(\underline{t}).$$

■

## 1.6 Examples of computed Poincaré series

The Poincaré series has been studied for several varieties. Campillo, Delgado and Gusein-Zade treated e.g. curves, rational surfaces and plane divisorial valuations, see for example [C,D,G-Z4], [C,D,G-Z8], [D,G-Z]. Ebeling and Gusein-Zade investigated the Poincaré series for quasi-homogeneous singularities, see [Eb], [Eb,G-Z1], [Eb,G-Z2]. Cutkosky showed that for normal surface singularities, from the Poincaré series it is possible to compute the intersection matrix and the arithmetic genera of the exceptional components ([Cu]).

We recall the ideas of the definition of the Poincaré series for curves and rational surfaces as these are the first ones that have been analysed.

Let $(C, o)$ be a germ of a curve and let $C =: \cup_{j=1}^r C_j$ be its decomposition into irreducible components. Consider a uniformisation $\phi_j : (\mathbb{C}, o) \to (C, o)$ of the branch $C_j$, $j \in J := \{1, \cdots, r\}$, and define a valuation $\nu_j$ for each branch as follows: for a germ $g \in \mathcal{O}_{C,o}$, set $\nu_j(g)$ equal to the power of the leading term in the power series decomposition of the germ $g \circ \phi_j$. If $g \circ \phi_j \equiv 0$, we assume $\nu_j(g)$ to be equal to $\infty$.

For irreducible plane curves, Campillo, Delgado and Gusein-Zade discovered that the Poincaré series coincides with the monodromy zeta function ([C,D,G-Z2]. They used the formula of A'Campo which provides an equivalent description for the monodromy zeta function in terms of an embedded resolution of the curve (see [A'C]). For reducible plane curves, they showed that the Poincaré series coincides with the Alexander polynomial ([C,D,G-Z4]). They related both via the formula of Eisenbud and Neumann which describes the Alexander polynomial of a curve in terms of en embedded resolution for the curve ([E,N]).

In higher dimensions one can not just parametrise the variety $X$. For curves however, note that the order of $t$ in $g \circ \phi_j(t)$ is the intersection multiplicity of $\{g = 0\}$ with the branch $C_j$. A natural generalisation consists then in considering a resolution $\pi : (\mathcal{X}, \mathcal{D}) \to (X, o)$ of the variety $X$. We denote the irreducible components of $\mathcal{D}$ by $E_j$, $j \in J$. For a function $g \in \mathcal{O}_{\mathcal{X},o}, g \neq 0$, let $\nu_j(g)$ be the multiplicity of $g \circ \pi$ along $E_j$, $j \in J$. The map $\nu_j : \mathcal{O}_{\mathcal{X},o} \to \mathbb{Z}_{\geq 0}$ defines a valuation on the field of quotients of the ring $\mathcal{O}_{\mathcal{X},o}$. These valuations define a multi-index filtration on the ring $\mathcal{O}_{\mathcal{X},o}$. Hence, they induce a Poincaré series.

Let $(\mathcal{S}, o)$ be a germ of an isolated rational surface singularity and let $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathcal{S}, o)$ be its minimal resolution. As the singularity is rational, the irreducible components $E_j$, $j \in J$, of the exceptional divisor $\mathcal{D}$ are isomorphic to the projective line $\mathbb{P}^1_{\mathbb{C}}$ (see for example [Li1]). Write $\mathcal{V}$ for the set $\{\nu_1, \cdots, \nu_r\}$ of valuations on $\mathcal{O}_{\mathcal{S},o}$ that are induced by the irreducible components $E_j$, $j \in J$.

Campillo, Delgado and Gusein-Zade computed the Poincaré series induced by the multi-index filtration defined by these valuations. Before giving the formula we introduce some notation. Let $S_{\mathcal{V}}$ be the semigroup of values of $\mathcal{S}$ with respect to $\mathcal{V}$. If $-k_j$ is the self-intersection number $E_j \cdot E_j$, $j \in J$, then Artin showed - see [Ar] - that $S_{\mathcal{V}}$ is the set of integer points $\underline{v} := (v_1, \cdots, v_r) \in \mathbb{Z}^r_{\geq 0}$ belonging to the cone that is defined by the inequalities

$$k_j v_j - \sum_{i: E_i \cap E_j = pt} v_i \geq 0, \qquad 1 \leq j \leq r.$$

In [P] it is shown that this cone is contained in the positive quadrant. Put $n_j$ the left hand side of the inequality and $s_j$ the number of indices $i$ for which $E_i$ and $E_j$ intersect. Let $\chi_{s,n}$ be the Euler characteristic of the $n$-th symmetric power of ($\mathbb{P}^1_{\mathbb{C}}$ without $s$ points). Then

$$P(\underline{t}) = \sum_{\underline{v} \in S} \left( \prod_{j=1}^{r} \chi_{s_j, n_j} \right) \underline{t}^{\underline{v}}.$$

# Chapter 2

# Toric geometry

## 2.1 Toric varieties

Toric varieties appear for the first time in the early seventies. In these days they were called torus embeddings. At first stage they were studied as compactifications of tori, what explains the terminology of torus embeddings. However, toric varieties are of very broad interest and that could maybe be neglected when using the older terminology. Ash, Mumford, Rapoport, Tai [A,M,R,T], Brylinski [Br], Danilov [Da], Demazure [Dem], Jurkiewitz [J], Kempf, Knudsen, Mumford, Saint-Donat [Ke,Kn,M,S], Miyake, Oda [Mi,O], Sturmfels [St], [Mill,St], Teissier [Te] and others made great contributions to the theory of toric varieties.

Toric varieties are relatively concrete algebraic varieties. The normal toric varieties are all Cohen-Macaulay, their singularities are rational, the Nash conjecture is true for them, etc. Their combinatorial character makes them very nice objects to work with. This explains why they are often used to provide examples to discover phenomena. Lots of tools in algebraic geometry can be translated to a combinatorial world from which one can pass again to the algebraic geometric side. It will mostly be in this way that we will use the bridge between both mathematical fields. On the other hand, for example for counting lattice points in polyhedra one walks in the other direction. Examples of the comfort that toric varieties offer can be found in cohomology and in resolution of singularities. Further on we will see the last one illustrated.

We refer to [O1] for more details and proofs.

There are different ways to define toric varieties. Classically they are introduced as follows.

**2.1 Definition** A *toric variety* of dimension $d$ is an irreducible variety $X$ which contains $T := (\mathbb{C}^*)^d$ as a Zariski open subset and such that the action of $(\mathbb{C}^*)^d$ on itself extends to an action on $X$.

**2.2 Example** $(\mathbb{C}^*)^d$ and $\mathbb{C}^d$ are clearly toric varieties. $\square$

**2.3 Example** Let $x_0, x_1, \cdots, x_d$ be homogeneous coordinates on $\mathbb{P}^d$. Via the map

$$\begin{array}{ccc} (\mathbb{C}^*)^d & \longrightarrow & \mathbb{P}^d \\ (t_1, \cdots, t_d) & \longmapsto & (1 : t_1 : \cdots : t_d) \end{array}$$

we identify $(\mathbb{C}^*)^d$ with the Zariski open subset $\mathbb{P}^d \setminus V(x_0 x_1 \cdots x_d)$ of $\mathbb{P}^d$. Considering the action $(t_1, \cdots, t_d) \cdot (x_0 : x_1 : \cdots : x_d) := (x_0 : t_1 x_1 : \cdots : t_d x_d)$, it follows that $\mathbb{P}^d$ is a toric variety. $\square$

Another way of looking at normal toric varieties is via fans. This combinatorial structure is extremely useful.

Let $M$ be a lattice and let $N$ be the dual space to $M$, say $N \cong \mathbb{Z}^d$. Then there is a natural bilinear map $M \times N \to \mathbb{Z} : (m, n) \mapsto \langle m, n \rangle$. Denote $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. A *rational finite polyhedral cone* $\sigma \subset N_{\mathbb{R}}$ is a cone generated by finitely many elements of $N$, so there exist $n_1, \cdots, n_r$ in $N$ such that

$$\sigma = \{\lambda_1 n_1 + \cdots + \lambda_r n_r \mid \lambda_1, \cdots, \lambda_r \in \mathbb{R}_{\geq 0}\}.$$

A rational finite polyhedral cone $\sigma$ is said to be *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$. We will just say *cone* to refer to a rational strongly convex finite polyhedral cone. The *dual cone* $\check{\sigma} \subset M_{\mathbb{R}}$ to $\sigma$ is defined as the set

$$\{m \in M \mid \langle m, x \rangle \geq 0, \forall x \in \sigma\}.$$

A *face* $\tau$ of $\sigma$ is an intersection $\{l = 0\} \cap \sigma$, where $l$ is a linear form which is nonnegative on $\sigma$. The *dimension* of $\sigma$ is the dimension of the smallest subspace of $N_{\mathbb{R}}$ containing $\sigma$.

*Gordan's Lemma* states that the semigroup $\overline{S} := \check{\sigma} \cap M$ is finitely generated. We then consider $\mathbb{C}[\overline{S}]$ as the $\overline{S}$-graded algebra $\oplus_{s \in \overline{S}} \mathbb{C}\chi^s$ and we denote the corresponding normal variety Spec $\mathbb{C}[\overline{S}]$ by $X_\sigma$.

A *fan* $\Sigma$ is a set of cones in $N_{\mathbb{R}}$ such that each face of a cone in $\Sigma$ is also a cone in $\Sigma$ and such that the intersection of two cones in $\Sigma$ is a face of each. From a fan $\Sigma$ the variety $X_\Sigma$ is obtained from the affine varieties $X_\sigma$, $\sigma$ in $\Sigma$, by gluing $X_\sigma$ and $X_\tau$ along the common open subvariety $X_{\sigma\cap\tau}$ for all $\sigma$ and $\tau$ in $\Sigma$.

If the torus $T = (\mathbb{C}^*)^d$ is mapped into the $d$-dimensional toric variety $X_\Sigma$ via the map $\phi$, then define for $\underline{b} := (b_1, \cdots, b_d) \in \mathbb{Z}^d$ the 1-*parameter subgroup* $u_{\underline{b}}$ as

$$\begin{array}{rcl} u_{\underline{b}} : \mathbb{C}^* & \longrightarrow & X_\Sigma \\ t & \longmapsto & \phi(t^{b_1}, \cdots, t^{b_d}). \end{array}$$

There is a one-to-one correspondence between the cones $\sigma$ of $\Sigma$ and the orbits of the torus action on $X_\Sigma$: an orbit $O$ corresponds to a cone $\sigma$ if and only if $\lim_{t\to 0} u_{\underline{b}}(t)$ exists and lies in $O$ for all $\underline{b}$ in the interior of $\sigma$. For a cone $\sigma$ and its corresponding orbit $orb(\sigma)$ holds that $\dim(\sigma)+\dim(orb(\sigma)) = d$.

The connection between fans and normal toric varieties is stated in the following result.

**2.4 Theorem** *Let $T_N$ be the torus given by the lattice $N$. There exists a bijection between the fans in $N_{\mathbb{R}}$ and the normal toric varieties that contain the torus $T_N$ as an open dense set.*

This correspondence can be extended to non-normal affine toric varieties. These are namely exactly the varieties of the form Spec $\mathbb{C}[S]$ with $S$ a strict semigroup in $\check{\sigma} \cap M$ that generates $\check{\sigma}$ as cone (see [O1]). Also in this case $S$ is finitely generated ([Sta]).

**2.5 Example** The cuspidal cubic $C := V(x^3 - y^2) \subset \mathbb{C}^2$ contains $\mathbb{C}^*$ via the map $t \mapsto (t^2, t^3)$ and $\mathbb{C}^*$ acts on $C$ via $t \cdot (x, y) := (t^2 x, t^3 y)$. Thus $C$ is an affine toric variety. The curve $C$ can also be written as Spec $\mathbb{C}[t^2, t^3]$. Hence the semigroup $S := \langle 2, 3\rangle \subset \mathbb{Z}$ is not saturated (a semigroup $S$ is *saturated* if one has for all $s \in S$ and for all $n \in \mathbb{Z}_{>0}$ that if $ns \in S$, then also $s \in S$) and the curve $C$ is not normal. $\square$

**2.6 Example (Matsumura)** The above correspondence is not true for non-normal toric varieties in general. The rational curve with a node obtained by identifying the origin and the point at infinity of the projective line $\mathbb{P}^1$ is a counterexample. Indeed, if the node would lie in an open affine toric variety, then this open set would meet both orbits of the rational curve. This means that the open set would be the whole rational curve which is not affine. □

## 2.2 Toric resolution of singularities

In 1964 Hironaka published the first proof that shows the existence of resolution of singularities in characteristic 0. He even shows that a resolution is possible allowing only centra of blowing-ups that are smooth and reduced. For toric varieties, when admitting centra of blowing-ups that are not necessarily smooth, reduced or orbits, one can visualise the resolution by means of the fan that describes the toric variety.

A cone is said to be *smooth* - also called *regular* or *simple* -, respectively *simplicial* if it is generated by a subset of a basis of $\mathbb{Z}^d$, respectively of $\mathbb{R}^d$. For a fan $\Sigma$ it holds that $X_\Sigma$ is smooth if and only if every $\sigma$ in $\Sigma$ is smooth. Let $\Sigma$, respectively $\Sigma'$, be a fan in $N$, respectively $N'$. Let $\phi : N'_\mathbb{R} \to N_\mathbb{R}$ be the map induced by the homomorphism of lattices $N' \to N$. If for every cone $\sigma'$ in $\Sigma'$ there exists a cone $\sigma$ in $\Sigma$ for which $\phi(\sigma') \subset \sigma$, then $\phi$ induces a morphism $\Phi : X_{\Sigma'} \to X_\Sigma$. Denote $|\Sigma|$ for the support of the toric variety $X_\Sigma$. The map $\Phi$ is proper if and only if $\phi^{-1}(|\Sigma|) = |\Sigma'|$. Suppose that $\Sigma'$ is a *refinement* or *subdivision* of $\Sigma$, i.e. each cone of $\Sigma$ is a union of cones in $\Sigma'$, then the morphism $X_{\Sigma'} \to X_\Sigma$ induced by the identity map on $N$ is birational (it is an isomorphism on the open torus $T_N$ contained in each) and proper. Remember that a projective birational morphism corresponds to the blowing-up in an ideal and vice versa. Thanks to the following theorem we can use the technique of subdividing fans to resolve singularities of toric varieties.

**2.7 Theorem** *For any toric variety $X_\Sigma$ there exists a refinement $\Sigma'$ of $\Sigma$ such that $X_{\Sigma'} \to X_\Sigma$ is a resolution of singularities.*

In particular, the resulting resolution is *equivariant*, i.e. $\pi(t{\cdot}x) = \pi(t)\pi(x)$, for all $t$ in the embedded torus $T$ in $X_{\Sigma'}$ and for all $x \in X_{\Sigma'}$.

Given a fan $\Sigma$ with associated normal toric variety $X_\Sigma$. For $\sigma \in \Sigma$, the primitive elements $n_1, \cdots, n_s$ of $N$ which lie on the rays of $\sigma$, are called the *fundamental generators* of $\sigma$. For $\tau \in \Sigma$ containing $\sigma$ as a face

- denoted $\tau > \sigma$ - and for $1 \leq i \leq s$, set $\tau_i$ the cone obtained from $\tau$ by replacing one of the fundamental generators $n_i$ of $\sigma$ by $n_0 := n_1 + \cdots + n_s$ and without changing the other fundamental generators. For successive cones $\sigma$, one can iterate this procedure. The minimal regular subdivision (i.e. all the cones in the fan are regular) obtained by proceeding in this way is called the *star subdivison* and thus gives a resolution of singularities. If $X_\Sigma$ is smooth, then the subdivision

$$\Sigma^* := (\Sigma \backslash \{\tau | \tau \in \Sigma, \tau > \sigma\}) \cup (\bigcup_{\tau \in \Sigma, \tau > \sigma} \{\varsigma | \varsigma \text{ face of } \tau_i \text{ for some } 1 \leq i \leq s\})$$

gives rise to the blowing-up of $X_\Sigma$ along the closed subvariety $\overline{\mathrm{orb}(\sigma)}$.

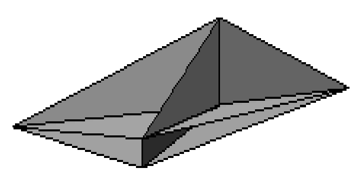
**2.8 Example** Consider the cone

$$\sigma := \langle (0,0,1), (1,0,-3), (0,1,-3), (1,1,-8) \rangle \subset \mathbb{R}^3.$$

This cone corresponds to the toric affine variety $X := V(xy^2 - zw^2) \subset \mathbb{C}^4$. Let $\Sigma_\sigma$ be the fan that contains exactly the faces of $\sigma$. A regular subdivision $\Sigma'$ of $\Sigma_\sigma$ is induced by adding the cones

$$\langle (0,0,1), (1,0,-3), (1,1,-7) \rangle, \langle (0,0,1), (0,1,-3), (1,1,-7) \rangle,$$
$$\langle (1,1,-8), (0,1,-3), (1,1,-7) \rangle, \langle (1,1,-8), (1,0,-3), (1,1,-7) \rangle$$

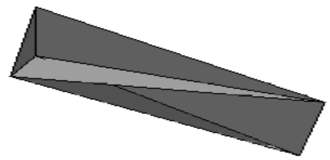
to the fan $\Sigma_\sigma$. Let $X_{\Sigma'}$ be the associated toric variety. As for a cone and its corresponding orbit holds that the sum of their dimensions equals $d$, it follows that the edge $(1,1,-7)$ corresponds to a divisor on $X_{\Sigma'}$. As this divisor is collapsed onto $X$, this resolution of singularities yields exactly one exceptional divisor. The newly introduced 2-dimensional cones in the fan $X_{\Sigma'}$ correspond to exceptional curves that are contained in the exceptional divisor.



Another resolution of singularities of $X$ is induced by adding the cones

$$\langle (0,0,1), (1,1,-8), (0,1,-3) \rangle \text{ and } \langle (0,0,1), (1,1,-8), (1,0,-3) \rangle.$$

This time there is no exceptional divisor created.



Note also that the star subdivision does not give the shortest way to obtain a resolution of singularities of $X$ in this example. $\square$

## 2.3 Valuations

Let $X := \text{Spec } \mathbb{C}[S]$ be a toric variety and let $\sigma \subset N_{\mathbb{R}}$ be such that $S$ generates $\check{\sigma}$ as cone. Without loss of generality, we assume that $M$ is also the group generated by the semigroup $S$. Let $\pi : X' \to X$ be an equivariant proper birational morphism. Let $D$ be an irreducible codimension 1 subvariety of the normal variety $X'$ and let $\nu_D$ be the induced valuation on $\mathbb{C}(X)$. The divisor $D$ defines the element $n_D$ of $N$ for which:

$$\langle m, n_D \rangle = \nu_D(\chi^m), \text{ for all } m \in M.$$

As $D$ is irreducible, $n_D$ is a primitive element in $\sigma \cap N$. Vice versa a primitive element $n$ in $\sigma \cap N$ defines a discrete valuation $\nu$ on $\mathbb{C}(X)$ by setting

$$\nu(\sum_{m \in F} a_m \chi^m) := min\{\langle m, n \rangle \mid m \in F, a_m \neq 0\},$$

where $F$ is a finite subset of $S$. In fact, one has $\nu = \nu_D$ for some $D$ in $X'$ for an appropriated $\pi : X' \to X$. Such valuations are usually called *monomial valuations*. For given monomial valuations $\nu_1, \cdots, \nu_r$ corresponding to primitive elements in $\sigma$ and for $\underline{v} = (v_1, \cdots, v_r) \in \mathbb{Z}^r$, the ideals

$$I(\underline{v}) = \{g \in \mathbb{C}[S] \mid \nu_i(g) \geq v_i, 1 \leq i \leq r\}$$

are monomial ideals, i.e. generated by elements of type $\chi^m$ for elements $m$ in $S$.

For an irreducible divisor $D$ on $X'$, the centre of the valuation $\nu_D$ in $X$ is the closure of the orbit that is associated to the unique face $\tau$ of $\sigma$ such that $\mathring{\tau}$ contains $n_D$.

**2.9 Example** Let $S := \mathbb{Z}^3_{\geq 0}$, so $X = \mathbb{C}^3$. We choose primitive elements $n_1 := (1,5,6)$ and $n_2 := (2,4,3)$ in $\sigma \cap N = \mathbb{Z}^3_{\geq 0}$. We denote the valuations corresponding to $n_1$ and $n_2$ by $\nu_1$ and $\nu_2$. It is clear that the ideals $I((v_1,v_2))$ with respect to $\nu_1$ and $\nu_2$ are monomial ideals, i.e. generated by monomials. One has

$$I((v_1,v_2)) = (x^a y^b z^c \mid a,b,c \in \mathbb{Z}_{\geq 0}, a+5b+6c \geq v_1 \text{ and } 2a+4b+3c \geq v_2).$$

For example (see also Example 3.3 and following lines)

$$\begin{aligned}
&I((6,6)) = (x^6, y^2, z^2, yz, xy, x^2z) \quad \text{and} \\
&I((-2,12)) = \\
&(x^6, y^3, z^4, x^4y, x^2y^2, x^5z, x^3z^2, x^2z^3, yz^3, y^2z^2, xyz^2, xy^2z, x^3yz).
\end{aligned}$$

$\square$

# Chapter 3

# Clusters of infinitely near points

## 3.1 Introduction

In the beginning of the twentieth century, Enriques developed a theory of infinitely near points to describe the geometry of plane curve singularities. A point $Q$ is said to be infinitely near to a point $P$ if it is contained in the preimage of the blowing-up in $P$. Together with Chisini he studied systems of plane curves which pass with assigned multiplicities through a fixed set of infinitely near points of a point of the plane ([En,Ch]). They proved that there exist such curves if and only if some inequalities (the so called 'proximity inequalities') are satisfied.

In the thirties Zariski formulated this theory in terms of local rings, valuation rings, valuation ideals and complete ideals, see [Z]. One of the main results in Zariski's theory is that any complete ideal in a two-dimensional regular local ring has a unique factorisation into simple (i.e. not the completion of the product of two proper ideals) complete ideals. The set of exponents which appear in the factorisation can be seen from the proximity inequalities for the linear system corresponding to the complete ideal. They are equal to the differences of both sides in these inequalities.

Later on, in the sixties, Lipman studied divisors with exceptional support. He extended the results of Enriques and Chisini and the theory by Zariski for rational surface singularities ([Li1]). The exponents in the unique factorisation were now rational numbers. In the eighties, he was able to recover a unique factorisation theorem in higher dimensions, see [Li2] and [Li3]. The exponents could now be negative. The result holds for finitely supported complete ideals in a regular local ring of any dimension. These ideals are supported at the closed point and there exists a finite sequence of blowing-ups in points that makes the ideal invertible.

In the eighties Casas described the singularities of the polar curve of a plane curve by studying the infinitely near points in which the irreducible components of the polar curve pass. The problem became equivalent to the decomposition of the completion of the Jacobian ideal in a product of powers of simple ideals, see [Ca1].

In the nineties Campillo, Gonzalez-Sprinberg and Lejeune-Jalabert came up with some new nice results ([C,G-S,L-J]). Of highest interest for us is their fully combinatorial characterisation of these systems in the case that the infinitely near points are 0-dimensional orbits in smooth toric varieties. Secondly, they provide a natural embedded resolution for whole sets of hypersurfaces, namely for those that are general in a finitely supported complete ideal.

In this chapter we introduce the terminology of infinitely near points, clusters etc. according to [C,G-S,L-J] and we comment the results that are of main interest for us.

## 3.2 Clusters and finitely supported complete ideals

Let $X$ be a nonsingular variety of dimension $d \geq 2$ and let $Z$ be a variety obtained from $X$ by a finite succession of point blowing-ups. A point $Q \in Z$ is said to be *infinitely near* to a point $P \in X$ if $P$ is in the image of $Q$; we write $Q \geq P$. A *constellation* is a finite sequence $\mathcal{C} := \{Q_0, Q_1, \cdots, Q_{r-1}\}$ of infinitely near points of $X$ such that $Q_0 \in X =: X_0$ and each $Q_j$ is a point on the variety $X_j$ obtained by blowing up $Q_{j-1}$ in $X_{j-1}$, $j \in J := \{1, \cdots, r\}$.

The relation '$\geq$' gives rise to a partial ordering on the points of a constellation. In the case that they are totally ordered, so $Q_{r-1} \geq \cdots \geq Q_0$, the constellation $\mathcal{C}$ is called a *chain*. For every $Q_j$ in $\mathcal{C}$, the subsequence $\mathcal{C}^j := \{Q_i \mid Q_j \geq Q_i\}$ of $\mathcal{C}$ is a chain. The integer $l(Q_j) := \#\mathcal{C}^j - 1$ is called the *level* of $Q_j$. In particular $Q_0$ has level 0. If no other point of

$\mathcal{C}$ has level 0 then $Q_0$ is called the *origin* of $\mathcal{C}$. We will always work with constellations that have an origin and we often will denote the origin of the constellation by $o$. If $Q_j \geq Q_i$ and $l(Q_j) = l(Q_i) + 1$, we will write $Q_j \succ Q_i$ or $j \succ i$.

For each $Q_i \in \mathcal{C}$, denote the exceptional divisor of the blowing-up in $Q_i$ by $B_i$ and its strict transform, respectively total transform at some intermediate stage (including the final stage) $X_j$, $i+1 \leq j \leq r-1$, by $E_i$, respectively $E_i^*$. If $Q_j \in E_i$, then one says that $Q_j$ is *proximate* to $Q_i$. This will be denoted as $Q_j \to Q_i$ or $j \to i$. As

$$E_i = E_i^* - \sum_{j \to i} E_j^*,$$

it follows that also $\{E_0^*, \cdots, E_{r-1}^*\}$ is a basis of the group of divisors with exceptional support $\oplus_{j=0}^{r-1} \mathbb{Z} E_j$.

A pair $\mathcal{A} := (\mathcal{C}, \underline{m})$ consisting of a constellation $\mathcal{C} := \{Q_0, \cdots, Q_{r-1}\}$ and a sequence $\underline{m} := (m_0, \cdots, m_{r-1})$ of nonnegative integers is called a *cluster*. One calls $m_j$ the *weight* of $Q_j$ in the cluster and we write $D(\mathcal{A}) := \sum_{j=0}^{r-1} m_j E_j^*$. Introducing the numbers $v_j$, $0 \leq j \leq r-1$, by setting

$$m_j := v_j - \sum_{j \to i} v_i,$$

allows us to write also $D(\mathcal{A}) = \sum_{j=0}^{r-1} v_j E_j$.

The idea of clusters is to express that a system of hypersurfaces is passing through the points of the constellation with (at least) the given multiplicities. Let $o$ be a point of $X$. Remember that the ideals of the form $I(\underline{v}) = \{g \in \mathcal{O}_{X,o} \mid \underline{\nu}(g) \geq \underline{v}\}$ are exactly the complete ideals (see Section 1.2). If we want that these ideals principalise by blowing up the points of the constellation, we require the ideals to be finitely supported. Formally, an ideal $I$ in $\mathcal{O}_{X,o}$ is called *finitely supported* if $I$ is primary for the maximal ideal $\mathsf{m}$ of $\mathcal{O}_{X,o}$ - so supported at the closed point - and if there exists a constellation $\mathcal{C}$ of infinitely near points of $X$ such that $I\mathcal{O}_{X(\mathcal{C})}$ is an invertible sheaf where $X(\mathcal{C})$ is the modification space, also called the *sky*. Remember that the invertible sheaves correspond bijectively to the Cartier divisors on $X(\mathcal{C})$. An infinitely near point $Q$ of $o$ is a *base point* of $I$ if $Q$ belongs to the constellation with the minimal number of points with the above property. We denote the constellation of base points of the finitely supported ideal $I$ by $\mathcal{C}_I$.

Let $I$ be an ideal in a local ring $(\mathcal{O}_{X,Q}, \mathsf{m})$, then the *order* $\mathrm{ord}_Q I$ *of* $Q$ *at* $I$ is defined as $\max\{n \mid I \subset \mathsf{m}^n\}$. One associates a cluster $\mathcal{A}_I := (\mathcal{C}_I, \underline{m})$ to $I$ as follows: for $j \in \{0, \cdots, r-1\}$, let $Q_j^-$ be the point in the constellation such that $Q_j \succ Q_j^-$ and denote its weight by $m_j^-$. The weight $m_j$ of $Q_j$ is defined by induction by setting:

1. $m_0 := \mathrm{ord}_{Q_0} I_{Q_0}$ with $I_{Q_0} := I$;
2. $m_j := \mathrm{ord}_{Q_j} I_{Q_j}$ with $I_{Q_j} := (x)^{-m_j^-} I_{Q_j^-} \mathcal{O}_{X,Q_j}$,

where $x$ is a generator of the principal ideal $\mathsf{m}_{Q_j^-}\mathcal{O}_{X,Q_j}$. Then the ideal sheaf $I\mathcal{O}_{X(\mathcal{C}_I)}$ is associated to $-D(\mathcal{A}_I)$.

**3.1 Example** Let us consider the ideal $I := (x^2y, x^2 - y^2) \subset \mathbb{C}[x, y]$. This ideal will turn out to be finitely supported. Via a principalisation we will find the cluster $\mathcal{A}_I$ associated to $I$.

We blow up in the origin $Q_0$ of $\mathbb{C}^2$ and we look in two affine charts to see the whole picture, we simply call them chart 1 and chart 2. The order of $Q_0$ in $I$ is $m_0 = 2$. We have $I_{Q_1} = (xy, 1 - y^2)$ in chart 1 where $Q_1$ has coordinates $(0, 1)$ in that affine chart. Note that the origin in chart 2 has not to be blown up so we can restrict ourselves to chart 1 to complete the principalisation. Doing the translation $y' := 1 - y$, makes $Q_1$ to be the origin of chart 1. Then $I_{Q_1} = (x - xy', 2y' - y'^2)$ and $m_1 = 1$. Blowing up in $Q_1$ gives rise to the ideal $(1 - xy', 2y' - xy'^2) = (1)$ in chart 1.1 and to $I_{Q_2} = (x - xy', 2 - y') = (x, 2 - y')$ in chart 1.2. Setting $y'' := 2 - y'$, we see that $m_2 = 1$. We blow up in $Q_2$ and we obtain a principal transform of $I$ in each affine chart, so $I$ is a finitely supported ideal and its constellation of base points consists of $\{Q_0, Q_1, Q_2\}$. The level of $Q_0$ is 0 and of the other two base points it is 1. We have $(m_0, m_1, m_2) = (2, 1, 1)$.
In chart 1.1 the equation of the corresponding Cartier divisor is $x^3$, in chart 1.2.1 it is $x^3$ and in chart 1.2.2 it is $x^2y^3$.

$\square$

If $\mathcal{C}$ is a constellation with origin at $o$, the cluster $\mathcal{A} := (\mathcal{C}, \underline{m})$ is called *idealistic* if there exists a finitely supported ideal $I$ in $\mathcal{O}_{X,o}$ such that $I\mathcal{O}_{X(\mathcal{C})}$ is the ideal sheaf associated to $-D(\mathcal{A})$. We call *galaxy* of $\mathcal{C}$ the set of idealistic clusters on $\mathcal{C}$. For an idealistic cluster $\mathcal{A}$, Lipman proved that there exists a unique finitely supported complete ideal $I_{\mathcal{A}}$ such that $I_{\mathcal{A}}\mathcal{O}_{X(\mathcal{C})} = \mathcal{O}_{X(\mathcal{C})}(-D(\mathcal{A}))$, namely that given by the direct image of $\mathcal{O}_{X(\mathcal{C})}(-D(\mathcal{A}))$ at $X$, see [Li2]. Note that the galaxy of a constellation is a semigroup. Indeed, if $\mathcal{A}_1 := (\mathcal{C}, \underline{m_1})$ and $\mathcal{A}_2 := (\mathcal{C}, \underline{m_2})$ are idealistic clusters, then also $\mathcal{A} := (\mathcal{C}, \underline{m_1} + \underline{m_2})$ is idealistic. One has $I_{\mathcal{A}} = I_{\mathcal{A}_1} * I_{\mathcal{A}_2}$, where $*$ denotes the completion of the product of ideals.

Zariski formulated the questions whether this semigroup generates a regular cone and how to characterise this semigroup. In dimension 2 these questions were totally answered by Chisini and Enriques. They showed that the semigroup was regular and they gave a combinatorial determination of the semigroup. In the following section, we introduce toric clusters and comment the answers to these questions of Zariski.

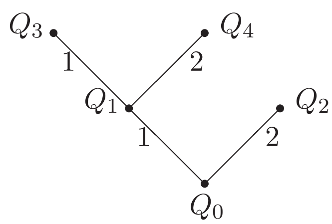
## 3.3 Toric clusters

Let $M$ be a $d$-dimensional lattice ($d \geq 2$), $N$ its dual lattice and $\sigma$ a regular cone in $N_{\mathbb{R}}$. Consider the smooth affine toric variety $X :=$ Spec $\mathbb{C}[\check{\sigma} \cap M]$.

A $d$-dimensional *toric constellation* of infinitely near points with origin $Q_0$ is a constellation $\mathcal{C} := \{Q_0, Q_1, \cdots, Q_{r-1}\}$ such that each $Q_j$ is a 0-dimensional orbit in the toric variety $X_j$ obtained by blowing up $Q_{j-1}$ in $X_{j-1}$, $1 \leq j \leq r-1$. In Section 2.2 we saw that blowing up in orbits of smooth varieties corresponds to making star subdivisions of the fan corresponding to the variety. In this way each blowing-up in a 0-dimensional orbit induces the creation of $d$ cones of dimension $d$ and thus of $d$ new 0-dimensional orbits. Hence, the choice of a point $Q_i$ in a toric chain is equivalent to the choice of an integer $a_i \in \{1, \cdots, d\}$, which determines a $d$-dimensional cone in the fan.

A tree with a root such that each vertex has at most $d$ following adjacent vertices is called a *$d$-ary tree*. The above observation makes that toric constellations can be represented in a nice way. It shows namely that there is a natural bijection between the set of $d$-dimensional toric constellations with origin and the set of finite $d$-ary trees with a root, with the edges labeled with positive integers not greater than $d$, such that two edges with the same source have different labels.

**3.2 Example**




Suppose $d = 3$ and $\mathcal{C}$ is the constellation pictured at the left. It represents the following resolution process: by blowing up in the origin $Q_0$ we get an exceptional variety $B_0 \cong \mathbb{P}^2$. In $B_0$ there are two points in which we blow up, namely $Q_1$ and $Q_2$. The labels indicate in which affine chart the points of the constellation are created. For example the point $Q_1$ is the origin of the affine chart induced by the edge going out of $Q_0$ with label 1, we shortly denote this affine chart by 1. After blowing up in $Q_1$ we get an exceptional variety $B_1 \cong \mathbb{P}^2$, where again we blow up in two points. The point $Q_3$ is the origin of the affine chart denoted by 1.1 and $Q_4$ is the origin of the affine chart 1.2.

□

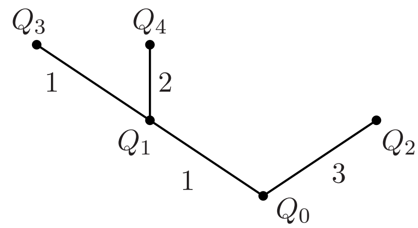
A cluster $\mathcal{A} := (\mathcal{C}, \underline{m})$ is called *toric* if the constellation $\mathcal{C}$ is toric. Let us have a look at the finitely supported ideals in the case of toric clusters. We refer to [Ke,Kn,M,S] for the proofs.

As we want the ideals to be supported in the 0-dimensional orbit, they should be invariant under the action of the torus and thus be monomial. To a monomial ideal $I$ one can associate a Newton polytope $\mathcal{N}$. It is the union of the compact faces of the convex hull of $m + \check{\sigma}$ as $m$ runs through the set of exponents of monomials in $I$. The facets of the Newton polytope correspond with the Rees valuations of $I$, i.e. the valuations induced by the irreducible components of the exceptional divisor of the normalised blowing-up $\overline{Bl_I X}$ of $I$ (see also Section 1.2). Furthermore, a monomial ideal is complete if and only if it contains every monomial whose exponent is a point of $(\mathcal{N} + \check{\sigma}) \cap M$.

Given a toric idealistic cluster $\mathcal{A}$, we can now find the unique finitely supported complete monomial ideal $I_{\mathcal{A}}$ associated to it. We show by the following example how one can do that.

### 3.3 Example

Consider the following toric cluster $\mathcal{A}$ in $\mathbb{C}^3$



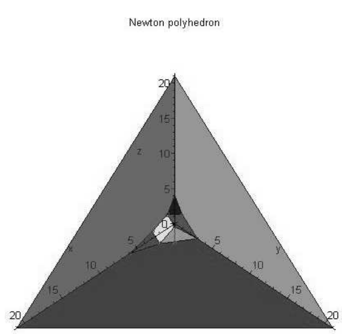

with

$$(m_0, m_1, m_2, m_3, m_4) = (3, 2, 1, 1, 1) \text{ or } (v_0, v_1, v_2, v_3, v_4) = (3, 5, 4, 6, 9).$$

We first determine the Rees valuations $\nu_j$, $0 \leq j \leq 4$, associated to the points $Q_j$ of the constellation. The order of a monomial $x^a y^b z^c$ along the first exceptional component $E_0$ is $a + b + c$. When blowing up in $Q_0$, we get a new point in chart 1 in which we blow up, namely $Q_1$. The total transform of $x^a y^b z^c$ in chart 1 is $x^{a+b+c} y^b z^c$. The order of $x^a y^b z^c$ along the second exceptional component $E_1$ is $a+2b+2c$. In this way we obtain that the Rees valuations are represented by the following vectors in the lattice $\mathbb{N}^3 = \sigma$:

$$\nu_0 \leftrightarrow (1,1,1) \quad \nu_1 \leftrightarrow (1,2,2) \quad \nu_2 \leftrightarrow (2,2,1) \quad \nu_3 \leftrightarrow (1,3,3) \quad \nu_4 \leftrightarrow (2,3,4).$$

Saying that a monomial $x^a y^b z^c$ passes through $Q_j$ is saying that $\nu_j(x^a y^b z^c) \geq v_j$, for $0 \leq j \leq 4$. The induced hyperplanes define a Newton polytope. We picture the Newton polyhedron $\mathcal{N} + \check{\sigma}$.




Now let $I_{\mathcal{A}}$ be the ideal generated by the monomials whose exponents are in this Newton polyhedron. $\square$

In the case of toric clusters, the galaxy can be completely combinatorially characterised. We first introduce the notion of 'linearly proximate' to formulate the characterisation.

Fix a point $Q$ in a toric constellation $\mathcal{C}$ of dimension $d$ and some non-negative integers $a, b$ and $t$ such that $1 \leq a, b \leq d$, $a \neq b$. Let $Q(a, b^t)$ be the terminal point of the chain with origin $Q$ coded by $(a, b, \cdots, b)$ where $b$ appears $t$ times. If $t = 0$, it is denoted by $Q(a)$. The point $Q(a, b^t)$ may not belong to $\mathcal{C}$. A point $P$ that is infinitely near to $Q$ is said to be *linearly proximate* to $Q$, if $P = Q(a, b^t)$, with $a, b$ and $t$ as above. We denote this relation by $P \twoheadrightarrow Q$. Then we have that $P$ is linearly proximate to $Q$ if and only if there exists a 1-dimensional orbit $\mathsf{l}$ in $B_Q$ such that $P$ belongs to the strict transform of the closure of $\mathsf{l}$ in $E_Q$. This explains the terminology.

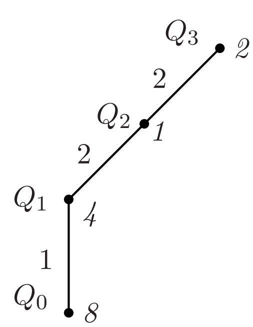
Denote $M_Q(a, b) := \sum_{t \geq 0} m_{Q(a,b^t)}$. Campillo, Gonzalez-Sprinberg and Lejeune-Jalabert show in [C,G-S,L-J] that a toric cluster $\mathcal{A}$ is idealistic if and only if for each point $Q$ of the constellation $\mathcal{C}$ and for each pair of integers $a$ and $b$ such that $a, b \in \{1, \cdots, d\}$, the following inequality is satisfied:

$$M_Q(a, b) + M_Q(b, a) \leq m_Q.$$

These inequalities are called the *linear proximity inequalities.* They generalise in dimension greater than 2 for toric clusters the inequalities that Enriques and Chisini obtained in dimension 2.

### 3.4 Example

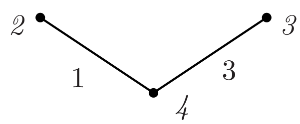
Consider the cluster $\mathcal{A}_1$




For example, the points with weights $Q_1, Q_2$ and $Q_3$ are all linearly proximate to the origin. This cluster does not satisfy the linear proximity inequalities (for example $M_{Q_2}(2, 1) + M_{Q_2}(1, 2) \leq m_{Q_2}$ is not satisfied).

However, there exists another cluster $\mathcal{A}_2$, namely the one with weights $(8,4,2,1)$ such that $I_{\mathcal{A}_1} = I_{\mathcal{A}_2}$. This can be seen by drawing the Newton polytope associated to the cluster. This new cluster is idealistic, hence $I_{\mathcal{A}_1}$ is finitely supported . $\square$

Given $\underline{v} \in \mathbb{Z}^r$, the previous example shows that $I(\underline{v})$ being finitely supported does not imply that $\underline{v}$ determines an idealistic cluster. It is an interesting question whether $I(\underline{v})$ is finitely supported for a given $\underline{v}$ in $\mathbb{Z}^r_{\geq 0}$. In dimension 2, for each $\underline{v} \in \mathbb{Z}^r$ one can find a vector $\underline{v'} \in \mathbb{Z}^r$ such that $I(\underline{v}) = I(\underline{v'})$ and such that the cluster determined by the divisor $\sum_{j=0}^{r-1} v'_j E_j$ is idealistic. The algorithm of Laufer gives a method to find this $\underline{v'}$, see [La]. In higher dimensions there exist examples of clusters for which there does not exist an idealistic cluster that determines the same ideal. The cluster




in dimension 3 is such an example. By drawing the Newton polytope we find the ideal that is associated to the cluster; it is not finitely supported.

In [C,G-S,L-J] Campillo, Gonzalez-Sprinberg and Lejeune-Jalabert also answer the question of Zariski whether the cone defined by the galaxy is regular in the toric case. They show that the cone is regular if and only if:

1. the tree is binary and
2. if there exist $Q \in \mathcal{C}$ and labels $a, b, c$ such that $a \neq b$ and $\{Q(a), Q(b), Q(a,c)\} \subset \mathcal{C}$, then $b = c$.

## 3.4 Idealistic clusters and embedded resolutions

In [C,G-S,L-J] it is shown that the canonical map from the sky of the constellation of base points of a finitely supported ideal $I$ to $X$ is an embedded resolution of the subvariety of $(X, o)$ defined by $i$ general enough elements in $I$, $1 \leq i \leq d$. Moreover, it gives the minimal desingularisation for a complete intersection surface $S$ defined by a general $(d-2)$-uple with respect to a cluster of dimension $d$. It is a composition of point blowing-ups, namely of those $Q \in \mathcal{C}$ with $m_Q \neq 1$.

**3.5 Example** Let us have a look at the cluster in Example 3.3. This cluster is idealistic and from the Newton polyhedron we can write down the corresponding finitely supported complete ideal $I(\mathcal{A})$.

$$I(\mathcal{A}) = (x^6, y^3, z^4, x^3y, x^2y^2, yz^2, y^2z, x^3z, xz^2, xyz).$$

The blowing-up of the constellation gives an embedded resolution for $h(x, y, z) := x^6 + y^3 + z^4 + x^3y + x^2y^2 + yz^2 + y^2z + x^3z + xz^2 - xyz$ .□

## 3.5 Idealistic clusters - intersection theoretical aspects

In this section we mention some facts about idealistic clusters, expressed in intersection theory, that will be useful for us. For more information, see for example [C,G-S,L-J].

If $\mathcal{A} = (\mathcal{C}, \underline{m})$ is an idealistic cluster of Spec $R$, then its associated divisor $-D(\mathcal{A})$ is numerically effective, i.e. $-D(\mathcal{A}) \cdot C \geq 0$ for all exceptional curves $C$ on $X(\mathcal{C})$. Let $d$ be the dimension of Spec $R$. Kleiman showed that it then also holds, for $1 \leq k \leq d-1$, that

$$(-D(\mathcal{A}))^k \cdot V \geq 0$$

for every $k$-dimensional exceptional subvariety $V$ of $X(\mathcal{C})$ (see [Kl]).

Let us now fix a point $Q$ of a toric constellation $\mathcal{C}$ and let $a, b \in \{1, \cdots, d\}$, $a \neq b$. When taking for $V$ the strict transform of the 1-dimensional orbit through $Q(a)$ and $Q(b)$ in $B_Q$, the above inequalities imply the linear proximity inequalities. If $V = E_Q$, we get $m_Q^{d-1} \geq \sum_{P \to Q} m_P^{d-1}$ and $(-D(\mathcal{A}))^{d-1} \cdot V = 0$ if and only if $m_Q^{d-1} = \sum_{P \to Q} m_P^{d-1}$.

Let $\overline{Bl_{I_\mathcal{A}} R}$ be the normalised blow-up of the ideal $I_\mathcal{A}$, then the map from the sky $X(\mathcal{C})$ to Spec $R$ factorises by $\overline{Bl_{I_\mathcal{A}} R}$. Let $\sigma$ be the morphism $X(\mathcal{C}) \to \overline{Bl_{I_\mathcal{A}} R}$ in this factorisation. Then $(-D(\mathcal{A}))^k \cdot V = 0$ if and only if $V$ contracts, i.e. dim $\sigma(V) <$ dim $V$. Now let us consider the case when $V = E_j$ for some index $j$ of a point $Q_j$ of the constellation, $1 \leq j \leq r$. Then the induced valuation $\nu_j$ is Rees for $I_\mathcal{A}$ if and only if its centre in $\overline{Bl_{I_\mathcal{A}} R}$ is a divisor. As the centre of $\nu_j$ in $\overline{Bl_I R}$ is $\sigma(E_j)$, we now have

$$E_j \text{ contracts under } \sigma$$
$$\Updownarrow$$
$$\nu_j \text{ is not a Rees valuation for } I_\mathcal{A}.$$

# Chapter 4

# Poincaré series of a toric variety

## 4.1 Introduction

In Chapter 1 we introduced the Poincaré series that we study in this thesis. We investigate this Poincaré series for affine toric varieties. The method that we use for its computation is based on the fact that their coordinate ring is a graduated ring.

In [C,D,G-Z2], Campillo, Delgado and Gusein-Zade showed that the Poincaré series of a plane curve equals the zeta function of monodromy. For the proof they use the formula of A'Campo which expresses the zeta function of monodromy of a curve in terms of data of an embedded resolution for the curve, see [A'C]. Also for rational surface singularities they obtained an analogous formula for the Poincaré series ([C,D,G-Z8]). In Section 4.3, we generalise this formula and proof that it is equal to the Poincaré series of an affine toric variety.

As an interesting example, in Section 4.4 we study the case of divisorial valuations on the ring $\mathcal{O}_{\mathbb{C}^d,o}(d \geq 2)$ that are created by toric constellations.

## 4.2 Computation of the Poincaré series

Let $M$ be a lattice with dual space $N := \mathbb{Z}^d$ and let $\sigma \subset N \otimes_{\mathbb{Z}} \mathbb{R}$ be a rational finite polyhedral strongly convex $d$-dimensional cone. Write $\check{\sigma}$ for the dual cone to $\sigma$. We consider a semigroup $S$ in $\check{\sigma} \cap M$ that generates

$\check{\sigma}$ as cone and we denote the induced toric variety Spec $(\mathbb{C}[S])$ by $X$.

Let $\pi : X' \to X$ be an equivariant proper birational morphism, $X'$ being another toric variety and let $\mathcal{V} := \{\nu_1, \cdots, \nu_r\}$ be an arbitrary set of divisorial monomial valuations such that $\mathcal{V} \subset \overset{\circ}{\sigma}$. We compute the Poincaré series

$$P(t_1, \cdots, t_r) := \frac{\prod_{j=1}^{r}(t_j - 1) \sum_{\underline{v} \in \mathbb{Z}^r} \dim(I(\underline{v})/I(\underline{v} + \underline{1}))\underline{t}^{\underline{v}}}{(t_1 \cdots t_r - 1)}$$

of the affine toric variety $X$ with respect to $\mathcal{V}$.

In what follows $\underline{\nu}$ stands for the vector $(\nu_1, \cdots, \nu_r)$ and by $\langle s, \underline{\nu} \rangle$ we mean the vector $(\langle s, \nu_1 \rangle, \langle s, \nu_2 \rangle, \cdots, \langle s, \nu_r \rangle)$. We define the monomial cone $C$ as the set of values that can be obtained by valuating monomials, i.e. $C := \{\underline{\nu}(m) \mid m \text{ monomial in } \mathbb{C}[S]\}$.

One has a $\mathbb{Z}$-linear map $\phi : M \longrightarrow \mathbb{Z}^r$ given by $\phi(m) := \langle m, \underline{\nu} \rangle$. This map induces the following map among Laurent series groups

$$\begin{array}{rcl} \Phi : \mathbb{Z}[[M]] = \mathbb{Z}[[u_1, \cdots, u_d, u_1^{-1}, \cdots, u_d^{-1}]] & \longrightarrow & \mathbb{Z}[[\mathbb{Z}^r]] = \mathbb{Z}[[t_1, \cdots, t_r, t_1^{-1}, \cdots, t_r^{-1}]] \\ \sum_i \lambda_i \underline{u}^m & \longmapsto & \sum_i \lambda_i \underline{t}^{\langle m, \underline{\nu} \rangle}. \end{array}$$

Notice that for each commutative group $\Gamma$ the Laurent series $\mathbb{Z}[[\Gamma]]$ is in fact a $\mathbb{Z}[\Gamma]$-module, and that the assignment of this module to $\Gamma$ is functorial.

In particular, for our given semigroup $S$, one has the multi-graded Poincaré series of commutative algebra $Q(\underline{u}) := \sum_{s \in S} \underline{u}^s$ which is an element of $\mathbb{Z}[[S]]$ and so can be interpreted as an element in $\mathbb{Z}[[M]]$. In fact, this multi-graded Poincaré series is usually expressed as a rational function having $Q(\underline{u})$ as power series expansion.

The Poincaré series in geometry $P(\underline{t})$ is an element in $\mathbb{Z}[[\mathbb{N}^r]]$ and so also an element of $\mathbb{Z}[[\mathbb{Z}^r]]$.

We now compute the Poincaré series $P$ and we obtain a relation between the Poincaré series $P$ and $Q$.

**4.1 Theorem** *The Poincaré series $P$ defined by the multi-index filtration induced by $\mathcal{V}$ and associated to the multi-graduation of $S$, is the image under $\Phi$ of the Poincaré series of commutative algebra of the semigroup $S$, i.e.*

$$P(\underline{t}) = \Phi(Q(\underline{u})).$$

*Proof.* For a set $A \subset \{i_1, \cdots, i_s\}$, let $\alpha_A$ be the function

$$\begin{array}{rcl} \alpha_A : \mathbb{Z}^s & \longrightarrow & \mathbb{Z}^s \\ \underline{v} & \longmapsto & \underline{v}' \end{array}$$

where $v'_i := v_i - 1$ if $i \in A$ and $v'_i := v_i$ if $i \notin A$.
Then, for $\underline{v} \in \mathbb{Z}^r$, the coefficient of $\underline{t}^{\underline{v}}$ in $\prod_{j=1}^r (t_j - 1) L(t_1, \cdots, t_r)$ is

$$(-1)^r \sum_{A \subset \{1, \cdots, r\}} (-1)^{\#A} \dim \frac{I(\alpha_A(\underline{v}))}{I(\alpha_A(\underline{v}) + \underline{1})}$$

and the coefficient of $\underline{t}^{\underline{v}}$ in $P(\underline{t})$ is

$$\begin{array}{ll} (-1)^{r+1} \displaystyle\sum_{A \subset \{1, \cdots, r\}} (-1)^{\#A} \dim \frac{I(\alpha_A(\underline{v}))}{I(\alpha_A(\underline{v}) + \underline{1})} & + \\ (-1)^{r+1} \displaystyle\sum_{A \subset \{1, \cdots, r\}} (-1)^{\#A} \dim \frac{I(\alpha_A(\underline{v}) - \underline{1})}{I(\alpha_A(\underline{v}))} & + \\ (-1)^{r+1} \displaystyle\sum_{A \subset \{1, \cdots, r\}} (-1)^{\#A} \dim \frac{I(\alpha_A(\underline{v}) - \underline{2})}{I(\alpha_A(\underline{v}) - \underline{1})} & + \quad \cdots \end{array}$$

This finite sum can be rewritten as

$$(-1)^{r+1} \sum_{A \subset \{1, \cdots, r\}} (-1)^{\#A} \dim \frac{\mathbb{C}[S]}{I(\alpha_A(\underline{v}) + \underline{1})}.$$

Every subset $A \subset \{1, \cdots, r\}$ can be written in a unique way as $A_1 \times A_2$, with $A_1 \subset \{2, \cdots, r\}$ and $A_2 \subset \{1\}$. We group the terms having the same component $A_1$ and we get

$$\begin{array}{l} \displaystyle\sum_{A \subset \{1, \cdots, r\}} (-1)^{\#A} \dim \frac{\mathbb{C}[S]}{I(\alpha_A(\underline{v}) + \underline{1})} = \\ \displaystyle\sum_{A \subset \{2, \cdots, r\}} (-1)^{\#A} \#\{\chi^s \mid \langle s, \nu_1 \rangle = v_1, \langle s, (\nu_2, \cdots, \nu_r) \rangle \geq \alpha_A(v_2, \cdots, v_r) + \underline{1}\}. \end{array}$$

We go on simplifying in the same way, so now we write every subset $A \subset \{2, \cdots, r\}$ as $A_1 \times A_2$, with $A_1 \subset \{3, \cdots, r\}$ and $A_2 \subset \{2\}$ and so on. At each step we group the terms having the same component $A_1$ and we obtain

$$\begin{aligned}
& \sum_{A \subset \{3,\cdots,r\}} (-1)^{\#A+1} \#\{\chi^s \mid \langle s, \nu_1\rangle = v_1, \langle s, \nu_2\rangle = v_2, \\
& \qquad \langle s, (\nu_3, \cdots, \nu_r)\rangle \geq \alpha_A(v_3, \cdots, v_r) + \underline{1}\} \\
= & \sum_{A \subset \{4,\cdots,r\}} (-1)^{\#A+2} \#\{\chi^s \mid \langle s, \nu_1\rangle = v_1, \langle s, \nu_2\rangle = v_2, \langle s, \nu_3\rangle = v_3, \\
& \qquad \langle s, (\nu_4, \cdots, \nu_r)\rangle \geq \alpha_A(v_4, \cdots, v_r) + \underline{1}\} \\
\vdots & \\
= & (-1)^{r-1} \#\{\chi^s \mid \langle s, \underline{\nu}\rangle = \underline{v}\}.
\end{aligned}$$

Hence the Poincaré series $P(\underline{t})$ is

$$\begin{aligned}
& \sum_{\underline{v} \in \mathbb{Z}^r} \#\{\chi^s \mid \langle s, \underline{\nu}\rangle = \underline{v}\} \underline{t}^{\underline{v}} \\
= & \sum_{s \in S} \underline{t}^{\langle s, \underline{\nu}\rangle} \\
= & \Phi(Q(\underline{u})).
\end{aligned}$$

■

In what follows we will write $N(\underline{v})$ when we want to refer to the number $\#\{s \in S \mid \langle s, \underline{\nu}\rangle = \underline{v}\}$. Since $\sigma$ is $d$-dimensional and strongly convex, one has that $N(\underline{v})$ is finite for every $\underline{v}$.

## 4.3 The Poincaré series à la A'Campo

For curves, rational surface singularities and plane divisorial valuations, there exists a description of the Poincaré series at the level of the modification space. Let $\mathcal{D} := \bigcup_{j=1}^{r} E_j$ be the exceptional variety with irreducible components $E_j, j \in J := \{1, \cdots, r\}$. We denote by $\overset{\circ}{E_j}$ the smooth part of the irreducible component $E_j$, i.e. without intersection points with all other components of the exceptional divisor. Let $M := -(E_i \circ E_j)$ be minus the intersection matrix of the components of the exceptional variety $\mathcal{D}$. Let $\nu_j$ be the discrete valuation on the local ring $\mathcal{O}_{X,o}$ induced by $E_j$.

The semigroup of values $S := \{\underline{\nu}(g) \mid g \in \mathcal{O}_{X,o}\}$ is exactly the set of vectors $\{\underline{v} \in \mathbb{Z}^r_{\geq 0} \mid \underline{v}M \geq \underline{0}\}$. For a topological space $E$, let $S^nE := E^n/S_n$ ($n \geq 0$) be the $n$-th symmetric power of the space $E$, i.e. the space of $n$-tuples of points of the space $E$ ($S^0E$ is a point).

Campillo, Delgado and Gusein-Zade construct the space

$$Y := \bigcup_{\{\underline{v} \in S\}} \left( \prod_{j=1}^{r} S^{n_j(\underline{v})} \overset{\circ}{E}_j \right),$$

where $\underline{v}M =: \underline{n}(\underline{v})$. For $g \in \mathcal{O}_{X,o}$, $g \neq 0$ and $\underline{v} := \underline{\nu}(g)$, the number $n_j(\underline{v})$ is equal to the intersection number of the strict transform of $g$ with $E_j$. Let $Y_{\underline{v}}$ be the connected component $\prod_{j=1}^{r} S^{n_j(\underline{v})} \overset{\circ}{E}_j$ of $Y$. They show that

$$P(\underline{t}) = \sum_{\underline{v} \in \mathbb{Z}^r} \chi(Y_{\underline{v}}) \underline{t}^{\underline{v}}. \tag{4.1}$$

In the case of curves and plane divisorial valuations, this description induces the elegant formula of the Poincaré series where the exponents can be written as the Euler characteristics of the smooth parts $\overset{\circ}{E}_j$.

In what follows we prove a generalised form of (4.1) for affine toric varieties. Let $M, N, \sigma$ and $S$ be as in Section 4.2. We consider the affine toric variety $X := \text{Spec } \mathbb{C}[S]$ and we take a regular subdivision of the cone $\sigma$. Let $\Sigma$ be the fan induced by this regular subdivision and call $X_\Sigma$ the toric variety associated to $\Sigma$. Let $\pi : X_\Sigma \to X$ be the proper birational map induced by the refinement.

Let $\tau$ be a regular $d$-dimensional cone in the refinement of $\sigma$. The primitive vectors $\{n_1, \cdots, n_d\}$ on the rays of $\tau$ form a basis for $\tau$. Let $\{r_1, \cdots, r_d\}$ be its dual basis, so $\langle r_i, n_j \rangle = \delta_{i,j}$, $1 \leq i, j \leq d$. We have that $\mathbb{C}[S]$ is a subring of the regular ring $\mathbb{C}[\check{\tau} \cap M] := \oplus_{t \in \check{\tau} \cap M} \mathbb{C}\underline{x}^t \cong: \mathbb{C}[y_1, \cdots, y_d]$. Under this isomorphism $\underline{x}^{r_i}$ is mapped to $y_i$. Hence, $\tau$ is giving rise to an affine chart $X_\tau := \text{Spec } \mathbb{C}[\check{\tau} \cap M]$ for $X_\Sigma$. In this way $X_\Sigma$ is being covered by affine charts that are induced by the regular $d$-dimensional cones in the subdivision of $\sigma$. For $1 \leq i \leq d$, let $D_i$ be the divisor on $X_\Sigma$ such that the vector in $N$ corresponding to the valuation $\nu_{D_i}$, is $n_i$. Then $D_i$ has equation $y_i$ in the chart $X_\tau$.

As the Poincaré series only permits valuations with centre in the 0-dimensional orbit, these valuations have to correspond to vectors in the interior of $\sigma$. As these rays not are rays of $\sigma$, they correspond to the exceptional divisors created by $\pi$. Let $\rho_1, \cdots, \rho_s$ be vectors in $\overset{\circ}{\sigma}$ such that the minimal subdivision of $\sigma$ that contains the rays through $\rho_1, \cdots, \rho_s$ is regular. Let $\mathcal{V}$ be a subset of the valuations that correspond to the primitive vectors on these rays, say $\mathcal{V} := \{\nu_1, \cdots, \nu_r\}$. Denote the corresponding exceptional varieties by $E_1, \cdots, E_r$.

If $g \in \mathcal{O}_{X,o}$, then we write $\hat{g}$ for the strict transform. For $\underline{v} \in \mathbb{Z}^r$, we define the set

$$\begin{aligned} \overset{\bullet}{D}_{\underline{v}} : \; &= \{\{\hat{g} = 0\} \cap \mathcal{D} \mid g \in \mathcal{O}_{X,o},\ \underline{\nu}(g) = \underline{v} \text{ and } \{\hat{g} = 0\} \text{ does not} \\ &\quad \text{contain any non-empty intersection } E_a \cap E_b, a, b \in J, a \neq b\}. \end{aligned}$$

Obviously to know $\overset{\bullet}{D}_{\underline{v}}$ it is sufficient to consider the elements $g$ in $\mathbb{P}F_{\underline{v}}$. We make a topological space of it as follows. We write $E_{\underline{v}}$ for the sum $\sum_{j=1}^{r} v_j E_j$ and $B$ for the line bundle associated to $E_{\underline{v}}$. The restriction $R$ of $\mathcal{O}_Z(-E_{\underline{v}}) \otimes B^{-1}$ to $\mathcal{D}$ is a line bundle and as $\mathcal{D}$ is a projective variety, the global sections of $R$ form a finite dimensional vector space. For $g \in F_{\underline{v}}$, the divisor $\hat{g} \cap \mathcal{D}$ is the divisor of zeroes of a global section of $R$. Then $\overset{\bullet}{D}_{\underline{v}}$ can be seen as a subset of the projectivisation of this vector space.

For toric varieties, we can give a more explicit description of the space $F_{\underline{v}}$. As the ideal $I(\underline{v})$ is a monomial ideal, one can see the space $F_{\underline{v}}$ as the set of functions $g = \sum_{i=1}^{s} \lambda_i m_i$ in $\mathcal{O}_{X,o}$ (the $m_i$ are monomials and the $\lambda_i, 1 \leq i \leq s$, are complex numbers different from 0) for which $\underline{\nu}(g) = \underline{v}$ and for which for all $i \in \{1, \cdots, s\}$ holds that $m_i \in I(\underline{v})$ and that there exists a $j \in J$ such that $\nu_j(m_i) = \nu_j(g)$. If a function $g$ with $\underline{\nu}(g) = \underline{v}$ has this form, we say that $g$ is in reduced form. We write $\mathrm{supp}(g)$ for the support of $g$, which is the set $\{m_i \mid 1 \leq i \leq s\}$.

Now let us fix $\underline{v} \in \mathbb{Z}^r$. Let $\mathcal{M}$ be the set of all monomials that can appear in the support of some $g$ in $\mathbb{P}F_{\underline{v}}$, so $\mathcal{M} := \{m \text{ monomial} \mid m \in I(\underline{v}) \text{ and } \exists j \in J : \nu_j(m) = v_j\}$. Note that $\mathcal{M}$ is a finite set. For a subset $L$ of $\mathcal{M}$, let $\underline{\nu}(L)$ be the vector $\underline{w} \in \mathbb{Z}^r$ with $w_j := \min\{\nu_j(m) \mid m \in L\}$, $j \in J$.

Now we can give a shorter proof of Theorem 1.2, obtained by exploiting the fact that the varieties that we consider are toric.

**4.2 Proposition**

$$P(\underline{t}) = \sum_{\underline{v}\in\mathbb{Z}^r} \chi(\mathbb{P}F_{\underline{v}})\underline{t}^{\underline{v}}.$$

*Proof.* We write $\mathbb{P}F_{\underline{v}}$ as a disjoint union:

$$\mathbb{P}F_{\underline{v}} = \bigcup_{L\subset\mathcal{M},\underline{\nu}(L)=\underline{v}} \{g \in \mathbb{P}F_{\underline{v}} \mid \mathrm{supp}(g) = L\}.$$

For $\Lambda_L := \{g \in \mathbb{P}F_{\underline{v}} \mid \mathrm{supp}(g) = L\}$, one has that $\chi(\Lambda_L) = \chi((\mathbb{C}^*)^k) = 0$ for some $k \in \mathbb{Z}_{>0}$ if $L$ is not a singleton, and $\chi(\Lambda_L) = 1$ if $L$ is a singleton. This gives us

$$\chi(\mathbb{P}F_{\underline{v}}) = N(\underline{v}).$$

■

We will construct a subspace $Z_{\underline{v}}$ of $\mathbb{P}F_{\underline{v}}$ that has the same Euler characteristic as $\mathbb{P}F_{\underline{v}}$ and such that there exists a homeomorphism of $Z_{\underline{v}}$ with $\overset{\bullet}{D}_{\underline{v}}$. Then from Proposition 4.2 it will follow:

**4.3 Theorem** *The Poincaré series $P(\underline{t})$ is equal to*

$$\sum_{\underline{v}\in\mathbb{Z}^r} \chi(\overset{\bullet}{D}_{\underline{v}})\underline{t}^{\underline{v}}.$$

We now start with the proof of Theorem 4.3. The obvious candidate for the space $Z_{\underline{v}}$ is the set

$$\begin{aligned} Z_{\underline{v}} \quad := \quad & \{g \in \mathbb{P}F_{\underline{v}} \mid \{\hat{g} = 0\} \text{ does not contain any non-empty} \\ & \text{intersection } E_a \cap E_b, a, b \in J, a \neq b\}. \end{aligned}$$

**4.4 Lemma**

$$\chi(Z_{\underline{v}}) = \chi(\mathbb{P}F_{\underline{v}}).$$

*Proof.* Let $g := \sum_{i=1}^{s} \lambda_i m_i$ be a function in reduced form and suppose that $E_a \cap E_b \neq \emptyset$ for some $a$ and $b$ in $J$, $a \neq b$. If $\{\hat{g} = 0\}$ contains $E_a \cap E_b$ then for all $\mu := (\mu_1, \cdots, \mu_s) \in (\mathbb{C}^*)^s$, also $\{\hat{g_{\underline{\mu}}} = 0\}$ contains $E_a \cap E_b$

where $g_{\underline{\mu}} := \sum_{i=1}^{s} \mu_i m_i$. This yields that

$$\begin{aligned}
\chi(Z_{\underline{v}}) &= \chi(\{g \in \mathbb{P}F_{\underline{v}} \mid g \text{ a monomial and } \{\hat{g}=0\} \text{ does not contain any} \\
&\qquad \text{non-empty intersection } E_a \cap E_b, a, b \in J, a \neq b\}) \\
&= \chi(\{g \in \mathbb{P}F_{\underline{v}} \mid g \text{ a monomial}\}) \\
&= N(\underline{v}) \\
&= \chi(\mathbb{P}F_{\underline{v}}).
\end{aligned}$$

■

Now we investigate the map

$$\begin{aligned}
\phi : Z_{\underline{v}} &\longrightarrow \overset{\bullet}{D}_{\underline{v}} \\
g &\longmapsto \{\hat{g}=0\} \cap \mathcal{D}.
\end{aligned}$$

The following lemma tells us how $\{\hat{g}=0\} \cap \mathcal{D}$ looks like.

**4.5 Lemma** *We consider $g := \sum_{i=1}^{s} \lambda_i m_i \in \mathbb{P}F_{\underline{v}}$, a function in reduced form. For $j \in J$, let $\Lambda_{g,j}$ be the set $\{m \in$ supp$(g) \mid \nu_j(m) = v_j\}$. Then the equation of $\{\hat{g}=0\} \cap E_j$ is $\sum_{m_i \in \Lambda_{g,j}} \lambda_i \hat{m}_i = 0$ in every affine chart $X_\tau$ where the intersection $\{\hat{g}=0\} \cap E_j$ is visible.*

*Proof.* If $\{\hat{g}=0\} \cap B_j \neq \emptyset$, then in an affine chart $X_\tau$ covering $B_j \cong \mathbb{P}^{d-1}$ it can be described by the equation $\sum_{i \in K_j} \lambda_i \hat{m}_i = 0$ where

$$\begin{aligned}
K_j &:= \{i \mid \nu_j(m_i) = \min\{\nu_j(m) \mid m \in \operatorname{supp}(g)\}\} \\
&= \{i \mid \nu_j(m_i) = v_j\} \\
&= \Lambda_{g,j}.
\end{aligned}$$

Now Lemma 4.5 follows directly. ■

Note that $\hat{m}$ can be equal to 1 in some affine chart covering $E_j$ $(j \in J)$. However, there exists always a $j \in J$ such that $\{\hat{m}=0\} \cap E_j \neq \emptyset$.

Let us have a look at the following example. Lemma 4.5 allows us to deduce quickly a necessary condition on a subset $S$ of $\mathbb{P}F_{\underline{v}}$ for the sets $\{g \in S\}$ and $\{\{\hat{g}=0\} \cap \mathcal{D} \mid g \in S\}$ to be in one-to-one correspondence.

**4.6 Example** Let $M = N := \mathbb{Z}^4$ and $S = \check{\sigma} \cap M := \mathbb{N}^4$. We choose the valuations represented by the following vectors in $\overset{\circ}{\sigma}$:

$$\begin{array}{lll} \nu_0 \leftrightarrow (1,1,1,1) & \nu_1 \leftrightarrow (1,2,2,2) & \nu_2 \leftrightarrow (2,1,2,2) \\ & \nu_3 \leftrightarrow (1,3,3,3) & \nu_4 \leftrightarrow (2,3,4,4) \end{array}$$

Note that these valuations are induced by a toric constellation in $\mathbb{C}^4$. Let $\underline{v}$ be $(6,11,10,14,18)$ and take $g(x,y,z,u) := y^2z^4 + x^3y^4 + x^{14}$. The values of the monomials in the support of $g$ are:

$$\nu(y^2z^4) = (6,12,10,18,22), \quad \nu(x^3y^4) = (7,11,10,15,18) \quad \text{and}$$

$$\nu(x^{14}) = (14,14,28,14,28).$$

Lemma 4.5 tells us that the equation of $\hat{g} \cap E_1$ in an affine chart where the intersection is visible, is $\widehat{y^2z^4}$. For $\hat{g} \cap E_2$ it is $\widehat{x^3y^4}$, for $\hat{g} \cap E_3$ it is $\widehat{y^2z^4} + \widehat{x^3y^4}$, for $\hat{g} \cap E_4$ it is $\widehat{x^{14}}$ and for $\hat{g} \cap E_5$ it is $\widehat{x^3y^4}$. It follows that the strict transform of $h(x,y,z,u) := y^2z^4 + x^3y^4 + \mu x^{14}$, with $\mu \neq 0$, has the same intersection with $\mathcal{D}$ as $\hat{g}$.

□

We formalise what Example 4.6 shows. As in Proposition 4.2, write $\mathbb{P}F_{\underline{v}}$ as the disjoint union $\bigcup_{L \subset \mathcal{M}, \underline{\nu}(L) = \underline{v}} \Lambda_L$. Let $L$ be a support appearing in this disjoint union. For $j \in J$, we define $\Lambda_{L,j}$ as the set of monomials $m$ in $L$ such that $\nu_j(m) = v_j$. The observation made in Example 4.6 shows that the map

$$\begin{array}{rcl} \phi_L : \{g \in \mathbb{P}F_{\underline{v}} \mid \operatorname{supp}(g) = L\} & \longrightarrow & \{\{\hat{g} = 0\} \cap \mathcal{D} \mid g \in \mathbb{P}F_{\underline{v}} \text{ and } \operatorname{supp}(g) = L\} \\ g & \longmapsto & \{\hat{g} = 0\} \cap \mathcal{D} \end{array}$$

certainly can not be a bijection if there exists a subset $D$ of $J$, with the following properties:

$$\exists a, b \in J : a \in D, \quad b \notin D, \quad (\cup_{d \in D} \Lambda_{L,d}) \cap (\cup_{d \notin D} \Lambda_{L,d}) = \emptyset. \qquad (4.2)$$

Indeed, if such a subset $D$ exists, then write $g := g_a + g_b$ where $\operatorname{supp}(g_a) = \cup_{d \in D} \Lambda_{L,d}$ and $\operatorname{supp}(g_b) = \cup_{d \notin D} \Lambda_{L,d}$. Then by Lemma 4.5 it follows that for $g_{\underline{\lambda}} := \lambda_a g_a + \lambda_b g_b$, the transforms $\{\hat{g_{\underline{\lambda}}} = 0\}$ and $\{\hat{g} = 0\}$ have the same intersection with $\mathcal{D}$ for all $\underline{\lambda} := (\lambda_a, \lambda_b) \in (\mathbb{C}^*)^2$. We claim that also the converse is true.

**4.7 Proposition** *Let $L \subset \mathcal{M}$ be as above. If for all $a, b \in J$, $a \neq b$, holds that there exists no subset $D$ with $a \in D$, $b \notin D$ and $(\cup_{d \in D} \Lambda_{L,d}) \cap (\cup_{d \notin D} \Lambda_{L,d}) = \emptyset$, then the map $\phi_L$ is a bijection.*

*Proof.*

1. First we show that for a monomial $m$ in $L$, if given $\{\hat{m} = 0\} \cap \mathcal{D}$, one can find $m$ again. So suppose $m$ in $L$. Take a $j \in J$ such that $\nu_j(m) = v_j$ and such that $\{\hat{m} = 0\} \cap E_j$ is visible in an affine chart in the final stadium, say in the chart $X_\tau$. The monomial $m$ can be written in a unique way as $\underline{x}^{\sum c_i r_i}$, for some nonnegative integers $c_1, \cdots, c_d$. Then the equation of the total transform of $m$ in the chart $X_\tau$ is $y_1^{\langle n_1, c_1 r_1 \rangle} \cdots y_d^{\langle n_d, c_d r_d \rangle} = \underline{y}^{\underline{c}}$. For $1 \leq i \leq d$, if $n_i$ defines a valuation of $\mathcal{V}$, say $\nu_{k_i}$, set then $h_i := v_{k_i}$. Otherwise let $h_i$ be 0. The strict transform of $m$ is then
$$y_1^{\langle n_1, c_1 r_1 \rangle - h_1} \cdots y_d^{\langle n_d, c_d r_d \rangle - h_d}.$$
Now suppose that $\hat{m} \cap E_j = y_1^{c'_1} \cdots y_d^{c'_d}$ is given. As $\underline{c} = \underline{c'} + \underline{h}$, we know the monomial $m$.

2. Let $g$ and $h$ be two functions in $\mathbb{P}F_{\underline{v}}$ with the same support $L$. Write $g := \sum_{i=1}^{s} \lambda_i m_i$ and $h := \sum_{i=1}^{s} \mu_i m_i$, with $\lambda_i$ and $\mu_i$ different from 0 ($1 \leq i \leq s$). Suppose that $\{\hat{g} = 0\} \cap \mathcal{D} = \{\hat{h} = 0\} \cap \mathcal{D}$ and that $\lambda_j \neq \mu_j$. For lack of a subset $D$ of $J$ with property (1) and because of part 1 of the proof, it follows that $\lambda_i / \mu_i = \lambda_j / \mu_j$, for all $i \in J$ and so $g = h$.

■

The functions in which we are interested are the functions $g$ such that $\{\hat{g} = 0\}$ does not contain any non-empty intersection $E_a \cap E_b$. They can be characterised as follows:

**4.8 Lemma** *Let $a$ and $b$ be different elements of $J$ such that $E_a \cap E_b \neq \emptyset$. Then*

$$\{\hat{g} = 0\} \textit{ contains } E_a \cap E_b$$
$$\Updownarrow$$
$$\textit{there is no } m \textit{ in supp}(g) \textit{ for which } \nu_a(m) = v_a \textit{ and } \nu_b(m) = v_b.$$

*Proof.* Suppose that there is no $m$ in $\mathrm{supp}(g)$ for which $\nu_a(m) = v_a$ and $\nu_b(m) = v_b$. In an affine chart where one sees $E_a \cap E_b$, one has

$$\hat{g} := \hat{g_a} + \hat{g}_b + \hat{r}, \quad E_a \leftrightarrow x_a = 0, \qquad E_b \leftrightarrow x_b = 0$$

where $g_a$ is the part of $g$ with $\mathrm{supp}(g_a) = \{m \in \mathrm{supp}(g) \mid \nu_a(m) = \nu_a(g)\}$, $g_b$ is the part of $g$ with $\mathrm{supp}(g_b) = \{m \in \mathrm{supp}(g) \mid \nu_b(m) = \nu_b(g)\}$ and $r$ is $g - g_a - g_b$. From Lemma 4.5 it follows that $\hat{g_b} + \hat{r} \in (x_a)$ and $\hat{g_a} + \hat{r} \in (x_b)$. Then also $\hat{g} \in (x_a, x_b)$ and hence $\{\hat{g} = 0\}$ contains $E_a \cap E_b$.

If $\{\hat{g} = 0\}$ contains $E_a \cap E_b$, there exists an affine chart in which one has

$$\hat{g} = x_a g_a + x_b g_b + x_a x_b g_r, \quad E_a \leftrightarrow x_a = 0, \qquad E_b \leftrightarrow x_b = 0,$$

with $g_a \notin (x_b)$ and $g_b \notin (x_a)$. If $m \in \mathrm{supp}(g)$ such that $\nu_a(m) = v_a$ and $\nu_b(m) = v_b$, Lemma 4.5 implies that $\hat{m} \in \mathrm{supp}(x_a g_a) \cap \mathrm{supp}(x_b g_b)$ what is impossible. ∎

Now let $g \in \mathbb{P}F_{\underline{v}}$ and let $L$ be the support of $g$. Taking the above characterisation into account, we see that if $\hat{g}$ does not contain $E_a \cap E_b$, then there exists no $D \subset J$ for which $a \in D$, $b \notin D$ and $(\cup_{d \in D} \Lambda_{L,d}) \cap (\cup_{d \notin D} \Lambda_{L,d}) = \emptyset$. Note that the other implication is false in general (see for example the constellation given above with $a = 1$ and $b = 2$).

We denote

$$Z_{\underline{v},L} := \{g \in Z_{\underline{v}} \mid \mathrm{supp}(g) = L\} \text{ and } \overset{\bullet}{D}_{\underline{v},L} := \{\{\hat{g} = 0\} \cap \mathcal{D} \in \overset{\bullet}{D}_{\underline{v}} \mid \mathrm{supp}(g) = L\}.$$

Then we can write $Z_{\underline{v}}$ as a disjoint union $\cup_L Z_{\underline{v},L}$ where for each $L$ holds that there is no subset $D$ of $J$ satisfying condition (4.2). Proposition 4.7 tells us that the map

$$\begin{aligned} \psi_L : Z_{\underline{v},L} \quad &\longrightarrow \quad \overset{\bullet}{D}_{\underline{v},L} \\ g \quad &\longmapsto \quad \{\hat{g} = 0\} \cap \mathcal{D} \end{aligned}$$

is a bijection and then Part 1 of the proof of Proposition 4.7 allows us to conclude that the map

$$\begin{aligned} \phi : Z_{\underline{v}} \quad &\longrightarrow \quad \overset{\bullet}{D}_{\underline{v}} \\ g \quad &\longmapsto \quad \{\hat{g} = 0\} \cap \mathcal{D} \end{aligned}$$

is bijective.

Now by Lemma 4.5 one can see that $\phi$ is a homeomorphism. Then we have that $\chi(Z_{\underline{v}}) = \chi(\overset{\bullet}{D}_{\underline{v}})$ which completes the proof of Theorem 4.3. ■

## 4.4 The Poincaré series of $\mathbb{C}^d$ induced by a toric constellation

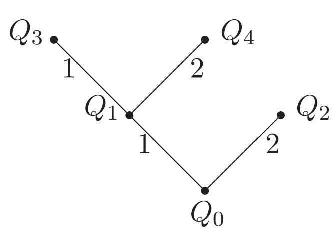
In this section we look at the particular case where $X$ is $\mathbb{C}^d$ endowed with the action of the torus $T \cong (\mathbb{C}^*)^d$ $(d \geq 2)$. We study the Poincaré series of $\mathbb{C}^d$ where the modification $\pi$ is given by a toric constellation $\mathcal{C}$ with origin.

Let $E_j$ $(j \in J)$ be the irreducible components of the exceptional divisor $\mathcal{D}$ created by blowing up the constellation $\mathcal{C} := \{Q_0, Q_1, \cdots, Q_{r-1}\}$ and let $\xi_j$ be the generic point of $E_j$ $(j \in J)$. Then $\mathcal{O}_{X,\xi_j}$ is a discrete valuation ring. We write $\nu_j$ for the induced valuation.

We denote the matrix of the linear system of equations $\langle s, \underline{\nu} \rangle = \underline{v}$ by $\mathcal{L}(\mathcal{C})$ and we denote the column vectors of $\mathcal{L}(\mathcal{C})$ by $\underline{v}_1, \cdots, \underline{v}_d$. Let $C$ be the cone in $\mathbb{R}^r_{\geq 0}$ generated by $\underline{v}_1, \cdots, \underline{v}_d$. Note that $C$ is the monomial cone associated to the constellation $\mathcal{C}$. Recall that the cone is regular if it can be generated by a part of a basis of $\mathbb{Z}^r$. If $\mathcal{L}(\mathcal{C})$ has rank $s$ and if $s < d$, then the cone is said to be *degenerate*.

**4.9 Example**

Consider the following toric constellation in $\mathbb{C}^4$.




The associated matrix $\mathcal{L}(\mathcal{C})$ is

$$\begin{cases} a + b + c + d = v_1 \\ a + 2b + 2c + 2d = v_2 \\ 2a + b + 2c + 2d = v_3 \\ a + 3b + 3c + 3d = v_4 \\ 2a + 3b + 4c + 4d = v_5 \end{cases}$$

and $C$ is the cone $\langle(1,1,2,1,2),(1,2,1,3,3),(1,2,2,3,4)\rangle \subset \mathbb{Z}^5_{\geq 0}$ which is obviously degenerate.

$\square$

To know the Poincaré series one can use Theorem 4.1. Let $\underline{v}_1, \cdots, \underline{v}_d$ be the column vectors of $\mathcal{L}(\mathcal{C})$. Then it follows by Theorem 4.1 that

$$P(\underline{t}) = \frac{1}{(1-\underline{t}^{\underline{v}_1})\cdots(1-\underline{t}^{\underline{v}_d})}.$$

One can also obtain $P(\underline{t})$ by computing the numbers $N(\underline{v}) = \#\{s \in \mathbb{N}^d \mid \langle s, \underline{\nu}\rangle = \underline{v}\}$ for each $\underline{v} \in \mathbb{Z}^r$. We determine them as later these values will be useful for us. The following proposition gives some properties of the cone $C$ which will allow us to compute the numbers $N(\underline{v})$.

**4.10 Proposition**

1. *$C$ is a regular cone;*

2. *$C$ is degenerate if and only if the number of different labels along the edges appearing in the constellation is less than or equal to $d-2$.*

*Proof.* If the number of different labels appearing in the constellation $\mathcal{C}$ is less than or equal to $d-2$, then at least two columns in $\mathcal{L}(\mathcal{C})$ are equal and $C$ is degenerate. Suppose that the number of different labels along the edges is bigger than $d-2$, say that $1, \cdots, d-1$ are labels appearing in the constellation and let $Q_1, \cdots, Q_{d-1}$ be points in the constellation such that $Q_i$ is a point with minimal level arising in an affine chart induced by the label $i$ and such that whenever $Q_i \geq Q_j$ and $Q_i \neq Q_j$, then $i > j$ $(i, j \in \{1, \cdots, d-1\})$.
The linear equations induced by the origin $Q_0$ and by $Q_1, \cdots, Q_{d-1}$ give rise to a linear system whose determinant can be supposed to be of the

form

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 2 & 2 & \cdots & \cdots & 2 \\ * & a_2-1 & a_2 & \cdots & \cdots & a_2 \\ * & * & a_3-1 & a_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & \cdots & * & a_{d-1}-1 & a_{d-1} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 \\ -1 & 0 & 0 & \cdots & \cdots & 0 \\ * & -1 & 0 & \cdots & \cdots & 0 \\ * & * & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & \cdots & * & -1 & 0 \end{vmatrix},$$

where $a_2, \cdots, a_{d-1}$ are integer numbers. As this determinant is equal to 1, the cone $C$ is regular and nondegenerate.
Now if $C$ is degenerate, and generated by $s := \text{rank}(\mathcal{L}(\mathcal{C}))$ vectors then there are $s-1$ different labels appearing in the constellation. In the same way as above we obtain a $(s \times s)$-determinant which is equal to 1 such that also in this case $C$ is regular. ■

Observe that it follows from the proof that saying that the cone $C$ is degenerate, is the same as saying that there is one column that appears at least twice in $\mathcal{L}(\mathcal{C})$ and that there are no other linear dependencies between the columns of $\mathcal{L}(\mathcal{C})$.

Notice also that it follows from the fact that a toric constellation is represented by a regular subdivision of the cone $\mathbb{R}^d_{\geq 0}$, that the monomial cone $C$ is regular.

- If $C$ is nondegenerate then $N(\underline{v}) = 1$ if $\underline{v} \in C$, else $N(\underline{v}) = 0$. Suppose that $\underline{v}_1, \cdots, \underline{v}_d$ are the column vectors of the linear system $\mathcal{L}(\mathcal{C})$. As $C$ is regular, we obtain again

$$P(\underline{t}) = \frac{1}{(1-\underline{t}^{\underline{v}_1}) \cdots (1-\underline{t}^{\underline{v}_d})}.$$

- If $C$ is degenerate, let then $\underline{v}_1, \cdots, \underline{v}_s$ be the $s$ different vectors that generate $C$ and let $\underline{v}_s$ be the vector that appears at least twice as column vector in $\mathcal{L}(\mathcal{C})$. As $C$ is regular, one can write each $\underline{v} \in C$ in a unique way as $\underline{v} = \lambda_1 \underline{v}_1 + \cdots + \lambda_{s-1}\underline{v}_{s-1} + \lambda \underline{v}_s$, for some $\lambda_i, \lambda \in \mathbb{Z}_{\geq 0}$ $(1 \leq i \leq s-1)$.
Setting $k := d - s + 1$, a simple calculation shows that

$$N(\underline{v}) = \binom{k+\lambda-1}{\lambda}$$

if $\underline{v} \in C$, else $N(\underline{v}) = 0$. Using that $\sum_{\lambda \in \mathbb{Z}_{\geq 0}} \binom{k+\lambda-1}{\lambda} x^\lambda = \frac{1}{(1-x)^k}$, one sees again that

$$P(\underline{t}) = \frac{1}{(1-\underline{t}^{\underline{v}_1}) \cdots (1-\underline{t}^{\underline{v}_{s-1}})(1-\underline{t}^{\underline{v}_s})^k}.$$

For Example 4.9, the Poincaré series is then

$$\frac{1}{(1-t_1t_2t_3^2t_4t_5^2)(1-t_1t_2^2t_3t_4^3t_5^3)(1-t_1t_2^2t_3^2t_4^3t_5^4)^2}.$$

□

Also Theorem 4.3 can be written in a more explicit way in the case of toric constellations in $\mathbb{C}^d$.

**4.11 Proposition** *Let $\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_s$ be the different columns of the matrix $\mathcal{L}(\mathcal{C})$ determined by the constellation. Then*

$$P(\underline{t}) = \frac{1}{(1-\underline{t}^{\underline{v}_1})^{\chi(\dot{D}_{\underline{v}_1})} \cdots (1-\underline{t}^{\underline{v}_s})^{\chi(\dot{D}_{\underline{v}_s})}}.$$

*Proof.* If the cone $C$ associated to $\mathcal{L}(\mathcal{C})$ is nondegenerate, then one has for each $\underline{v}_i$ $(1 \leq i \leq s)$ that $\chi(\dot{D}_{\underline{v}_i}) = N(\underline{v}_i) = 1$. If $C$ is degenerate then we have for all but one $\underline{v}_i$ $(1 \leq i \leq s)$ that $\chi(\dot{D}_{\underline{v}_i}) = N(\underline{v}_i) = 1$. For the column $\underline{v}$ that appears more than once, one gets $\chi(\dot{D}_{\underline{v}}) = N(\underline{v}) = k$, with $k := d - s + 1$. ■

As a consequence of Theorem 4.3 and Proposition 4.11, we obtain that the value $\chi(\dot{D}_{\underline{v}})$ can be calculated from the values $\chi(\dot{D}_{\underline{v}_1}), \cdots, \chi(\dot{D}_{\underline{v}_s})$ and $k$.

## 4.5 Poincaré series for a toric complete intersection

In the previous section we saw that the Poincaré series of $\mathbb{C}^d$ with respect to a set of valuations that is induced by a toric constellation has a cyclotomic form. Obviously we can replace $\mathbb{C}^d$ by any regular toric affine variety.

In this section we give an application of Theorem 4.1. By a result in algebra it will follow that for a complete toric intersection, its Poincaré series is cyclotomic.

The setting is the following. Let $M$ be a lattice with dual space $N := \mathbb{Z}^d$ and let $\sigma \subset N \otimes_{\mathbb{Z}} \mathbb{R}$ be a rational finite polyhedral strongly convex $d$-dimensional cone. Write $\check{\sigma}$ for the dual cone to $\sigma$. We consider a semigroup $S$ in $\check{\sigma} \cap M$ that generates $\check{\sigma}$ as cone and for which the group generated by $S$ is $M$. We denote the induced toric variety Spec $\mathbb{C}[S]$ by $X$. Let $\{s_1, \cdots, s_l\}$ be a system of generators for $S$. Consider the embedding of $X$ in the affine space $\mathbb{C}^l$ induced by:

$$\begin{array}{rcl} \varepsilon : \mathbb{C}[x_1, \cdots, x_l] & \rightarrow & \mathbb{C}[S] \\ x_i & \mapsto & \chi^{s_i}. \end{array}$$

We define the degree $\deg(x_i) := s_i$. Then the kernel $I(X) :=\ker(\varepsilon)$ of the $\mathbb{C}$-algebra morphism $\varepsilon$ is $S$-graded. Sturmfels explains that $I(X)$ is then generated by binomials of the form $\underline{x}^{\alpha_j} - \underline{x}^{\beta_j}$, with $\deg(\underline{x}^{\alpha_j}) = \deg(\underline{x}^{\beta_j}) = q_j$ and $\mathrm{supp}(\underline{x}^{\alpha_j}) \cap \mathrm{supp}(\underline{x}^{\beta_j}) = \emptyset$, where $1 \leq j \leq g$, $g \in \mathbb{Z}_{\geq 0}$ and the $q_j \in S$ are unique (see [St]).

Obviously $g \geq l - d$. When $g = l - d$, then $X$ is said to be a toric *complete intersection*. This property is independent of the chosen system of generators for $S$. In [Mill,St] one can find a proof of the following theorem.

**4.12 Theorem** *If the semigroup $S$ defines an affine toric complete intersection, then its Poincaré series $Q(\underline{u}) = \sum_{s \in S} \underline{u}^s$ is equal to*

$$\frac{\prod_{j=1}^{g}(1 - \underline{u}^{q_j})}{\prod_{i=1}^{l}(1 - \underline{u}^{s_i})}.$$

Applying Theorem 4.1, we now get the following result (we use the same notation as before).

**4.13 Corollary** *Let $n_1, \cdots, n_r$ be elements in $\overset{\circ}{\sigma}$, then*

$$P(t_1, \cdots, t_r) = \frac{\prod_{j=1}^{g}(1 - t_1^{\langle q_j, n_1 \rangle} \cdots t_r^{\langle q_j, n_r \rangle})}{\prod_{i=1}^{l}(1 - t_1^{\langle s_i, n_1 \rangle} \cdots t_r^{\langle s_i, n_r \rangle})}.$$

# Chapter 5

# The zeta function of monodromy and the topological zeta function

Since the $19th$ century many zeta functions have been defined and investigated. Some examples are the Riemann zeta function, the Weil zeta functions and the Igusa zeta function. Many other zeta functions are mentioned on [Wa].

In this thesis we investigate some aspects of the zeta function of monodromy and of the topological zeta function. We first give a short introduction to these zeta functions, according to the aim of this thesis. In the first two sections of this chapter, we introduce these zeta functions. A very mysterious thing is going on here; these zeta functions seem to be related. The monodromy conjecture states this prediction, see Section 5.3.

## 5.1 The zeta function of monodromy

Let $f$ be a complex polynomial in $d$ variables.We assume that $f(0) = 0$. Take $\epsilon > 0$ small enough such that the open ball $B_\epsilon$ with radius $\epsilon$ around the origin intersects the fibre $f^{-1}(0)$ transversally. Then choose $\epsilon \gg \eta > 0$ such that for $t$ in the disc $D_\eta \subset \mathbb{C}$ around the origin, the fibre $f^{-1}(t)$ intersects $B_\epsilon$ transversally. Write $X := f^{-1}(D_\eta) \cap B_\epsilon$, $X_t := f^{-1}(t) \cap B_\epsilon$ for $t \in D_\eta$ and $D^*_\eta := D_\eta \setminus \{0\}$ for the pointed disc.

Milnor showed that $f_{|_{X\setminus X_0}} : X \setminus X_0 \to D^*_\eta$ is a locally trivial fibration, see [Mil]. A fibre $X_t$ of this bundle is called *Milnor fibre*.

Let $t$ be a point in $D^*_\eta$ and consider the loop $\gamma : [0,1] \to D_\eta : t \mapsto \eta e^{2\pi i t}$ in $D^*_\eta$ encircling the origin once counterclockwise. Since $f_{|_{X\setminus X_0}}$ is a locally trivial fibration, the loop $\gamma$ lifts to a diffeomorphism $h$ of the Milnor fibre $X_\delta$, which is well determined up to homotopy. In this way $\gamma$ induces an automorphism $h^* : H^n(X_\delta, \mathbb{C}) \to H^n(X_\delta, \mathbb{C})$, $n \geq 0$, that is called the *monodromy transformation*.

**5.1 Definition** The *zeta function of monodromy* at the origin $\zeta_f$ associated to the polynomial $f$ is

$$\zeta_f(t) := \prod_{n\geq 0} (\det(id^* - th^*; H^n(X_\delta, \mathbb{C})))^{(-1)^{(n+1)}}.$$

If $f$ determines an isolated singularity, the cohomology groups $H^n(X_\delta, \mathbb{C})$ are all $\{0\}$ except for $n = 0$ and $n = d-1$. For an isolated singularity, the zeta function then becomes

$$\zeta_f(t) = \frac{\det(id^* - th^*; H^{d-1}(X_\delta, \mathbb{C}))^{(-1)^d}}{1-t}. \tag{5.1}$$

Very often one uses the alternative formula for the zeta function of monodromy provided by A'Campo ([A'C]). It describes the zeta function of monodromy in terms of an embedded resolution of the hypersurface $f^{-1}(0)$.

Let $\pi : Y \to \mathbb{C}^d$ be an embedded resolution of singularities of $f^{-1}\{0\}$. We write $E_j, j \in S$, for the irreducible components of $\pi^{-1}(f^{-1}\{0\})$ and we denote by $N_j$ and by $\nu_j - 1$ the multiplicities of $E_j$ in the divisor on $X$ of $f \circ \pi$ and $\pi^*(dx_1 \wedge \ldots \wedge dx_d)$ respectively. In Chapter 3 we met already the notion of $N_j$ and used there the notation $v_j$ for it. These notations are very standard in their own domain, so here we also use the notation according to the topic.

The couples $(N_j, \nu_j), j \in S$, are called the numerical data of the embedded resolution $(X, \pi)$. We denote also $\overset{\circ}{E_j} := E_j \setminus (\cup_{i\in S\setminus\{j\}} E_i)$, for $j \in S$. Further we write $\chi(\cdot)$ for the topological Euler–Poincaré characteristic.

Suppose that $f = 0$ has only one isolated singularity. Then we may suppose that $\pi$ is an isomorphism outside the inverse image of the origin. Say that the $E_j$, $j \in \{1, \cdots, r\}$, are the irreducible exceptional components of $\pi^{-1}(\{0\})$.

**5.2 Theorem** *(A'Campo)*

$$\zeta_f(t) = \prod_{j=1}^{r} (1 - t^{N_j})^{-\chi(\overset{\circ}{E}_j)}.$$

## 5.2 The topological zeta function

In 1992 Denef and Loeser created a new zeta function which they called the topological zeta function because of the topological Euler–Poincaré characteristic turning up in it. Roughly said, the topological zeta function $Z_{top,f}$ associated to a polynomial function $f : \mathbb{C}^d \to \mathbb{C}$ (or to the germ $f : (\mathbb{C}^d, 0) \to (\mathbb{C}, 0)$ of a holomorphic function) is a function containing information that we can pick out of every chosen embedded resolution of $f^{-1}\{0\} \subset \mathbb{C}^d$. They introduced it in [De,L1] in the following way.

Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function and let $\pi : X \to \mathbb{C}^d$ be an embedded resolution of singularities of $f^{-1}\{0\}$. We write $E_j, j \in S$, for the irreducible components of $\pi^{-1}(f^{-1}\{0\})$. Let $(N_j, \nu_j), j \in J$, be the numerical data of the embedded resolution $(X, \pi)$. For $I \subset S$ we denote also $E_I := \cap_{i \in I} E_i$ and $\overset{\circ}{E}_I := E_I \setminus (\cup_{j \notin I} E_j)$.

**5.3 Definition** The local *topological zeta function associated to* $f$ is the rational function in one complex variable

$$Z_{top,f}(s) := \sum_{I \subset S} \chi(\overset{\circ}{E}_I \cap \pi^{-1}\{0\}) \prod_{i \in I} \frac{1}{N_i s + \nu_i}.$$

For a polynomial function $f$ there is a global version replacing $\overset{\circ}{E}_I \cap \pi^{-1}\{0\}$ by $\overset{\circ}{E}_I$. When we do not specify, we mean the local one.

Denef and Loeser proved that every embedded resolution gives rise to the same function, so the topological zeta function is a well-defined singularity invariant (see [De,L1]). Once the motivic Igusa zeta function was introduced, they proved this result alternatively in [De,L2] by showing that this more general zeta function specialises to the topological one.

In [De,L1] Denef and Loeser also give a formula for the topological zeta function in terms of Newton polyhedra.

Let $f \in \mathbb{C}[x_1, \cdots, x_d]$ be a non-constant polynomial vanishing in the origin. Write $\underline{x}^{\underline{k}} := x_1^{k_1} \cdots x_d^{k_d}$ and $f := \sum_{\underline{k} \in \mathbb{N}^d} c_{\underline{k}} \underline{x}^{\underline{k}}$. The *support of* $f$ is $\mathrm{supp}(f) := \{\underline{k} \in \mathbb{N}^d \mid c_{\underline{k}} \neq 0\}$. The *global Newton polyhedron* $\Gamma_{gl}$ of $f$ is the convex hull of $\mathrm{supp}(f)$ and the *Newton polyhedron* $\Gamma$ of $f$ is the convex hull of $\Gamma_{gl} + \mathbb{R}^d_{\geq 0}$. A *face* of $\Gamma$ is defined as the intersection $\{l = 0\} \cap \Gamma$ for a linear form $l$ which is nonnegative on $\Gamma$. The *dimension of a face* $\tau$ is the dimension of the smallest subspace containing $\tau$. For a face $\tau$ of $\Gamma$ we write $f_\tau := \sum_{\underline{k} \in \tau} c_{\underline{k}} \underline{x}^{\underline{k}}$. A polynomial $f$ is called *nondegenerate with respect to* $\Gamma$ if for every compact face $\tau$ of $\Gamma$, the polynomials $f_\tau$ and $\partial f_\tau / \partial x_i$ have no common zeroes in $(\mathbb{C}^*)^d$, $1 \leq i \leq d$.

For $\underline{a} \in \mathbb{R}^d_+$, set $N(\underline{a}) := \min_{\underline{x} \in \Gamma} \underline{a} \cdot \underline{x}$ and $\nu(a) := \sum_{i=1}^d a_i$. The sets $F(\underline{a}) := \{\underline{x} \in \Gamma \mid \underline{a} \cdot \underline{x} = N(\underline{a})\}$ are faces of $\Gamma$. To a face $\tau$ of $\Gamma$ one associates a cone $\tau'$ that is the closure of the set $\{\underline{a} \in \mathbb{R}^d_+ \mid F(\underline{a}) = \tau\}$. It has dimension $d - \mathrm{dim}(\tau)$. In particular, if $\mathrm{dim}(\tau) = d - 1$ then $\tau'$ is a ray, say $\tau' := \underline{r}\mathbb{R}_{\geq 0}$ for some $\underline{r} \in \mathbb{N}^d$, and the equation of the hyperplane through $\tau$ is then $\underline{r} \cdot \underline{x} = N(\underline{r})$.

Let $\gamma$ be the convex hull in $\mathbb{R}^d$ of a subset of $\mathbb{Z}^d$. Denote $\mathrm{Aff}(\gamma)$ for the affine space spanned by $\gamma$. Let $\omega_\gamma$ be the volume form on $\mathrm{Aff}(\gamma)$ such that a parallelepiped spanned by a lattice basis of $\mathbb{Z}^d \cap \mathrm{Aff}(\gamma)$ has volume 1. For a face $\tau$ of $\Gamma$ we set $\mathrm{Vol}(\tau) = 1$ if $\mathrm{dim}(\tau) = 0$. Otherwise we define $\mathrm{Vol}(\tau)$ as the volume of $\tau \cap \Gamma_{gl}$ for the volume form $\omega_\tau$. For an $l$-dimensional simplicial cone $C := \underline{r_1}\mathbb{R}_{\geq 0} + \cdots + \underline{r_l}\mathbb{R}_{\geq 0} \subset \mathbb{R}^d_{\geq 0}$, with $\underline{r_1}, \cdots, \underline{r_l}$ primitive vectors in $\mathbb{N}^d$, the *multiplicity* $\mathrm{mult}(C)$ of $C$ is the volume of the parallelepiped spanned by $\underline{r_1}, \cdots, \underline{r_l}$ for the volume form $\omega_C$. The multiplicity $\mathrm{mult}(C)$ also equals the greatest common divisor of the determinants of all $l \times l$ matrices obtained by omitting columns from the matrix with rows $\underline{r_1}, \cdots, \underline{r_l}$.

Let now $C$ be a simplicial cone in $\mathbb{R}^d_{\geq 0}$ with primitive linearly independent generators $\underline{r_1}, \cdots, \underline{r_l} \in \mathbb{N}^d$. To $C$ we associate the rational function

$$J_C(s) := \frac{\mathrm{mult}(C)}{\prod_{i=1}^l (N(\underline{r_i})s + \nu(\underline{r_i}))}.$$

Now for an arbitrary face $\tau$ of $\Gamma$, a rational function denoted $J_\tau(s)$ is defined as follows:

1. If $\tau = \Gamma$, then put $J_\tau(s) := 1$;

2. else choose a decomposition of $\tau'$ in simplicial cones $C_i$, $1 \leq i \leq n$, of dimension $l := \mathrm{dim}(\tau')$ such that $\mathrm{dim}(C_i \cap C_j) < l$ if $i \neq j$. Then set $J_\tau(s) := \sum_{i=1}^n J_{C_i}(s)$.

Denef and Loeser show that the function $J_\tau$ does not depend on the chosen decomposition of $\tau'$ and they prove the following description for the topological zeta function.

**5.4 Theorem** *If $f$ is nondegenerate with respect to $\Gamma$, then*

$$Z_{top,f}(s) = \sum_{\tau \text{ vertex of } \Gamma} J_\tau(s) + \frac{s}{s+1} \sum_{\substack{\tau \text{compact} \\ \text{face of } \Gamma, \\ dim(\tau)\geq 1}} (-1)^{dim(\tau)}(_{dim}\tau)!Vol(\tau)J_\tau(s).$$

A very remarkable fact about the topological zeta function $Z_{top,f}$ of a polynomial $f$, is that a lot of the numbers $-\nu_j/N_j$, called the *candidate poles*, are not poles of $Z_{top,f}$. We will say more about that in Chapter 6.

## 5.3 The monodromy conjecture

The monodromy conjecture was first stated for the Igusa zeta function. When Denef and Loeser introduced the topological zeta function, an analogous version of the monodromy conjecture arose.

One calls $\alpha$ an *eigenvalue of monodromy of $f$* if $\alpha$ is an eigenvalue for some $t \cdot id^* - h^* : H^n(X_\delta, \mathbb{C}) \to H^n(X_\delta, \mathbb{C})$ at some $b \in f^{-1}\{0\}$. The zeta functions $\zeta_f$ determine all the eigenvalues of the monodromy transformations. For isolated singularities this follows from Formula 5.1. Denef showed that this is also true for non-isolated singularities, see [De2]. In other words, the zeroes and poles of $\zeta_f$ are the eigenvalues of monodromy of $f$.

The monodromy conjecture relates the poles of the topological zeta function $Z_{top,f}$ with the eigenvalues of monodromy of $f$.

**5.5 Conjecture** (*Monodromy conjecture*) If $s$ is a pole of $Z_{top,f}$, then $e^{2\pi is}$ is an eigenvalue of monodromy of $f$ for some point of the hypersurface $f = 0$.

Loeser proved the conjecture for plane curves. He gave also a proof for a particular class of functions in higher dimensions, one of the conditions is that the polynomial should be nondegenerate with respect to its Newton polyhedron ([L1, L2]). Also Artal-Bartolo, Cassou-Noguès, Luengo and Melle-Hernández, Rodrigues and Veys provided results about the conjecture.

# Chapter 6

# On the poles of the topological zeta function

## 6.1 Introduction

Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function and consider an embedded resolution of singularities $\pi : X \to \mathbb{C}^d$ of $f^{-1}\{0\}$. We write $E_j, j \in S$, for the irreducible components of $\pi^{-1}(f^{-1}\{0\})$ and $(N_j, \nu_j), j \in S$, for the numerical data of the embedded resolution $(X, \pi)$. From the definition of the topological zeta function associated to $f$

$$Z_{top,f}(s) := \sum_{I \subset S} \chi(\overset{\circ}{E}_I \cap \pi^{-1}\{0\}) \prod_{i \in I} \frac{1}{N_i s + \nu_i},$$

it follows that the poles are part of the set $\{-\nu_j/N_j \mid j \in S\}$; therefore the $-\nu_j/N_j$ are called the candidate poles. Notice that the poles are negative rational numbers.

The poles of the topological zeta function of $f$ are interesting numerical invariants. Various conjectures relate them to the eigenvalues of the local monodromy of $f$, see for example [De,L1].

A related numerical invariant of $f$ at $0 \in \mathbb{C}^d$ is its *log canonical threshold* $c_0(f)$ which is by definition the supremum of the set

$$\{c \in \mathbb{Q} \mid \text{ the pair } (\mathbb{C}^d, c \operatorname{div} f) \text{ is log canonical in a neighbourhood of } 0\}.$$

It is described in terms of the embedded resolution as $c_0(f) = \min\{\nu_j/N_j \mid 0 \in h(E_j), j \in S\}$ (see [Ko2, Proposition 8.5]). It was studied in various papers of Alexeev, Cheltsov, Ein, de Fernex, Kollár, Kuwata, M$^c$Kernan, Mustaţă, Park, Prokhorov, Reid, Shokurov and others. Especially the sets $\mathcal{T}_d := \{c_0(f) \mid f \in \mathbb{C}[x_1, \ldots, x_d]\}$, with $d \in \mathbb{Z}_{>0}$, show up in interesting conjectures, see [Al], [Ko], [Ku], [M$^c$K,Pr], [Pr] and [Sh].

In the context of the topological zeta function, one similarly studies the set

$$\mathcal{P}_d := \{s_0 \mid \exists f \in \mathbb{C}[x_1, \ldots, x_d] : Z_{top,f}(s) \textit{ has a pole in } s_0\}.$$

The case $d = 1$ is trivial: $\mathcal{P}_1 = \{-1/i \mid i \in \mathbb{Z}_{>0}\}$.

From now on we assume that $d \geq 2$. A more or less obvious lower bound for $\mathcal{P}_d$ is $-(d-1)$, see [Se1, Section 2.4]. In [Se,Ve], Segers and Veys studied the 'smallest poles' for $d = 2$ and $d = 3$. They showed that $\mathcal{P}_2 \cap (-\infty, -\frac{1}{2}) = \{-\frac{1}{2} - \frac{1}{i} \mid i \in \mathbb{Z}_{>1}\}$ and that $\mathcal{P}_3 \cap (-\infty, -1) = \{-1 - \frac{1}{i} \mid i \in \mathbb{Z}_{>1}\}$. They expected that this could be generalised to

$$\mathcal{P}_d \cap (-\infty, -\frac{d-1}{2}) = \{-\frac{d-1}{2} - \frac{1}{i} \mid i \in \mathbb{Z}_{>1}\}, \quad \text{for all } d \in \mathbb{Z}_{>1}.$$

In particular, they predicted that the lower bound $-(d-1)$ could be sharpened to $-d/2$. This better bound was proven by Segers in [Se2].

We show for all $d \geq 4$ that $\{-(d-1)/2 - 1/i \mid i \in \mathbb{Z}_{>1}\} \subset \mathcal{P}_d$, and as main result we prove that *any* rational number in the remaining interval $[-(d-1)/2, 0)$ is a pole of some topological zeta function.

**6.1 Theorem** For $d \geq 2$ we have $[-(d-1)/2, 0) \cap \mathbb{Q} \subset \mathcal{P}_d$.

With the Thom-Sebastiani principle [De,L3], $x_1^i + x_2^2 + \cdots + x_d^2$ is the obvious candidate to have $-(d-1)/2 - 1/i$ as a pole of its associated topological zeta function. For the theorem, the key is to find a suitable family of polynomials.

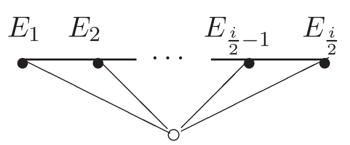
We will put the useful information of the resolution into a diagram, which is called the dual intersection graph. It is obtained as follows. One associates a vertex to each exceptional component in the embedded resolution (represented by a dot) and to each component of the strict transform of $f^{-1}\{0\}$ (represented by a circle). One also associates to each intersection an edge, connecting the corresponding vertices. The fact that $E_j$ has numerical data $(N_j, \nu_j)$ is denoted by $E_j(N_j, \nu_j)$.

If the strict transform of $f^{-1}\{0\}$ is irreducible, we will denote it by $E_0$. Let $E_j$ be an exceptional variety and let $E_i$, $i \in S$, be the components that intersect $E_j$ in $X$. We set $\alpha_i := \nu_i - (\nu_j/N_j)N_i$ for $i \in S$; these numbers appear in the calculation of the residue of $Z_{top,f}$ in $-\nu_j/N_j$.

## 6.2 The set $\{-(d-1)/2-1/i \mid i \in \mathbb{Z}_{>1}\}$ is a subset of $\mathcal{P}_d$

***Embedded resolution for $x_1^i+x_2^2+\cdots+x_d^2=0$, $d \geq 4$, with $i$ even***

After blowing up $i/2$ times in the origin, we get an embedded resolution for $f$. We present the dual intersection graph for $i \neq 2$.




$E_1(2, n)$
$E_2(4, 2d-1)$
$E_{i/2-1}(i-2, (d-1)(i/2-2)+d)$
$E_{i/2}(i, (d-1)(i/2-1)+d)$

The exceptional variety $E_{i/2}$ gives the candidate pole $-(d-1)/2-1/i$ in which we are interested. If $i \neq 2$, its residue is

$$\frac{1}{N_{\frac{i}{2}}} \left( \chi(E_{I_1}^\circ) + \chi(E_{I_2}^\circ)\frac{1}{\alpha_{\frac{i}{2}-1}} + \chi(E_{I_3}^\circ)\frac{1}{\alpha_0} + \chi(E_{I_4}^\circ)\frac{1}{\alpha_0\alpha_{\frac{i}{2}-1}} \right),$$

where

$$I_1 := \{\frac{i}{2}\}, I_2 := \{\frac{i}{2}, \frac{i}{2}-1\}, I_3 := \{\frac{i}{2}, 0\}, I_4 := \{\frac{i}{2}, \frac{i}{2}-1, 0\}.$$

The Euler–Poincaré characteristics $\chi(E_{I_k}^\circ)$, $1 \leq k \leq 4$, are put in Table 1. These are easily computed since $E_{i/2} \cong \mathbb{P}^{d-1}$, and $E_{i/2-1}$ and $E_0$ intersect $E_{i/2}$ in a hyperplane and a smooth quadric, respectively.

| $\chi(E_{I_k}^\circ)$ | d odd | d even |
|---|---|---|
| $k=1$ | $1$ | $-1$ |
| $k=2$ | $0$ | $1$ |
| $k=3$ | $0$ | $2$ |
| $k=4$ | $d-1$ | $d-2$ |

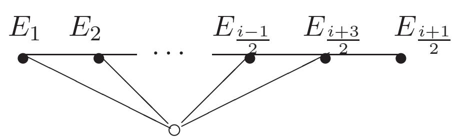
*Table 1*

Using that $\alpha_0 = (3-d)/2 - 1/i$ and $\alpha_{i/2-1} = 2/i$, some easy calculations yield that the residue is nonzero, for all $d \in \mathbb{N}$, $d \geq 4$.

If $i = 2$, we blow up just once in the origin to get an embedded resolution. By using $\alpha_0 = \frac{2-d}{2}, \chi(E_{I_1}^\circ) = 0(d \text{ even}), \chi(E_{I_1}^\circ) = 1(d \text{ odd})$, we conclude that also here the residue is nonzero.

***Embedded resolution for $x_1^i + x_2^2 + \cdots + x_d^2 = 0$, $d \geq 4$, with $i$ odd***

After blowing up $(i+1)/2$ times in the origin, followed by blowing up once more in $D := E_{(i+1)/2} \cap E_{(i-1)/2} \cong \mathbb{P}^{d-2}$, we get an embedded resolution with the following dual intersection graph.




$E_1(2, n)$
$E_2(4, 2d-1)$
$E_{(i-1)/2}(i-1, (d-1)(i-3)/2 + d)$
$E_{(i+1)/2}(i, (d-1)(i-1)/2 + d)$
$E_{(i+3)/2}(2i, (d-1)i + 2)$

The last exceptional variety has $-(d-1)/2 - 1/i$ as candidate pole. The relevant subsets in the computation of the residue are

$$I_1 := \{\frac{i+3}{2}\}, I_2 := \{\frac{i+3}{2}, 0\}, I_3 := \{\frac{i+3}{2}, \frac{i+1}{2}\}$$
$$I_4 := \{\frac{i+3}{2}, \frac{i-1}{2}\}, I_5 := \{\frac{i+3}{2}, \frac{i-1}{2}, 0\}.$$

Here $E_{(i+3)/2}$ is a $\mathbb{P}^1$-bundle over $D$. For $k = 2, 3, 4$ we have that $E_{I_k} \cong D$ and $E_{I_5}$ is a smooth quadric. With the Euler–Poincaré characteristics of Table 2 and $\alpha_0 = (3-d)/2 - 1/i$, $\alpha_{(i-1)/2} = 1/i$ and $\alpha_{(i+1)/2} = (d-1)/2$, we find that the residue is nonzero, for all $d \geq 4$.

| $\chi(E^\circ_{I_k})$ | d odd | d even |
|---|---|---|
| $k=1$ | $0$ | $-1$ |
| $k=2$ | $0$ | $1$ |
| $k=3$ | $d-1$ | $d-1$ |
| $k=4$ | $0$ | $1$ |
| $k=5$ | $d-1$ | $d-2$ |

*Table 2*

Throwing together these results we obtain

$$\{-\frac{d-1}{2}-\frac{1}{i}\mid i\in\mathbb{Z}_{>1}\}\subset\mathcal{P}_d.$$

Now that we checked this expectation, we proceed proving the theorem.

*Remark.*— Notice that $m\in\mathcal{P}_{d-1}$ implies that $m\in\mathcal{P}_d$. Indeed, any polynomial $f$ in $d-1$ variables can be considered as a polynomial in $d$ variables. An embedded resolution for $f^{-1}\{0\}\subset\mathbb{C}^{d-1}$ induces the obvious analogous one for $f^{-1}\{0\}\subset\mathbb{C}^d=\mathbb{C}^{d-1}\times\mathbb{C}$ and, since $\chi(\mathbb{C})=1$, the two associated topological zeta functions are equal. From this observation it follows that it is sufficient to prove that $[-(d-1)/2,-(d-2)/2)\cap\mathbb{Q}\subset\mathcal{P}_d$. As we showed in this section that $-(d-1)/2$ is contained in $\mathcal{P}_{d-1}$ and thus in $\mathcal{P}_d$, we restrict ourselves in the next sections to the subset $(-(d-1)/2,-(d-2)/2)\cap\mathbb{Q}$.

## 6.3 The set $(-1/2,0)\cap\mathbb{Q}$ is a subset of $\mathcal{P}_2$

Considering how candidate poles look like in the formula of the topological zeta function written in terms of newton polyhedra, the number $-(b+2)/(2a+2b)$ seems to appear as a candidate pole of the topological zeta function associated to the nondegenerate polynomial $f(x,y):=x^a(x^b+y^2)$, where $a$ and $b$ are positive integers.

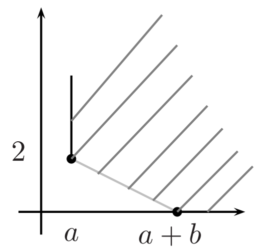
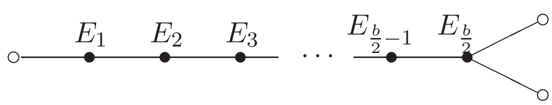



*Figure 1:* $\Gamma$ *of* $f(x,y) := x^a(x^b+y^2)$

An easy computation yields:

**6.2 Lemma** When $a$ and $b$ run through $2\mathbb{Z}_{>0}$, $a \neq 2$, the quotient $-(b+2)/(2a+2b)$ takes all rational values in $(-1/2, 0)$.

Taking the lemma into account, the functions $f(x,y) := x^a(x^b+y^2)$, where $a, b \in 2\mathbb{Z}_{>0}$ and $a \neq 2$, could be a pretty nice choice to obtain all desired poles. Easy calculations give the following dual resolution graph for $f$.




$E_1(a+2, 2)$
$E_2(a+4, 3)$
$E_3(a+6, 4)$
$E_{b/2-1}(a+b-2, b/2)$
$E_{b/2}(a+b, b/2+1)$

Because $E_{b/2}$ is intersected three times by other components, Theorem 4.3 in [Ve2] allows us to conclude that $-(b+2)/(2a+2b)$ is a pole of $Z_{top,f}$.

## 6.4 The set $(-(d-1)/2, -(d-2)/2) \cap \mathbb{Q}$ is a subset of $\mathcal{P}_d$, $\quad d \geq 3$

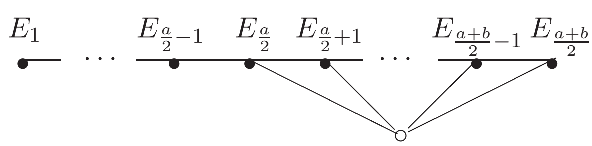
As this set is a translation by $-1/2$ of expected poles in dimension $d-1$, the Thom-Sebastiani principle in [De,L3] is again the motivation why we consider

$$f(x_1, \ldots, x_d) := x_d^2 + \cdots + x_3^2 + x_1^a(x_1^b + x_2^2),$$

where $a \in 2\mathbb{Z}_{>0}$ and $a \neq 2$, to reach the set $(-(d-1)/2, -(d-2)/2) \cap \mathbb{Q}$.

### *Embedded resolution for $z^2 + x^a(x^b + y^2)$*

Let us first explain in dimension 3 which embedded resolution we choose for $z^2 + x^a(x^b + y^2)$ ($a, b \in 2\mathbb{Z}_{>0}$, $a \neq 2$). We first blow up in the singular locus $\{x = z = 0\}$ of $f$ and further always in the singular locus of the strict transform; the first $a/2$ times this is an affine line and the last $b/2$ times it is a point. This is the special case for $d = 3$ in Table 3.
The dual intersection graph looks as follows.




$E_1(2,2)$
$E_2(4,3)$
$E_{a/2}(a, a/2+1)$
$E_{a/2+1}(a+2, a/2+3)$
$E_{a/2+2}(a+4, a/2+5)$
$E_{(a+b)/2}(a+b, a/2+b+1)$

The candidate pole given by the last exceptional surface, $E_{(a+b)/2}$, is equal to

$$-\frac{a/2+b+1}{a+b} = -\frac{b+2}{2a+2b} - \frac{1}{2},$$

and thus covers all rational numbers in $(-1, -1/2)$ if $a$ and $b$ run over $2\mathbb{Z}_{>0}$ and $a \neq 2$.

### *Embedded resolution for $x_d^2 + \cdots + x_3^2 + x_1^a(x_1^b + x_2^2)$, $d > 3$*

The sequence of blowing-ups in Table 3 yields an embedded resolution for

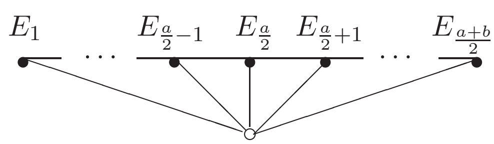
$$f(x_1, \ldots, x_d) := x_d^2 + \cdots + x_3^2 + x_1^a(x_1^b + x_2^2),$$

based on the previous one for $d = 3$.

| number $j$ of blowing-up | centre blowing-up | equation strict transform in relevant chart |
|---|---|---|
| 1 | $x_1 = x_3 = x_4 = \cdots = x_d = 0$ | $x_d^2 + \cdots + x_3^2 + x_1^{a-2}(x_1^b + x_2^2)$ |
| 2 | $x_1 = x_3 = x_4 = \cdots = x_d = 0$ | $x_d^2 + \cdots + x_3^2 + x_1^{a-4}(x_1^b + x_2^2)$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $a/2$ | $x_1 = x_3 = x_4 = \cdots = x_d = 0$ | $x_d^2 + \cdots + x_3^2 + x_1^b + x_2^2$ |
| $a/2+1$ | $(0, 0, \ldots, 0)$ | $x_d^2 + \cdots + x_3^2 + x_1^{b-2} + x_2^2$ |
| $a/2+2$ | $(0, 0, \ldots, 0)$ | $x_d^2 + \cdots + x_3^2 + x_1^{b-4} + x_2^2$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $(a+b)/2$ | $(0, 0, \ldots, 0)$ | $x_d^2 + \cdots + x_3^2 + 1 + x_2^2$ |

*Table 3*

The dual intersection graph here looks as follows.




$E_1(2, d-1)$
$E_2(4, 2d-3)$
$E_{a/2}(a, (d-2)a/2+1)$
$E_{a/2+1}(a+2, (d-2)a/2+d)$
$E_{(a+b)/2}(a+b, (d-2)(a+b)/2+b/2+1)$

Now $-\nu_{(a+b)/2}/N_{(a+b)/2}$ is equal to

$$-\frac{a/2+b+1+((a+b)/2)(d-3)}{a+b} = -\frac{b+2}{2a+2b} - \frac{d-2}{2},$$

which covers the interval $(-(d-1)/2, -(d-2)/2) \cap \mathbb{Q}$ when $a$ and $b$ vary in $2\mathbb{Z}_{>0}$ with $a \neq 2$.

***The rational number $-\nu_{(a+b)/2}/N_{(a+b)/2}$ is a pole of $Z_{top,f}$***

For all $d \geq 3$ and $f(x_1, \ldots, x_d) := x_d^2 + \cdots + x_3^2 + x_1^a(x_1^b + x_2^2)$, we calculate the residue of $Z_{top,f}$ in $-\nu_{(a+b)/2}/N_{(a+b)/2}$. Observe that if $(a+b)/(2+b) \in \mathbb{Z}$, the exceptional variety $E_{(a+b)/(2+b)}$ induces the same candidate pole as $E_{(a+b)/2}$. The other exceptional varieties always give

rise to other candidate poles.
The subsets playing a role in the contribution of $E_{(a+b)/(2+b)}$ to the residue are

$$I_1 := \{\frac{a+b}{2+b}\}, I_2 := \{\frac{a+b}{2+b}, \frac{a+b}{2+b} - 1\}, I_3 := \{\frac{a+b}{2+b}, \frac{a+b}{2+b} + 1\},$$
$$I_4 := \{\frac{a+b}{2+b}, 0\}, I_5 := \{\frac{a+b}{2+b}, \frac{a+b}{2+b} - 1, 0\}, I_6 := \{\frac{a+b}{2+b}, \frac{a+b}{2+b} + 1, 0\}.$$

Notice that if $d = 3$, $E_{(a+b)/(2+b)}$ does not intersect $E_0$.

We have that $E_{(a+b)/(2+b)}$ is isomorphic to the cartesian product of $\mathbb{A}^1$ and the blowing-up of $\mathbb{P}^{d-2}$ in a point. It is also easy to describe the whole intersection configuration on $E_{(a+b)/(2+b)}$.

| $\chi(E^\circ_{I_k})$ | d odd | d even |
|---|---|---|
| $k = 1$ | $0$ | $0$ |
| $k = 2$ | $1$ | $0$ |
| $k = 3$ | $1$ | $0$ |
| $k = 4$ | $0$ | $0$ |
| $k = 5$ | $d-3$ | $d-2$ |
| $k = 6$ | $d-3$ | $d-2$ |

*Table 4*

Using the relevant Euler–Poincaré characteristics of Table 4 and that $\alpha_{(a+b)/(2+b)-1} = 1/i$, $\alpha_{(a+b)/(2+b)+1} = -1/i$, we see that $E_{(a+b)/(2+b)}$ does not give any contribution to the residue in $-\nu_{(a+b)/2}/N_{(a+b)/2}$. Alternatively, this is implied by [Ve1, Proposition 6.5]. This means we only have to take the contribution of $E_{(a+b)/2}$ into account.

To compute this contribution the relevant subsets for the summation in the formula of the topological zeta function are

$$I_1 := \{\frac{a+b}{2}\}, I_2 := \{\frac{a+b}{2}, \frac{a+b}{2} - 1\},$$
$$I_3 := \{\frac{a+b}{2}, 0\}, I_4 := \{\frac{a+b}{2}, \frac{a+b}{2} - 1, 0\}.$$

The Euler–Poincaré characteristics $\chi(E^\circ_{I_k})$, $1 \leq k \leq 4$, are the same as those given in Table 1 and we have $\alpha_0 = -((d-4)a+(d-3)b+2)/(2(a+b))$

and $\alpha_{(a+b)/2-1} = (2-a)/(a+b)$.

As the residue then is equal to

$$\frac{(-2+3a+2b)(da-2a-b+db+2)}{(-2+a)(a+b)(da-4a+2+db-3b)} \quad \text{for } d \text{ odd and}$$

$$\frac{(2+b)(da-2a-b+db+2)}{(-2+a)(a+b)(da-4a+2+db-3b)} \quad \text{for } d \text{ even,}$$

we find that $-(\nu_{(a+b)/2})/(N_{(a+b)/2}) = -(b+2)/(2a+2b) - (d-2)/2$ is a pole of $Z_{top,f}$.

We conclude that $(-(d-1)/2, -(d-2)/2) \cap \mathbb{Q} \subset \mathcal{P}_d$, for all $d \geq 3$.

## 6.5 Some remarks

(1) Instead of achieving this result with the method of resolution of singularities one can find the poles of the topological zeta function of the polynomials

$$x_d^2 + \cdots + x_3^2 + x_1^a(x_1^b + x_2^2) \quad \text{and} \quad x_d^2 + \cdots + x_2^2 + x_1^i$$

with the help of Newton polyhedra. Indeed, we can write down the topological zeta function for these polynomials using the formula of Denef and Loeser, see Theorem 5.4. For example if $f(x_1, \ldots, x_d) := x_d^2 + \cdots + x_3^2 + x_1^a(x_1^b + x_2^2)$, where $a$ and $b$ are positive even integers and $a \neq 2$, put $A := (a+b)s + 1 + b/2 + (d-2)(a+b)/2$ and $B := as + 1 + (d-2)a/2$. We get

$$\begin{aligned} Z_{top,f}(s) &= (d-1)\frac{b}{2AB} + \frac{1}{A} + (d-2)\frac{a}{2B} \\ &+ \frac{s}{s+1}\left(\sum_{d=1}^{d-1}\binom{d-2}{d+1}\left(\frac{a}{2B} + \frac{b}{2AB}\right)(-2)^d \right. \\ &\left. + \sum_{d=1}^{d-1}\binom{d-1}{d}\frac{1}{A}(-2)^d + \sum_{d=1}^{d-2}\binom{d-2}{d}\frac{b}{2AB}(-2)^d\right). \end{aligned}$$

Handling the problem in this way leads to the same results. One just has to be careful with the dual cones of some faces, namely those that are not a rational simplicial cone.

(2) With a similar definition of $\mathcal{P}_d$ in each case, the same results hold for local and global versions of the motivic zeta function, the Hodge zeta function and Igusa's zeta function. Indeed, the results for the topological zeta function imply the results for those 'finer' zeta functions.

# Chapter 7

# The topological zeta function for surfaces resolved by a toric constellation

## 7.1 Introduction

Given a polynomial $f$ in $d$ variables over $\mathbb{C}$, its topological zeta function $Z_{top,f}$ can be calculated by computing an embedded resolution. If $f$ is nondegenerate with respect to its Newton polyhedron, then there exists also the formula for $Z_{top,f}$ in terms of its Newton polyhedron, see Theorem 5.4. We now restrict to surfaces for which there exists an embedded resolution of singularities realised by the blowing-up of a toric constellation that is the constellation of base points of a finitely supported ideal (see Section 3.4). We show that, directly from the tree that represents the toric constellation, one can read all information needed to write down the topological zeta function. To obtain this formula, we use some basic theory of geometry of curves on surfaces. The well-developed formulas for genera and for the topological Euler-Poincaré characteristic for curves on surfaces make that surfaces are very nice varieties to develop a formula for. This formula offers the advantage that it can be written down immediately from the tree and that it could be a tool to prove (or to unmask) the monodromy conjecture for this specific case.

## 7.2 Curves on surfaces and blowing-ups

First we recall some principles of intersection theory for curves on surfaces and some formulas for the Euler-Poincaré characteristic that we will use to calculate the topological zeta function. We refer to [Ha] and [Sha] for the proofs.

Let $X$ be a complex nonsingular projective surface. Denote by $\mathrm{Div}(X)$ the group of divisors on $X$ (note that Weil and Cartier divisors then coincide) and by $\mathrm{Pic}(X)$ the group of invertible sheaves up to isomorphism. In this context $\mathrm{Pic}(X)$ is isomorphic to the group of divisors modulo linear equivalence.

There is a unique pairing $\mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$, denoted by $C \cdot D$ for any two divisors $C, D$ such that

1. if $C$ and $D$ are nonsingular curves meeting transversally, then $C \cdot D$ is the number of points on $C \cap D$;
2. it is symmetric, i.e. $C \cdot D = D \cdot C$;
3. it is additive, i.e. $(C_1 + C_2) \cdot D = C_1 \cdot D + C_1 \cdot D$;
4. it depends only on the linear equivalence classes, i.e. if $C_1$ is linear equivalent with $C_2$, then $C_1 \cdot D = C_2 \cdot D$.

If $C$ and $D$ are curves on $X$ having no common irreducible component and if $P \in C \cap D$, then the *intersection multiplicity* $(C \cdot D)_P$ *of* $C$ *and* $D$ *at* $P$ is defined as the length of the module $\mathcal{O}_{P,X}/(f, g)$ where $f$ and $g$ are local equations of $C$ and $D$ at the point $P$. Then $C \cdot D = \sum_{P \in C \cap D} (C \cdot D)_P$.

Let $P$ be a point of $X$ and let $\pi : \tilde{X} \to X$ be the blowing-up in $P$. Denote by $E$ the exceptional curve on $\tilde{X}$ created by this blowing-up. The natural maps $\pi^*$: $\mathrm{Pic}(X) \to \mathrm{Pic}(\tilde{X})$ and $\mathbb{Z} \to \mathrm{Pic}(\tilde{X})$ defined by $1 \mapsto 1 \cdot E$ give rise to an isomorphism $\mathrm{Pic}(\tilde{X}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$. The intersection theory on $\mathrm{Pic}(\tilde{X})$ is determined by the rules:

1. If $C, D \in \mathrm{Pic}(X)$, then $(\pi^* C) \cdot (\pi^* D) = C \cdot D$;
2. if $C \in \mathrm{Pic}(X)$, then $(\pi^* C) \cdot E = 0$;
3. $E^2 = -1$;

4. if $\pi_*$: $\mathrm{Pic}(\tilde{X}) \to \mathrm{Pic}(X)$ denotes the projection on the first factor and if $C \in \mathrm{Pic}(X)$ and $D \in \mathrm{Pic}(\tilde{X})$, then $(\pi^* C) \cdot D = C \cdot \pi_*(D)$.

Let $C$ be a curve on $X$ and let $P$ be a point of multiplicity $\mathrm{mult}_P(C) =: r$ on $C$. If $\pi : \tilde{X} \to X$ is the blowing-up with centre $P$, then $\pi^* C = \hat{C} + rE$, with $\hat{C}$ the strict transform of $C$. If $\Pi : Z \to X$ is a composition of blowing-ups then we also denote $\hat{C}$ for the strict transform of $C$ at some intermediate stage.

Direct consequences of the above properties are:

1. $r = \hat{C} \cdot E$;

2. if $\Pi : Z \to X$ is a composition of blowing-ups such that $\hat{C}$ and $\hat{D}$ are nonsingular, then
$$(C \cdot D)_P = \mathrm{mult}_P(\hat{C})\mathrm{mult}_P(\hat{D}) + \sum_{Q \to P} \mathrm{mult}_Q(\hat{C})\mathrm{mult}_Q(\hat{D}).$$

Let $C$ be a projective curve and let $P_C$ be the Hilbert polynomial of $C$. The *arithmetic genus* $p_a(C)$ of $C$ is defined as $1 - P_C(0)$. The *geometric genus* $g(C)$ of $C$ is defined as $\dim_{\mathbb{C}} \Gamma(C, \omega_C)$, where $\omega_C$ is the canonical sheaf of the curve $C$. For a projective nonsingular curve these notions coincide.

If $C$ is a curve of degree $d$ in $\mathbb{P}^2$, then
$$p_a(C) = (d-1)(d-2)/2.$$

If $P$ is a point of multiplicity $r$ on $C \subset X$ and if $\pi : \tilde{X} \to X$ is the blowing-up with centre $P$, then
$$p_a(\hat{C}) = p_a(C) - \frac{r(r-1)}{2}.$$

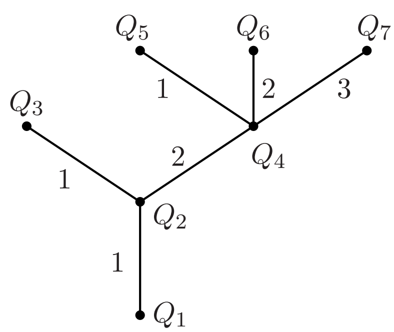
For a nonsingular projective curve $C$ it holds that
$$\chi(C) = 2 - 2p_a(C).$$

## 7.3 Computation of the numbers $\chi(E_I^\circ)$

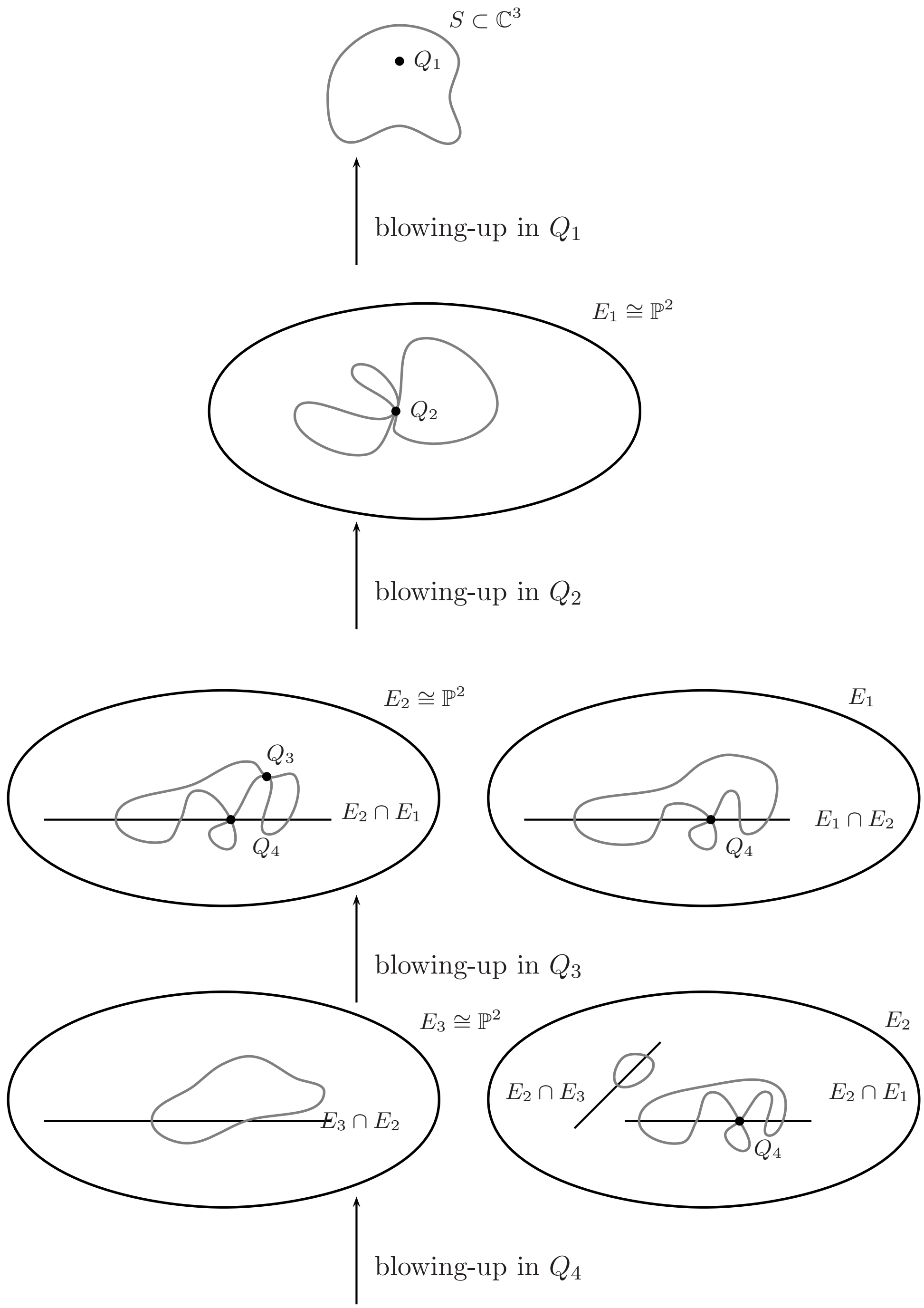
We consider an idealistic toric cluster in $\mathbb{C}^3$ and a complex polynomial $f$ in three variables in a finitely supported ideal such that the cluster gives an embedded resolution for the surface $S := V(f) \subset \mathbb{C}^3$. To determine the topological zeta function of $f$, we determine the numbers $\chi(\overset{\circ}{E}_I)$. We will denote the curves $\hat{S} \cap E_i$ by $C_i$, whatever the stage is in which we look at $\hat{S}$.

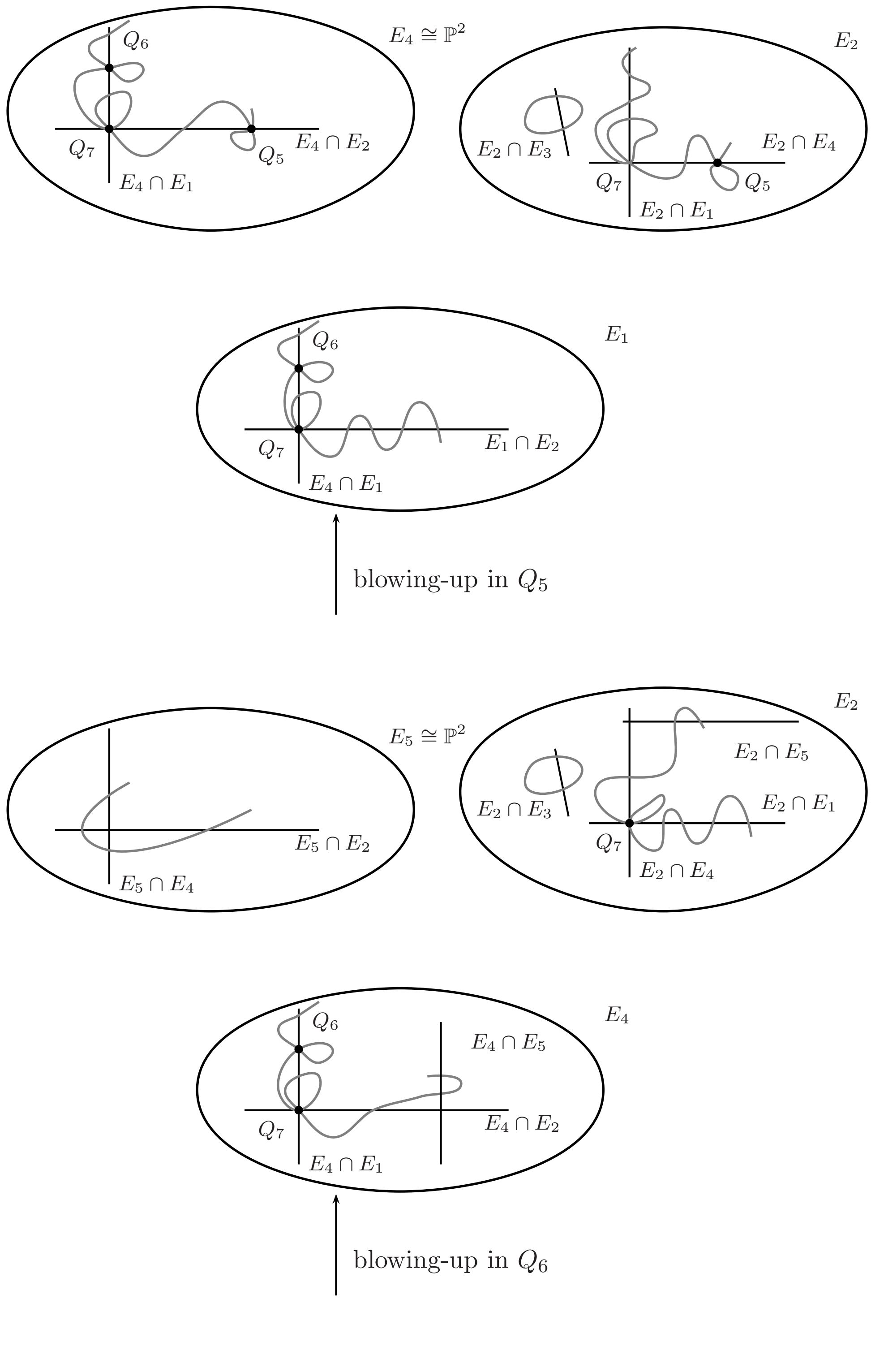
First of all, notice that when blowing up in a point of multiplicity $m$ on $S$ and $E$ being the created exceptional divisor, the curve $\hat{S} \cap E$ has degree $m$. Another important observation is that if $Q \in E$, then the multiplicity of $Q$ on $\hat{S} \cap E$ is equal to the multiplicity of $Q$ on $\hat{S}$. This follows from the fact that $E$ is transversal to $\hat{S}$ in $Q$. If not, the singularities could never be resolved by blowing up in points.

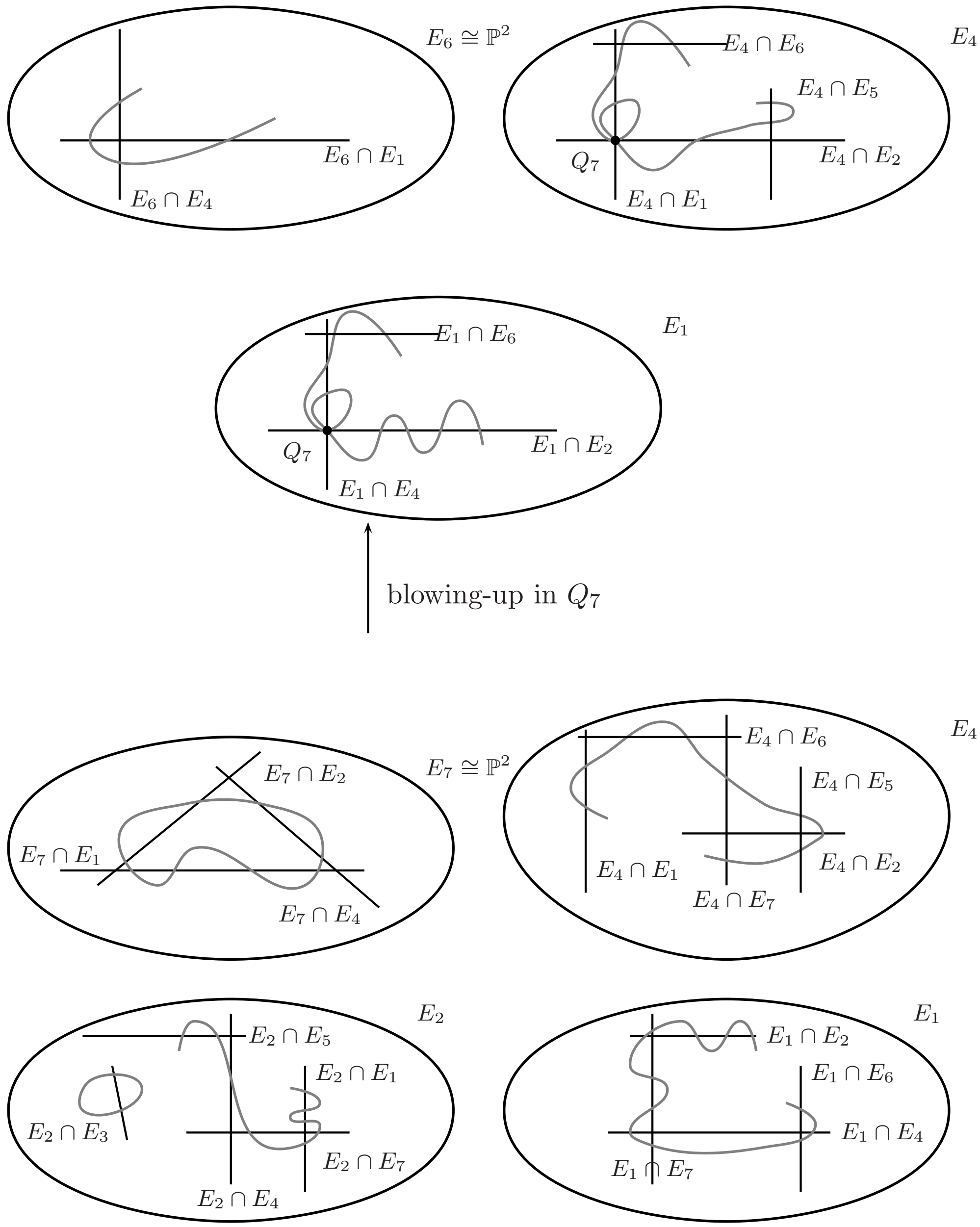
**7.1 Example** Consider the toric constellation represented by the following tree.




Let $S$ be a surface in $\mathbb{C}^3$ for which the above toric constellation gives an embedded resolution. We follow the resolution process and we picture the intersections that are relevant in the calculation of the numbers $\chi(\overset{\circ}{E}_I)$. These pictures are not meant to represent a realistic constellation, remember that the multiplicities of the points in the constellation should satisfy the proximity equalities. The gray curve pictured in the ambient $E_j$ represents the curve $C_j$.


$S \subset \mathbb{C}^3$
$Q_1$
blowing-up in $Q_1$
$E_1 \cong \mathbb{P}^2$
$Q_2$
blowing-up in $Q_2$
$E_2 \cong \mathbb{P}^2$
$Q_3$
$E_2 \cap E_1$
$Q_4$
$E_1$
$E_1 \cap E_2$
$Q_4$
blowing-up in $Q_3$
$E_3 \cong \mathbb{P}^2$
$E_3 \cap E_2$
$E_2$
$E_2 \cap E_3$
$E_2 \cap E_1$
$Q_4$
blowing-up in $Q_4$


$E_4 \cong \mathbb{P}^2$
$Q_6$
$Q_7$
$Q_5$
$E_4 \cap E_2$
$E_4 \cap E_1$
$E_2$
$E_2 \cap E_3$
$Q_7$
$Q_5$
$E_2 \cap E_4$
$E_2 \cap E_1$
$E_1$
$Q_6$
$Q_7$
$E_1 \cap E_2$
$E_4 \cap E_1$
blowing-up in $Q_5$
$E_5 \cong \mathbb{P}^2$
$E_5 \cap E_2$
$E_5 \cap E_4$
$E_2$
$E_2 \cap E_5$
$E_2 \cap E_3$
$E_2 \cap E_1$
$Q_7$
$E_2 \cap E_4$
$E_4$
$Q_6$
$E_4 \cap E_5$
$Q_7$
$E_4 \cap E_2$
$E_4 \cap E_1$
blowing-up in $Q_6$


$E_6 \cong \mathbb{P}^2$
$E_6 \cap E_1$
$E_6 \cap E_4$
$E_4$
$E_4 \cap E_6$
$E_4 \cap E_5$
$E_4 \cap E_2$
$Q_7$
$E_4 \cap E_1$
$E_1$
$E_1 \cap E_6$
$Q_7$
$E_1 \cap E_2$
$E_1 \cap E_4$
blowing-up in $Q_7$
$E_7 \cong \mathbb{P}^2$
$E_7 \cap E_2$
$E_7 \cap E_1$
$E_7 \cap E_4$
$E_4$
$E_4 \cap E_6$
$E_4 \cap E_5$
$E_4 \cap E_1$
$E_4 \cap E_2$
$E_4 \cap E_7$
$E_2$
$E_2 \cap E_5$
$E_2 \cap E_1$
$E_2 \cap E_3$
$E_2 \cap E_7$
$E_2 \cap E_4$
$E_1$
$E_1 \cap E_2$
$E_1 \cap E_6$
$E_1 \cap E_4$
$E_1 \cap E_7$

□

We now proceed to the computation of the $\chi(\overset{\circ}{E}_I)$. We will write $m_j$ for the multiplicity of the point $Q_j$ on $\hat{S}$ and $E_0$ for the strict transform $\hat{S}$. We will also use some notations that are typical in the context of clusters, we refer to Section 3.2 for them.

1. **$I := \{0, i, j\}$ with $0 < i < j$ and $j \to i$.**

   From the number of intersection points of $C_j$ and $E_i$ in $E_j \cong \mathbb{P}^2$, we subtract the number of points in which we will blow up. Then we get $\chi(\overset{\circ}{E}_I) = m_j - \sum_{\substack{k \succ j \\ k \to i}} (C_j \cdot (E_i \cap E_j))_{Q_k}$. We can conclude

   $$\chi(\overset{\circ}{E}_I) = m_j - \sum_{\substack{k \to i \\ k \twoheadrightarrow j}} m_k.$$

   Note that this number is positive because the cluster is satisfying the proximity inequalities as it is idealistic.

2. **$I := \{i, j, k\}$ with $0 \neq i < j < k, k \to i$ and $k \to j$.**

   The contribution to $\chi(\overset{\circ}{E}_I)$ comes from the intersection point of $E_i \cap E_j \cap E_k$ unless it is a point in which we will blow up. We can express this as follows:

   $$\chi(\overset{\circ}{E}_I) = 1 - \#\{l \mid l \to i, l \to j \text{ and } l \to k\}.$$

3. **$I := \{0, i\}$ with $0 \neq i$.**

   We look at $E_i$ in the final stage. There we have to subtract from $E_0 \cap E_i$ the intersection points with the other exceptional components.

   $$\chi(\overset{\circ}{E}_I) = \chi(C_i) - \sum_{j \to i} \chi(E_0 \widehat{\cap \overset{\circ}{E}_i \cap} E_j) - \sum_{i \to j} \chi(E_0 \widehat{\cap \overset{\circ}{E}_i \cap} E_j).$$

   We have $\chi(C_i) = 2 - 2p_a(C_i)$ for the nonsingular $C_i$ that can be irreducible or reducible. This leads to the formula

   $$\begin{aligned} \chi(\overset{\circ}{E}_I) &= m_i(3 - m_i) + \sum_{j \to i} m_j(m_j - 1) \\ &\quad - \sum_{j \to i} (m_j - \sum_{\substack{k \to i \\ k \twoheadrightarrow j}} m_k) - \sum_{i \to j} (m_i - \sum_{\substack{k \to j \\ k \twoheadrightarrow i}} m_k). \end{aligned}$$

4. $I := \{i, j\}$ **with** $0 \neq i < j, j \to i$**.**

   We compute the contribution from the configuration in $E_j \cong \mathbb{P}^2$.

$$\begin{aligned}\chi(\overset{\circ}{E}_I) &= 2 - \left(\chi(E_0 \overset{\circ}{\widehat{\cap E_i \cap}} E_j) + \#P_{ij} + \#Q_{ij} - \#M_{ij}\right) \\ &= 2 - (m_j - \sum_{\substack{k\to i\\ k\twoheadrightarrow j}} m_k) - \#P_{ij} - \#Q_{ij} + \#M_{ij},\end{aligned}$$

   with

$$\begin{aligned}P_{ij} &:= \{k \mid k \succ j, k \to i\} \\ Q_{ij} &:= \{k \mid k \neq i, j \to k\} \\ M_{ij} &:= \{k \mid k \succ j, k \to i \text{ and } \exists l : l \neq i, k \to l \text{ and } j \to l\}.\end{aligned}$$

5. $I := \{i\}$ **with** $i \neq 0$**.**

   We look in $E_i \cong \mathbb{P}^2$ and find

$$\begin{aligned}\chi(\overset{\circ}{E}_I) &= 3 - \left(\chi(E_0 \overset{\circ}{\cap} E_i) + \#P_i + 2\#Q_i - \binom{\#Q_i}{2}\right) \\ &= 3 + m_i(m_i - 3) - \sum_{j\to i} m_j(m_j - 1) \\ &\quad + \sum_{j\to i}(m_j - \sum_{\substack{k\to i\\ k\twoheadrightarrow j}} m_k) + \sum_{i\to j}(m_i - \sum_{\substack{k\to j\\ k\twoheadrightarrow i}} m_k) - \\ &\quad \#P_i - 2\#Q_i + \binom{\#Q_i}{2},\end{aligned}$$

   with

$$\begin{aligned}Q_i &:= \{k \mid i \to k\} \\ P_i &:= \{k \mid k \succ i \text{ and } \nexists l : i \to l \text{ and } k \to l\}.\end{aligned}$$

6. For $I$ not of the form of one of the sets described above, $\chi(\overset{\circ}{E}_I) = 0$.

## 7.4 The numerical data

Also the numerical data are completely determined by the tree.
We get the numbers $N_i$ via the recursive formula

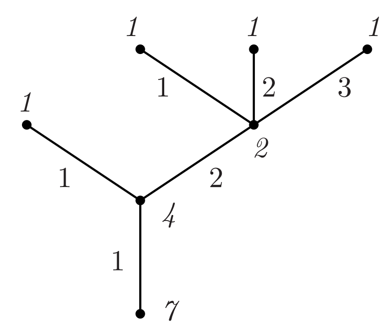
$$N_i = m_i + \sum_{i \to j} N_j.$$

For the $\nu_i$, we find

$$\nu_i = \sum_{i \to j} (\nu_j - 1) + 3.$$

## 7.5 Example

We take the constellation of Example 7.1 with the following multiplicities:




The above constellation gives an embedded resolution for the function

$$\begin{aligned} h(x,y,z) \quad := \quad & x^{12} + y^7 + z^7 + x^9 y + x^9 z + x^4 y^4 + x^3 z^4 + xyz^5 + xy^5 z + x^2 yz^4 \\ & + x^2 y^4 z + x^3 yz^3 + x^4 y^2 z^2 + x^4 y^3 z + x^5 yz^2 + x^5 y^2 z + x^7 yz. \end{aligned}$$

Some of the Euler-Poincaré characteristics needed to write down the topological zeta function are

$$\chi(E_0 \cap \overset{\circ}{\widehat{E_1}} \cap E_4) = 0, \qquad \chi(E_1 \cap \overset{\circ}{\widehat{E_2}} \cap E_4) = 0,$$

$$\chi(\overset{\circ}{\widehat{E_0 \cap E_1}}) = -17, \qquad \chi(\overset{\circ}{\widehat{E_1 \cap E_7}}) = -1$$

and so we get its topological zeta function:

$$\frac{200323s^4 + 289778s^3 + 150448s^2 + 32376s + 2295}{(39s+17)(20s+9)(s+1)(7s+3)(11s+5)}.$$

$\square$

# Chapter 8

# On monodromy for surfaces resolved by a toric constellation

## 8.1 Introduction

Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function and consider an embedded resolution of singularities $\pi : X \to \mathbb{C}^d$ of $f^{-1}\{0\}$ that is an isomorphism outside the inverse image of 0. We write $E_j, j \in S$, for the irreducible components of $\pi^{-1}(f^{-1}\{0\})$ and $(N_j, \nu_j), j \in S$, for the numerical data of the embedded resolution $(X, \pi)$. Suppose that $f = 0$ has only one isolated singularity and say that the $E_j$, $j \in J := \{1, \cdots, r\}$, are the exceptional irreducible components of $\pi^{-1}(\{0\})$. Recall that A'Campo showed that the zeta function of monodromy $\zeta_f(t)$ of $f$ is equal to

$$\zeta_f(t) = \prod_{j=1}^{r} (1 - t^{N_j})^{-\chi(\mathring{E}_j)}.$$

The monodromy conjecture states that a pole $s$ of the topological zeta function $Z_{top,f}$ associated to $f$ induces an eigenvalue $e^{2\pi i s}$ of monodromy of $f$.

Now fix a candidate pole $s := -\nu_j/N_j$, $j \in J$, of $Z_{top,f}$. We write $\nu_j/N_j$ as $a/b$ such that $a$ and $b$ are coprime and we define the set $J_b := \{j \in J \mid \; b \mid N_j\}$.

It follows from A'Campo's formula that

$$\begin{gathered} e^{2\pi i s} \text{ is a zero or pole of } \zeta_f \\ \Updownarrow \\ \textstyle\sum_{j\in J_b} \chi(\overset{\circ}{E}_j) \neq 0. \end{gathered}$$

In this chapter we consider generic surfaces for which there exists an embedded resolution of singularities realised by the blowing-up of a toric constellation that is the constellation of base points of a finitely supported ideal (see Section 3.4). In general there can be a lot of cancelations which make that $\sum_{j\in J_b} \chi(\overset{\circ}{E}_j) = 0$. We investigate this phenomenon partially in this specific context, in particular we study when $\chi(\overset{\circ}{E}_j) < 0$ for some $j \in J$. We do that in a geometrical way.

Let $I \subset \mathbb{C}[x, y, z]$ be a finitely supported monomial ideal and let $S$ be a surface given by a general element in $I$ such that we are in the context of Section 3.4. We will write $\hat{S}$ for its strict transform, whatever the stage is. We will denote the curves $E_j \cap \hat{S} \subset E_j \cong \mathbb{P}^2$ by $C_j$, $j \in J$, and their strict transforms by $\hat{C}_j$.

In Section 8.2 we provide some results that we will often apply in the search for situations where $\chi(\overset{\circ}{E}_j) < 0$. In Section 8.3 we pass to the computation of the numbers $\chi(\overset{\circ}{E}_j)$. Sometimes an example illustrates the configuration. The curve $C_j$ is then represented by the gray curve. An application is given in Section 8.4. We show that for an exceptional component $E_j$ with $\chi(\overset{\circ}{E}_j) > 0$ it holds that $e^{-2\pi i\nu_j/N_j}$ is an eigenvalue of monodromy of $f$. As a corollary we obtain that a candidate pole $-\nu_j/N_j$ of $Z_{top,f}$ of order 1 that is a pole of order 1 induces an eigenvalue of monodromy of $f$.

## 8.2 Preliminary results

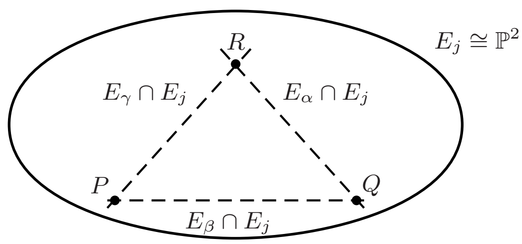
**8.1 Lemma** *$x, y$ or $z$ can not be a common factor in the equation of $C_j$.*

*Proof.* Suppose that $x$ is a factor in the equation of $C_j$. Then the monomial ideal $I$ would not be finitely supported. Indeed, at the moment that, say $x$ appears as common factor in the equation of $C_j$, the strict transform of $C_j$ in some affine chart takes the form $xg_x + zg_z$, for some polynomials $g_x, g_z$ in $\mathbb{C}[x, y, z]$. It is now obvious that the ideal can not be principalised by blowing up a toric constellation. ∎

**8.2 Lemma** *The curve $C_j \subset \mathbb{P}^2$ is generically reducible if and only if its equation is of the form $h(m, n) = 0$ with $h$ a homogeneous polynomial in two variables of degree at least 2, and $m$ and $n$ two monomials.*

*Proof.* The claim follows from [Bo,Deb,Na, Thm. 1.2]. ∎

Notice also that the setting of toric constellations in dimension 3 implies that the configuration in $E_j \cong \mathbb{P}^2$ is as in the picture below.




$E_\alpha \cap E_j$, $E_\beta \cap E_j$ and $E_\gamma \cap E_j$ are the exceptional lines - they are coordinate lines - that can appear and the points $P$, $Q$ and $R$ are the only points in which it is allowed to blow up.

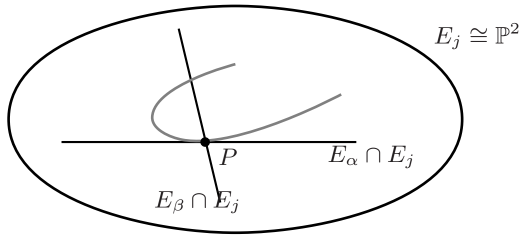
## 8.3 Determination of the cases in which appears a negative $\chi(E_j^\circ)$

**8.3 Lemma** *If the equation of $C_j$ contains three variables, then $\chi(E_j^\circ) \geq$* 0.

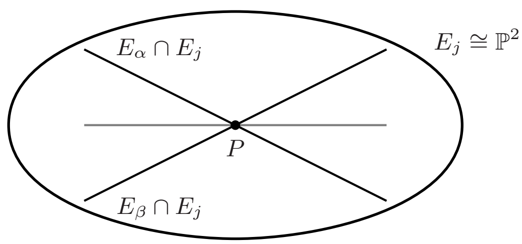
*Proof.* Suppose that $\chi(E_j^\circ) < 0$. We split cases according to whether $C_j$ is irreducible or not.

1. If the curve $C_j$ is irreducible, then $\chi(C_j) \leq 2$ and if $\chi(E_j^\circ) < 0$, the configuration in $E_j$ should consist of $C_j$ and two lines such that these three curves intersect in exactly one point, say $P$. Moreover $\chi(C_j)$ should then be equal to 2.
   When the degree $d$ of $C_j$ is at least 2 and if one of the lines would

not be a principal tangent line to $C_j$ in $P$, then this line would intersect $C_j$ in another point. This is a contradiction.



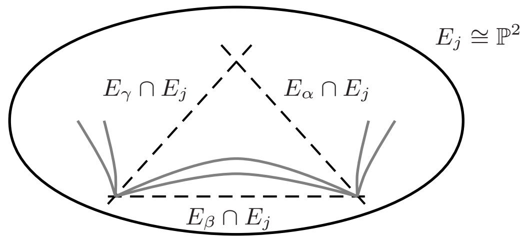

If follows that both lines should be principal tangent lines. Hence, $C_j$ should be analytically reducible in $P$, and consequently $\chi(C_j) < 2$.
So suppose now that $\deg(C_j) = 1$ as in the figure below.




In an affine chart where one can see $P$, one can write $E_\alpha \leftrightarrow x = 0$, $E_\beta \leftrightarrow y = 0$ and then $C_j$ should have an equation of the form $c_x x + c_y y = 0$ with $c_x$ and $c_y$ complex numbers. This contradicts the fact that the equation of $C_j$ contains the three variables. Hence, $\chi(\overset{\circ}{E}_j) \geq 0$.

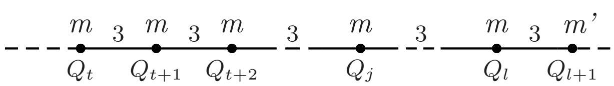
2. Suppose now that the curve $C_j$ is reducible. Lemma 8.2 implies that the equation of $C_j$ is of the form $h(m, n) = 0$ with $h$ a homogeneous polynomial in two variables of degree at least 2, and $m$ and $n$ two monomials. Then by Lemma 8.1 one easily verifies that the pair of monomials $(m, n)$ must be of the form $(x^a y^b, z^{a+b})$. We may suppose moreover that $a$ and $b$ are coprime (otherwise we change $h$). The decomposition in irreducible components of $C_j$ is then

$$\prod_{i=1}^{k}(x^a y^b - c_i z^{a+b}) \quad \text{with the } c_i \in \mathbb{C}^*.$$




As the curves $x^a y^b - c_i z^{a+b} = 0$ are rational with Euler characteristic 2, we get $\chi(\overset{\circ}{E}_j) = 0$ or $\chi(\overset{\circ}{E}_j) = 1$, depending on the position and the number of exceptional components that are present. ∎

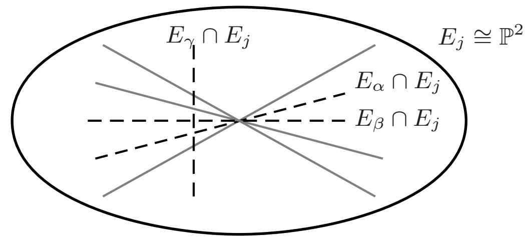
We continue to search for cases in which appears an exceptional component $E_j$ for which $\chi(\overset{\circ}{E}_j) < 0$. It follows by Lemma 8.1 that the equation of $C_j$ can not contain just one variable. The remaining case to investigate is when the equation of $C_j$ contains exactly two variables, say $x$ and $y$. As $C_j$ then has a homogeneous equation in two variables, say of degree $m$, it follows that the curve $C_j$ consists of $m$ lines having exactly one point in common. From Lemma 8.1 it follows that $x^m$ and $y^m$ certainly appear in the equation of $C_j$. The subchain in which $Q_j$ is contained is then of the form




In this chain $Q_t$ is the point with the lowest level for which an edge with label 3 is leaving and that also has multiplicity $m$. We suppose that $Q_l$ is the point in the chain with the highest level for which its multiplicity is equal to $m$. The point $Q_j$ can be equal to $Q_t$ and $Q_{l+1}$ can be absent. As $C_j$ is of degree $m$, the point $Q_j$ has multiplicity $m$ on the surface. Obviously the multiplicity of $Q_{j+1}$ is also $m$.

We will now study $\chi(\overset{\circ}{E}_j)$.

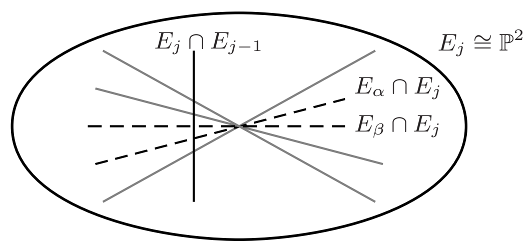
- If $Q_j = Q_t$, then the configuration in $E_j \cong \mathbb{P}^2$ is as in the picture




If the exceptional line $E_\gamma \cap E_j$ is present, then $\chi(\overset{\circ}{E}_j)$ is always equal to 0. Suppose now that $E_\gamma \cap E_j$ does not appear. This means that $Q_t$ is moreover the point with the lowest level in the chain from which an edge with label 3 is leaving.

1. If $Q_j$ is the origin of the constellation, then $\chi(\overset{\circ}{E}_j) = 2 - m$.
2. If there is exactly one point, say $Q_\alpha$, for which $j \to \alpha$, then $\chi(\overset{\circ}{E}_j) = 1 - m$.
3. Finally, if there exist two points, say $Q_\alpha$ and $Q_\beta$, for which $j \to \alpha$ and $j \to \beta$, then $\chi(\overset{\circ}{E}_j) = -m$.

- If $j \in \{t+1, \cdots, l-1\}$, then the configuration in $E_j$ is as in the following picture




and then $\chi(\overset{\circ}{E}_j) = 0$.

**Conclusion:**

*$\chi(\overset{\circ}{E}_j) < 0$ if and only if the configuration in $E_j \cong \mathbb{P}^2$ consists at least of three lines - possibly exceptional - that are all going through the same point.*

## 8.4 Application

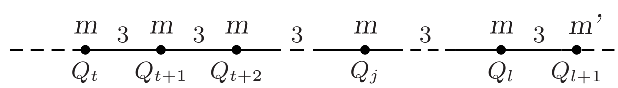
We can use the result obtained in the previous section to confirm a part of the monodromy conjecture in this particular setting.

Let $f$ be a complex polynomial in three variables representing a surface that suits in the context of the previous sections. Fix a candidate pole $s := -\nu_j/N_j$ of $Z_{top,f}$. We write $\nu_j/N_j$ as $a/b$ such that $a$ and $b$ are coprime. Remember that

$$\begin{gathered} e^{2\pi i s} \text{ is a zero or pole of } \zeta_f \\ \Updownarrow \\ \textstyle\sum_{i \in J_b} \chi(\mathring{E}_i) \neq 0. \end{gathered}$$

We observe the following.

**8.4 Lemma** *Let $\chi(\mathring{E}_t) < 0$ such that we are in the situation*




*where $Q_t$ is the point in the chain with the lowest level for which an edge with label 3 is leaving and where $Q_l$ is the point in this chain with the highest level for which its multiplicity is equal to $m$.*

1. *If a set $J_b$ contains the index $t$, then it also contains the indices in $\{t+1, \cdots, l\}$.*
2. *If $\frac{\nu_l}{N_l} = \frac{c}{d}$ with $c$ and $d$ coprime, then $t \notin J_d$.*

*Proof.* If we denote the numerical data of $E_t$ by $(N, \nu)$, then, independently of the number of points $Q_s$ for which $t \to s$, one easily computes that the numerical data for $i \in \{t+1, \cdots, l\}$ are

$$E_i((i-t+1)N, (i-t+1)\nu - (i-t)).$$

Now the first assertion follows immediately.
To see the second claim, suppose that $t \in J_d$. Then $d \mid N$ which implies that

$$l - t + 1 | (l-t+1)\nu - (l-t).$$

This contradiction closes the proof. ∎

We can now give an application of the characterisation of the negative $\chi(\mathring{E}_j)$.

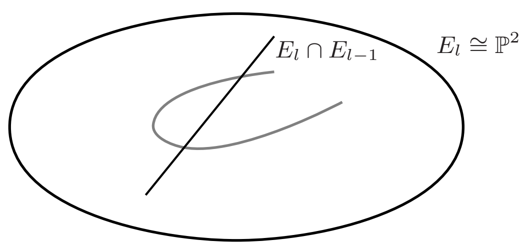
**8.5 Theorem** *If $\chi(\overset{\circ}{E_j}) > 0$, then $e^{-2\pi i \frac{\nu_j}{N_j}}$ is an eigenvalue of monodromy of $f$.*

*Proof.* Suppose that $E_j$ is an exceptional component such that $\chi(\overset{\circ}{E_j}) > 0$. To prove that $e^{-2\pi i\nu_j/N_j}$ is an eigenvalue of monodromy of $f$, we show that $e^{-2\pi i\nu_j/N_j}$ is a pole of $\zeta_f$. We write $\nu_j/N_j$ as $a/b$ with $a$ and $b$ coprime. If $J_b$ does not contain an index $t$ for which $\chi(\overset{\circ}{E_t}) < 0$, then there is nothing to verify. So suppose now that $\chi(\overset{\circ}{E_t}) < 0$ and that $t \in J_b$. From Lemma 8.4 it follows that $E_j \neq E_l$ and that $l \in J_b$. We will show that $\chi(\overset{\circ}{E_t}) + \chi(\overset{\circ}{E_l}) \geq 0$. Let us therefore study the configuration in $E_l \cong \mathbb{P}^2$.
The equation of $C_{l-1}$ is of the form $c_0x^m + c_1x^{m-1}y + \cdots + c_my^m$, with $c_k$ a complex number ($0 \leq k \leq m$) and $c_0$ and $c_m$ different from 0. It follows from Lemma 8.1 that the equation of $C_l$ is then equal to

$$c_0x^m + c_1x^{m-1}y + \cdots + c_my^m + zg(x,y,z)$$

for some polynomial $g \in \mathbb{C}[x,y,z]$. In particular we have that $C_l$ cannot be of the form as in Lemma 8.2 and thus that it is irreducible. Hence $\chi(C_l) \leq 2$.

1. If $Q_t$ is the origin of the constellation, then the configuration in $E_l \cong \mathbb{P}^2$ is like in the figure



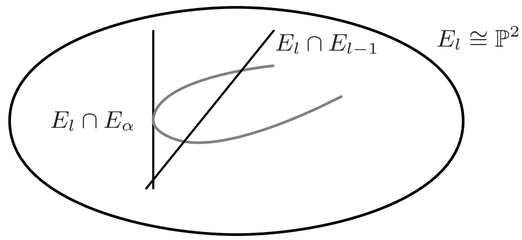

The curve $C_l$ can be singular. We get

$$\begin{aligned}\chi(\overset{\circ}{E_l}) &= \chi(E_l) - (\chi(C_l) + (\chi(E_{l-1}) - m)) \\ &\geq 3 - (2 + (2 - m)) \\ &= m - 1\end{aligned}$$

and

$$\chi(\overset{\circ}{E_t}) + \chi(\overset{\circ}{E_l}) \geq (2-m) + (m-1) = 1.$$

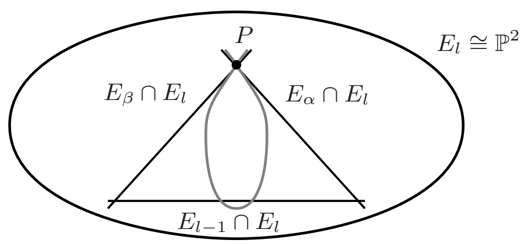
2. Suppose that there exists exactly one point, say $Q_\alpha$, for which $t \to \alpha$. Then $\chi(\overset{\circ}{E}_l)$ is minimal if $C_l$ and $E_l \cap E_\alpha$ intersect only in one point.




Again we find $\chi(\overset{\circ}{E}_l) \geq m - 1$. Then

$$\chi(\overset{\circ}{E}_t) + \chi(\overset{\circ}{E}_l) \geq (1 - m) + (m - 1) = 0.$$

3. When there exist two points, say $Q_\alpha$ and $Q_\beta$, to which $Q_t$ is proximate, then $\chi(\overset{\circ}{E}_l)$ is minimal when $E_\alpha, E_\beta$ and $C_l$ intersect in exactly one point, say in $P$.




Moreover, to have $\chi(\overset{\circ}{E}_l) = m - 1$, one should also require that $\chi(C_l) = 2$. If $m \geq 2$, then $E_\alpha \cap E_l$ and $E_\beta \cap E_l$ are principal tangent lines to $C_l$ in $P$ but then $\chi(C_l) \leq 1$. If $m = 1$, then the equation of $C_l$ must be $x + c_y y + c_z z = 0$ for some $c_y, c_z \in \mathbb{C}^*$. But then $C_l$ intersects the three coordinate lines in three different points and $\chi(\overset{\circ}{E}_l) = 1$. It follows that we always have that $\chi(\overset{\circ}{E}_l) \geq m$ and thus

$$\chi(\overset{\circ}{E}_t) + \chi(\overset{\circ}{E}_l) \geq -m + m = 0.$$

This study permits us to conclude that $\sum_{i \in J_b} \chi(\overset{\circ}{E}_i) > 0$. Hence, $e^{-2\pi i \frac{\nu_j}{N_j}}$ is an eigenvalue of monodromy of $f$.

■

**8.6 Corollary** *If $-\nu_j/N_j$ is a candidate pole of $Z_{top,f}$ of order 1 that is a pole, then $e^{-2\pi i\nu_j/N_j}$ is an eigenvalue of monodromy of $f$.*

*Proof.* In [Ve3] it is shown that then there exists an exceptional component $E_k$ such that $\nu_k/N_k = \nu_j/N_j$ and such that $\chi(\overset{\circ}{E}_k) > 0$. The result follows now immediately from Theorem 8.5. ■

Actually Veys shows in [Ve3] that if $E_j$ is created by blowing up a point and if $\chi(\overset{\circ}{E}_j) < 0$, that then the contribution of $E_j$ to the residue of $-\nu_j/N_j$ for $Z_{top,f}$ is equal to 0. In the following corollary we give an elementary proof of this result in this very particular setting.

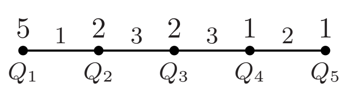
**8.7 Corollary** *If $f$ is nondegenerate with respect to its Newton polyhedron and if $\chi(\overset{\circ}{E}_j) < 0$, then $E_j$ does not give rise to a pole of $Z_{top,f}$.*

*Proof.* Indeed, if $\chi(\overset{\circ}{E}_j) < 0$, then we find that

$$m_j^2 = \sum_{i \to j} m_i^2.$$

This means that the valuation induced by $E_j$ is not Rees (see Section 3.5) and hence does not give rise to a facet in the Newton polyhedron of the polynomial $f$ (see Section 3.3). If $f$ is nondegenerate with respect to its Newton polyhedron, then it is shown in [De,L1] that the poles of $Z_{top,f}$ are induced by the facets of its Newton polyhedron. Hence, if $\chi(\overset{\circ}{E}_j) < 0$, then $E_j$ does not give rise to a pole. ■

**8.8 Example** The surface with equation $f := x^7+y^5+z^5+x^3y^2+x^4z^2 = 0$ has an embedded resolution given by the constellation




For the $\chi(\overset{\circ}{E}_j)$ we find

$$\chi(\overset{\circ}{E}_1) = 9, \quad \chi(\overset{\circ}{E}_2) = -1, \quad \chi(\overset{\circ}{E}_3) = 1, \quad \chi(\overset{\circ}{E}_4) = -1, \quad \chi(\overset{\circ}{E}_5) = 1.$$

The numerical data are

$$E_1(5,3), \quad E_2(7,5), \quad E_3(14,9), \quad E_4(20,13), \quad E_5(40,25).$$

Our results show that $E_1, E_3$ and $E_5$ give rise to an eigenvalue of monodromy of $f$.
For the topological and the monodromy zeta function we get

$$Z_{top,f}(s) = \frac{9(106s^2 + 107s + 25)}{5(s+1)(14s+9)(8s+5)}$$

and

$$\zeta_f(t) = \frac{(1-t^7)(1-t^{20})}{(1-t^5)^9(1-t^{14})(1-t^{40})}.$$

$\square$

# Appendix A

# On Poincaré series for hypersurfaces

## A.1 Introduction

There exist many Poincaré series for a hypersurface $H$ in an ambient variety $X$. One tries to find valuations such that the associated Poincaré series is an interesting object. Very nice results in this direction were obtained e.g. for plane curves, for quasi-homogeneous polynomials and for normal surface singularities.

In this chapter we consider some ways to define Poincaré series for hypersurfaces; we situate the existing examples in this outline. In Section A.3 we mention some interesting relations to investigate; we recall known results and we give a new result for toric hypersurfaces.

## A.2 Poincaré series for hypersurfaces

Curves have exactly one resolution and normal surface singularities have a minimal resolution. In this sense curves and normal surface singularities are special. In general, one can consider an arbitrary resolution or just a proper birational morphism and one uses the induced valuations to define a Poincaré series.

In concrete, to obtain a Poincaré series for a hypersurface, one can proceed in one of the following two ways.

- Let $H$ be a hypersurface embedded in an ambient variety $X$ by the map $\epsilon : \mathcal{O}_X \to \mathcal{O}_H$ and let $h = 0$ be the equation of $H$. We first make the following observation. There is a one-to-one correspondence between the ideals of $\mathcal{O}_H$ and the ideals of $\mathcal{O}_X$ containing $h$; hence a Poincaré series on $X$ induces a Poincaré series on $H$ and vice versa. Notice that under this correspondence, an ideal defined by valuations as in Section 1.3 does not have to induce an ideal defined by valuations. Still the ideals define a multi-indexfiltration and in general, one calls the series induced in this way Poincaré series.

  In particular, let $\{\nu_1, \cdots, \nu_r\}$ be a set of discrete valuations on $\mathcal{O}_{X,o}$ defining the ideals

  $$M(\underline{v}) := \{g \in \mathcal{O}_{X,o} \mid \nu_j(g) \geq v_j, 1 \leq j \leq r\}, \quad \underline{v} \in \mathbb{Z}^r.$$

  Let $J(\underline{v}) := M(\underline{v}) + (h)$ and let $I(\underline{v}) := \epsilon(J(\underline{v}))$. Then the Poincaré series determined by the ideals $J(\underline{v})$ and the one determined by the ideals $I(\underline{v})$ coincide. We will denote the Poincaré series induced by the ideals $M(\underline{v})$ by $P_X$ and the 'embedded' Poincaré series induced by the ideals $I(\underline{v})$ by $P_H^i$.

- Secondly, valuations on $\mathcal{O}_{H,o}$ induce a Poincaré series for $H$. This series is mostly used when one considers a resolution for $H$. Sometimes a resolution is immediately given, for example for curves, normal surfaces and for toric varieties. If not, one can consider an embedded resolution $\pi : Z \to X$ for the hypersurface $H$ in the ambient variety $X$. Let $\mathcal{D}$ be the exceptional variety with irreducible components $E_j, j \in \{1, \cdots, r\}$, created by $\pi$. Let $\hat{H}$ be the strict transform of the hypersurface $H$ and let $\hat{h} = 0$ be its equation. If $F_{j,1}, \cdots, F_{j,k_j}$ are the irreducible components of $E_j \cap \{\hat{h} = 0\}$ and if $\xi_{F_{j,i}}$ denotes the generic element of $F_{j,i}$ $(1 \leq j \leq r, 1 \leq i \leq k_j)$, then one takes the valuations induced by the discrete valuation rings $\mathcal{O}_{\hat{H},\xi_{F_{j,i}}}$ $(1 \leq j \leq r,\ 1 \leq i \leq k_j)$. These valuations define ideals $\mathcal{I}(\underline{v})$ in $\mathcal{O}_{H,o}$; we write $\mathcal{P}_H$ for the induced Poincaré series on $H$. The discrete valuations induced by the rings $\mathcal{O}_{Z,\xi_{E_j}}$, $1 \leq j \leq r$, define ideals $\mathcal{J}(\underline{v}) \subset \mathcal{O}_{X,o}$ and hence a Poincaré series $\mathcal{P}_X$ on the ambient variety $X$.

### A.1 Examples

1. For curves and rational surface singularities, the Poincaré series is introduced by using the valuations of the minimal resolution (see Section 1.6 for their definition). They coincide with the essential valuations, i.e. the valuations that appear in each resolution. They define ideals $\mathcal{I}(\underline{v})$. In this way we get the Poincaré series $\mathcal{P}_{\mathcal{C}}$ for the curve $\mathcal{C}$ and $\mathcal{P}_{\mathcal{S}}$ for the rational surface singularity $\mathcal{S}$.
2. In Chapter 4 we studied the Poincaré series for an affine toric variety. A regular subdivision determines a resolution for the toric variety. We can use the corresponding valuations to define ideals $\mathcal{I}(\underline{v})$ and we obtain a Poincaré series for the affine toric variety.
3. In [C,G-S,L-J] it is shown that the canonical map from the sky of the constellation of base points of a finitely supported ideal $I$ to $X$ is an embedded resolution of the subvariety of $(X, o)$ defined by $i$ general enough elements in $I$, $1 \leq i \leq d$. This particular resolution gives us valuations that define ideals $\mathcal{I}(\underline{v})$ that we can use to define a Poincaré series for the hypersurface. At the same time we get ideals $\mathcal{J}(\underline{v})$ which permit us to define an ambient Poincaré series $\mathcal{P}_X$.
4. In 1976 Varchenko showed how one can compute the zeta function of monodromy via the Newton polyhedron (see [Va]). In particular he has proven that the toric morphism corresponding to a regular subdivision of the normal fan (see Section 5.2 for the construction of the normal fan to the Newton polygon) gives an embedded resolution for the polynomials that are nondegenerate with this Newton polyhedron. Taking valuations in the normal fan gives ideals $M(\underline{v}) \subset \mathcal{O}_X$ and a Poincaré series $P_X$ of the ambient variety $X$. As explained above, the ideals $M(\underline{v})$ induce ideals $I(\underline{v}) \subset \mathcal{O}_H$ and hence a Poincaré series $P_H$ for the hypersurface $H$.
   Ebeling uses this fact when he constructs his Poincaré series for quasi-homogeneous singularities. The Newton polygon of a quasi-homogeneous polynomial is obviously contained in a hyperplane, maybe in infinitely many hyperplanes. The normal vectors to these hyperplanes form the normal cone. Ebeling picks out one valuation in this normal cone and studies the induced Poincaré series.
5. As the normal fan can always be constructed, one can use this construction for an arbitrary hypersurface. Consider now a hypersurface resolved by blowing up a toric constellation, as in Example 3.

Remember that the facets of the Newton polyhedron correspond to the Rees valuations of the corresponding finitely supported complete ideal. Hence, the rays of the normal fan are given by the Rees valuations. The monomial valuations of the toric constellation provide a regular subdivision of the normal fan. □

## A.3 Relations between these Poincaré series and relations with other functions

In this section we ask some interesting questions about these Poincaré series. In the formulation of these questions, we suppose that it is about a case where the question is relevant (where the mentioned Poincaré series exist).

1. How is $P_X$ or $\mathcal{P}_X$ related with $P_H$ or with $\mathcal{P}_H$?
2. Are $P_H$ and $\mathcal{P}_H$ essentially the same? Notice that $\mathcal{P}_H$ can be a function in more variables than $P_H$. One could repeat valuations in $P_H$ to compare.
3. What does $P_H$ or $\mathcal{P}_H$ tell about the singularity?

We recall some known results about curves, normal surface singularities, in particular rational surface singularities, for quasi-homogeneous singularities and we finish with a new result about toric hypersurfaces and general hypersurfaces with respect to a toric constallation.

As mentioned before, the Poincaré series for a curve $\mathcal{C}$ is the Poincaré series induced by the valuations appearing in its minimal resolution. We could wonder what the Poincaré series $\mathcal{P}_\mathcal{C}$ is when it is induced by an arbitrary embedded resolution for the curve. In [C,D,G-Z4] it is shown that $\mathcal{P}_\mathcal{C}$ coincides with the 'standard' Poincaré series for the curve $\mathcal{C}$ when considering an embedded resolution of the plane curve $\mathcal{C} \subset \mathbb{C}^2$. Via the introduced integral with respect to the Euler characteristic Campillo, Delgado and Gusein-Zade also show that the Poincaré series $\mathcal{P}_\mathcal{C}$, with respect to an embedded resolution of the curve in a rational surface singularity, equals the 'standard' Poincaré series for the curve.

They investigate the relation between $\mathcal{P}_\mathcal{C}$ and $\mathcal{P}_\mathcal{S}$ for an embedded resolution of a curve $\mathcal{C}$ in a rational surface singularity $\mathcal{S}$. We use the notation of Section 4.3. For $\underline{v} \in \mathbb{Z}^r$, let $\overset{\bullet}{D}_{\underline{v}}$ be the set

$$\{\{\hat{g} = 0\} \cap \mathcal{D} \mid g \in \mathcal{O}_{\mathcal{S},o},\ \underline{\nu}(g) = \underline{v} \text{ and } \{\hat{g} = 0\} \text{ does not contain any non-empty intersection } E_a \cap E_b, a, b \in J, a \neq b\}.$$

It was shown that the Poincaré series $\mathcal{P}_{\mathcal{S}}$ is equal to

$$\sum_{\underline{v} \in \mathbb{Z}^r} \chi(\overset{\bullet}{D}_{\underline{v}}) \underline{t}^{\underline{v}}.$$

Equivalently, for $\underline{v} \in \mathbb{Z}^r$, let $\overset{\circ}{D}_{\underline{v}}$ be the set

$$\{\{\hat{g} = 0\} \cap \mathcal{D} \mid g \in \mathcal{O}_{\mathcal{S},o},\ \underline{\nu}(g) = \underline{v} \text{ and } \{\hat{g} = 0\} \text{ does not contain any non-empty intersection } E_a \cap E_b, a, b \in J, a \neq b \text{ and no } F_{j,j_k}\}.$$

Campillo, Delgado and Gusein-Zade proved that then $\mathcal{P}_{\mathcal{C}}$ is equal to

$$\sum_{\underline{v} \in \mathbb{Z}^r} \chi(\overset{\circ}{D}_{\underline{v}}) \underline{t}^{\underline{v}}.$$

It has been shown that for irreducible plane curves the Poincaré series coincides with the zeta function of monodromy, for reducible plane curve singularities the Poincaré series is the same as the Alexander polynomial. This result is extremely interesting because this illustrates how the Poincaré series determines the germ $(C, o) \subset (\mathbb{C}^2, o)$ up to topological equivalence ([Wal],[Y]).
We remark that Campillo, Delgado and Gusein-Zade studied the relation between both series when given an embedded resolution for the curve $\mathcal{C}$ in a *rational surface singularity* $\mathcal{S}$ because they can use then the Artin criterion which in the context of rational singularities ensures that there is one-to-one correspondence between divisors on $\mathcal{D}$ - being points - and functions $g \in \mathcal{O}_{\mathcal{S},o}$.

Recall that the Poincaré series for irreducible plane curves - in that case there is one valuation involved - coincides with the zeta function of monodromy. Ebeling and Gusein-Zade studied the Poincaré series for quasi-homogeneous singularities with respect to one valuation. When one wants to search for relations with for example the zeta function of monodromy, one should ask the polynomials to be nondegenerate with respect to their Newton polyhedron. Then, for quasi-homogeneous nondegenerate singularities, Ebeling and Gusein-Zade showed that this Poincaré series is the same as the Saito dual of the zeta function of monodromy (see [Eb,G-Z1]).

For normal surface singularities Cutkosky proved that the Poincaré series determines the intersection matrix and the arithmetic genera of the exceptional components ([Cu]).

In Section 4.3 of this thesis, we considered the Poincaré series for affine toric varieties. We showed that $P_X$, $X$ being an affine toric variety, is equal to

$$\sum_{\underline{v}\in\mathbb{Z}^r} \chi(\overset{\bullet}{D}_{\underline{v}})\underline{t}^{\underline{v}}.$$

Let us now have a look at toric affine hypersurfaces. We consider $H = \operatorname{Spec} \mathbb{C}[\check{\sigma}\cap M] \subset \mathbb{C}^{d+1}$ a toric affine hypersurface, $M$ being a lattice of rank $d$. Let $\{s_1, \cdots, s_{d+1}\}$ be a system of generators of $S = \check{\sigma}\cap M$ and let

$$\begin{array}{rcl} \varepsilon : \mathbb{C}[x_1, \cdots, x_{d+1}] & \to & \mathbb{C}[S] \\ x_i & \mapsto & \chi^{s_i}. \end{array}$$

induce the embedding of $H$ in $\mathbb{C}^{d+1}$. We set deg $(x_i) = s_i$, $1 \leq i \leq d+1$. The hypersurface $H$ is given by a binomial of the form $h = \underline{x}^\alpha - \underline{x}^\beta$, with $\deg(\underline{x}^\alpha) = \deg(\underline{x}^\beta)$ and $\operatorname{supp}(\underline{x}^\alpha)\cap\operatorname{supp}(\underline{x}^\beta) = \emptyset$. Toric hypersurfaces are a particular example of quasi-homogeneous polynomials. The compact Newton polygon is given by a segment $\tau$, connecting $\alpha$ and $\beta$. We denote the normal cone to this segment by $\tau'$ and the hypersurface passing through $\tau'$ by $N'$. The equation of the hyperplane passing through the normal cone is $\sum_{i=1}^{d+1}(\alpha_i - \beta_i)x_i = 0$.

A finite set of valuations in $\sigma$ gives a Poincaré series $\mathcal{P}_H$ for $H$. On the other hand valuations in $\tau'$ give rise to ambient ideals $M(\underline{v}) \subset \mathbb{C}^{d+1}$ and hence to a Poincaré series $P_H$.

**A.2 Theorem**

1. *The cones* $\sigma$ *and* $\tau'$ *are isomorphic.*
2. $P_H = \mathcal{P}_H$.

*Proof.*

1. We show that there is an isomorphism $\theta : N_{\mathbb{R}} \to N'$ that maps $\sigma$ to $\tau'$. Let $n \in N_{\mathbb{R}}$, and let $a = (a_1, \cdots, a_{d+1})$ be the vector such that $a_i = \langle s_i, n\rangle$, $1 \leq i \leq d+1$. Then obviously $\sum_{i=1}^{d+1}(p_i - q_i)a_i = 0$ such that $a \in N'$. For the opposite way, take $a \in N'$. The equality $\sum_{i=1}^{d+1}(p_i - q_i)a_i = 0$ implies that there exists a $n \in N_{\mathbb{R}}$ such that $a_i = \langle s_i, n\rangle$, $1 \leq i \leq d+1$. This $n$ is then unique. As $\sigma$ is the dual cone to $\check{\sigma}$, it follows that $\sigma$ maps to $\tau'$.

2. The Poincaré series $P_H$ with respect to valuations $a^1, \cdots, a^r \in \overset{\circ}{\tau'}$ is induced by the ideals

$$J(\underline{v}) = (\underline{x}^\lambda \mid \langle \lambda, a^j\rangle \geq v_j, 1 \leq j \leq r) + (h).$$

As $\langle \lambda, a^j\rangle = \langle s, n^j\rangle$, where $s = \sum_{i=1}^{d+1} \lambda_i s_i$, it follows that $\underline{x}^\lambda \in J(\underline{v})$ if and only if $\varepsilon(\underline{x}^\lambda) = \chi^s$, with $\langle s, n^j\rangle \geq v_j$, $1 \leq j \leq r$.
The Poincaré series $\mathcal{P}_H$ with respect to the corresponding valuations $n^j$ is induced by the ideals

$$\mathcal{I}(\underline{v}) = (\chi^s \mid \langle s, n^j\rangle \geq v_j, 1 \leq j \leq r).$$

It now follows that both Poincaré series coincide. ■

**A.3 Corollary** *The Poincaré series* $P_H = \mathcal{P}_H$ *can be computed via the Newton polyhedron.*

*Proof.* This is an immediate consequence of Theorem A.2 and Corollary 4.13. ■

Some other interesting questions: Which information is contained in $\chi(\overset{\circ}{D}_{\underline{v}})$? Does it contain topological properties about the singularity? Does there exist a choice of valuations adapted to the variety, such that its Poincaré series is a very interesting tool in singularity theory? Are essential valuations or arc space valuations good candidates? The Saito dual is defined for one valuation, can one define a Saito dual for more valuations? Is the Poincaré series a candidate?

# Appendix B

# Nederlandse samenvatting

In dit proefschrift bestuderen we enkele klassieke functies in de singulariteitentheorie, met name de Poincaréreeks, de topologische zetafunctie en de zetafunctie van monodromie.

Campillo, Delgado en Gusein-Zade voerden deze Poincaréreeks in voor een algebraïsche variëteit $X$ met betrekking tot een eindige verzameling van discrete valuaties $\nu_1, \cdots, \nu_r : \mathcal{O}_{X,o} \to \mathbb{Z} \cup \{\infty\}$. Voor $\underline{v} \in \mathbb{Z}^r$ definiëren ze de idealen

$$I(\underline{v}) := \{g \in \mathcal{O}_{X,o} \mid \nu_j(g) \geq v_j, 1 \leq j \leq r\}$$

en verkrijgen zo een multi-indexfiltratie op de ring $\mathcal{O}_{X,o}$. De dimensies $d(\underline{v}) := \dim(I(\underline{v})/I(\underline{v}+\underline{1}))$ worden in een Poincaréreeks $P$ samengebracht.

$$P(t_1, \cdots, t_r) := \frac{\prod_{j=1}^{r}(t_j - 1) \sum_{\underline{v} \in \mathbb{Z}^r} d(\underline{v}) \underline{t}^{\underline{v}}}{(t_1 \cdots t_r - 1)}.$$

Hierbij is $\underline{t} := (t_1, \cdots, t_r)$ en $\underline{v} := (v_1, \cdots, v_r)$. Deze Poincaréreeks werd onder andere reeds bestudeerd voor krommen, rationale oppervlaktesingulariteiten en voor quasi-homogene polynomen.

We observeren dat deze Poincaréreeks slechts goed gedefinieerd is wanneer het centrum van de valuaties het maximaal ideaal is van de lokale ring $\mathcal{O}_{X,o}$ en we bestuderen deze Poincaréreeks dan in het bijzonder voor affiene torische variëteiten.

**Stelling 1** *Als* $X = Spec\ (\mathbb{C}[S])$, *dan is de Poincaréreeks van* $X$ *met betrekking tot de valuaties* $\{\nu_1, \cdots, \nu_r\}$

$$P(\underline{t}) = \sum_{\underline{v} \in \mathbb{Z}^r} \#\{\chi^s \mid \langle s, \underline{\nu} \rangle = \underline{v}\} \underline{t}^{\underline{v}}.$$

Definiëren we nu

$$\begin{array}{rcl} \Phi : \mathbb{Z}[[u_1, \cdots, u_d, u_1^{-1}, \cdots, u_d^{-1}]] & \longrightarrow & \mathbb{Z}[[t_1, \cdots, t_r, t_1^{-1}, \cdots, t_r^{-1}]] \\ \sum_i \lambda_i \underline{u}^m & \longmapsto & \sum_i \lambda_i \underline{t}^{\langle m, \underline{\nu} \rangle}, \end{array}$$

dan kunnen we ook schrijven

$$P(\underline{t}) = \Phi(Q(\underline{u}))$$

waarbij $Q(\underline{u}) := \sum_{s \in S} \underline{u}^s$ de Poincaréreeks is van $X$ in commutatieve algebra.

Campillo, Delgado en Gusein-Zade hebben voor krommen, rationale oppervlaktesingulariteiten en vlakke divisoriële valuaties een beschrijving gegeven voor de Poincaréreeks op het niveau van de modificatieruimte.

Noteer $\mathcal{D} := \bigcup_{j=1}^r E_j$ voor de exceptionele variëteit met irreducibele componenten $E_j, j \in J := \{1, \cdots, r\}$, en $\overset{\circ}{E}_j$ voor het glad deel van de irreducibele component $E_j$, i.e. zonder intersectiepunten met alle andere componenten van de exceptionele divisor. Zij $M := -(E_i \circ E_j)$ min the intersectiematrix van de componenten van $\mathcal{D}$. Zij $\nu_j$ de discrete valuatie op de lokale ring $\mathcal{O}_{X,o}$ geïnduceerd door $E_j$. De waardensemigroep $S := \{\underline{\nu}(g) \mid g \in \mathcal{O}_{X,o}\}$ is precies de verzameling van vectoren $\{\underline{v} \in \mathbb{Z}^r_{\geq 0} \mid \underline{v}M \geq \underline{0}\}$. Voor een topologische ruimte $E$, zij $S^nE := E^n/S_n$ $(n \geq 0)$ de $n$-de symmetrische macht van de ruimte $E$, i.e. de ruimte van de $n$-tallen van punten van de ruimte $E$ ($S^0E$ is een punt). Campillo, Delgado en Gusein-Zade construeren de ruimte

$$Y := \bigcup_{\{\underline{v} \in S\}} \left( \prod_{j=1}^r S^{n_j(\underline{v})} \overset{\circ}{E}_j \right),$$

waarbij $\underline{v}M =: \underline{n}(\underline{v})$. Voor $g \in \mathcal{O}_{X,o}$, $g \neq 0$ en $\underline{v} := \underline{\nu}(g)$ is $n_j(\underline{v})$ gelijk aan het intersectiegetal van de strikt getransformeerde van $g$ met $E_j$. Zij $Y_{\underline{v}}$ de samenhangingscomponent $\prod_{j=1}^r S^{n_j(\underline{v})} \overset{\circ}{E}_j$ van $Y$. Ze tonen aan dat

$$P(\underline{t}) = \sum_{\underline{v} \in \mathbb{Z}^r} \chi(Y_{\underline{v}}) \underline{t}^{\underline{v}}.$$

We veralgemenen dit door punten op de kromme te interpreteren als divisoren en we tonen de uitspraak aan voor affiene torische variëteiten. Voor $\underline{v} \in \mathbb{Z}^r$ definiëren we de verzameling

$$\begin{aligned} \overset{\bullet}{D}_{\underline{v}} : \; = \; & \{\{\hat{g} = 0\} \cap \mathcal{D} \mid g \in \mathcal{O}_{X,o},\ \underline{\nu}(g) = \underline{v} \text{ en } \{\hat{g} = 0\} \text{ bevat} \\ & \text{geen niet-ledige doorsnede } E_a \cap E_b, a, b \in J, a \neq b\}. \end{aligned}$$

**Stelling 2** *De Poincaréreeks $P(\underline{t})$ is gelijk aan*

$$\sum_{\underline{v} \in \mathbb{Z}^r} \chi(\overset{\bullet}{D}_{\underline{v}}) \underline{t}^{\underline{v}}.$$

We bepalen een expliciete formule voor de *Poincaréreeks $P$ wanneer $X = \mathbb{C}^d$ en de betrokken valuaties geïnduceerd zijn door een torische constellatie*. Deze heeft een cyclotomische vorm. Inderdaad, indien $X$ een complete intersectie is, dan is de Poincaréreeks $Q$ in commutatieve algebra cyclotomisch. Zo kunnen we via Stelling 1 onmiddellijk besluiten dat de *Poincaréreeks $P$ voor complete intersecties cyclotomisch is*. Een dergelijke vorm van de Poincaréreeks voor hyperoppervlakken sluit een relatie met de zetafunctie van monodromie alvast niet uit. Voor vlakke irreducibele krommesingulariteiten is de Poincaréreeks gelijk aan de zetafunctie van monodromie en voor quasi-homogene polynomen valt de Poincaréreeks met betrekking tot één valuatie samen met het Saito-duaal van de zetafunctie van monodromie.

We bespreken een aantal mogelijkheden om Poincaréreeksen voor hyperoppervlakken te definiëren en we hebben hierbij speciaal aandacht voor de *Poincaréreeks gedefinieerd via de Newtonpolyheder*. We tonen aan dat deze Poincaréreeks voor torische hyperoppervlakken gelijk is aan de Poincaréreeks die we berekend hebben in Hoofdstuk 4. Het zou interessant zijn de Poincaréreeks voor een willekeurig hyperoppervlak expliciet uit te drukken met behulp van de Newtonpolyheder en op zoek te gaan naar een multivariate versie van het Saito-duaal. Zo kan er misschien een verband ontdekt worden tussen de Poincaréreeks en de zetafunctie van monodromie, als de Poincaréreeks al niet zou samenvallen met het multivariate Saito-duaal... Aangezien de schrijfwijze $\sum_{\underline{v} \in \mathbb{Z}^r} \chi(\overset{\bullet}{D}_{\underline{v}}) \underline{t}^{\underline{v}}$ specialiseert tot de reeds bekende gevallen, is het misschien ook interessant te onderzoeken wat voor informatie bevat is in $\chi(\overset{\bullet}{D}_{\underline{v}})$. Wordt de topologie van de singulariteit hierdoor bepaald?

Ten tweede bestuderen we enkele aspecten rond de topologische zetafunctie en de zetafunctie van monodromie. De topologische zetafunctie werd ingevoerd door Denef en Loeser en is geassocieerd aan de kiem $f : (\mathbb{C}^d, 0) \to (\mathbb{C}, 0)$ van een holomorfe functie $f$ waarvoor $f(0) = 0$.

Zij $\pi : X \to \mathbb{C}^d$ een ingebedde resolutie van singulariteiten van $f^{-1}\{0\}$. We schrijven $E_j, j \in S$, voor de irreducibele componenten van $\pi^{-1}(f^{-1}\{0\})$ en $N_j$ en $\nu_j - 1$ voor de respectievelijke multipliciteiten van $f \circ g$ en $g^*(dx_1 \wedge \cdots \wedge dx_d)$ langs $E_j$. De $(N_j, \nu_j), j \in J$, worden de numerische data van de ingebedde resolutie $(X, \pi)$ genoemd. Voor een deelverzameling $I \subset S$ noteren we $E_I := \cap_{i \in I} E_i$ en $\overset{\circ}{E}_I := E_I \setminus (\cup_{j \notin I} E_j)$. Verder schrijven we $\chi(\cdot)$ voor de topologische Euler–Poincaré karakteristiek.

**Definitie** De lokale *topologische zetafunctie geassocieerd aan* $f$ is de rationale functie in één complexe variabele

$$Z_{top,f}(s) := \sum_{I \subset S} \chi(\overset{\circ}{E}_I \cap \pi^{-1}\{0\}) \prod_{i \in I} \frac{1}{N_i s + \nu_i}.$$

Denef en Loeser toonden aan dat elke ingebedde resolutie aanleiding geeft tot dezelfde functie.

De kandidaatpolen van de topologische zetafunctie zijn de rationale getallen $-\nu_j/N_j$. Het is opvallend dat de meeste kandidaatpolen geen pool zijn. Zij

$$\mathcal{P}_d := \{s_0 \mid \exists f \in \mathbb{C}[x_1, \ldots, x_d] \, : \, Z_{top,f}(s) \text{ heeft een pool in } s_0\}.$$

Segers en Veys toonden aan dat $\mathcal{P}_2 \cap (-\infty, -\frac{1}{2}) = \{-\frac{1}{2} - \frac{1}{i} \mid i \in \mathbb{Z}_{>1}\}$ en dat $\mathcal{P}_3 \cap (-\infty, -1) = \{-1 - \frac{1}{i} \mid i \in \mathbb{Z}_{>1}\}$. We tonen nu het het volgende resultaat aan.

**Stelling 3** *Voor elke* $d \geq 4$ *is* $\{-(d-1)/2 - 1/i \mid i \in \mathbb{Z}_{>1}\} \subset \mathcal{P}_d$ *en is elk rationaal getal in het interval* $[-(d-1)/2, 0)$ *een pool voor een topologische zetafunctie.*

Om dit aan te tonen pikken we polynomen uit waarvoor hun topologische zetafuncties precies deze lijst van polen geven.

De polen van de topologische zetafunctie spelen ook een hoofdrol in de monodromieconjectuur. Deze voorspelt een verband tussen de polen van de topologische zetafunctie en de eigenwaarden van monodromie. Wanneer $f = 0$ een geïsoleerde singulariteit heeft, dan kunnen we veronderstellen dat de bovenstaande ingebedde resolutie $\pi$ een isomorfisme is buiten het inverse beeld van de oorsprong. Als de $E_j$, $j \in \{1, \cdots, r\}$, de irreducibele componenten zijn van $\pi^{-1}(\{0\})$, dan toonde A'Campo aan dat de zetafunctie van monodromie $\zeta_f$ te schrijven is als

$$\zeta_f(t) = \prod_{j=1}^{r} (1 - t^{N_j})^{-\chi(\overset{\circ}{E}_j)}.$$

De eigenwaarden van monodromie zijn de nulpunten en polen van de zetafunctie van monodromie. De monodromieconjectuur beweert het volgende:

> *Als $s$ een pool is van $Z_{top,f}$, dan is $e^{2\pi is}$ een eigenwaarde van monodromie van $f$ voor een punt op het hyperoppervlak $f = 0$.*

De monodromieconjectuur is in dimensie 2 volledig bewezen door Loeser. We bestuderen de conjectuur in een specifieke context in dimensie 3. We beschouwen er een monomiaal ideaal met eindige drager. Campillo, Gonzalez-Sprinberg en Lejeune-Jalabert toonden aan de de opblazing van de geassocieerde torische constellatie van basispunten een ingebedde resolutie geeft voor een generiek element uit dat ideaal. We bepalen de *topologische zetafunctie onmiddellijk uit de gegeven cluster*. We berekenen hiervoor de Eulerkarakteristieken $\chi(\overset{\circ}{E}_I)$ voor $I \subset S$. In het bijzonder verkrijgen we een explitiete formule voor $\chi(\overset{\circ}{E}_i)$.

We fixeren nu een kandidaatpool $s := -\nu_j/N_j$, $j \in J$, van $Z_{top,f}$. We schrijven $\nu_j/N_j$ als $a/b$ zodat $a$ en $b$ copriem zijn en we definiëren de verzameling $J_b := \{j \in J \mid \; b \mid N_j\}$. Uit A'Campo's formule volgt:

$$\begin{gathered} e^{2\pi is} \text{ is een nulpunt of pool van } \zeta_f \\ \Updownarrow \\ \textstyle\sum_{j \in J_b} \chi(\overset{\circ}{E}_j) \neq 0. \end{gathered}$$

Daarom zijn we bijzonder geïnteresseerd in $\chi(\overset{\circ}{E}_i)$. De combinatorische formule die we verkrijgen wordt nog niet uitgebuit in deze thesis. Via een meetkundige manier verklaren we een gedeeltelijk fenomeen achter de monodromieconjectuur, steeds in deze specifieke context. Concreet, we onderzoeken wanneer $\chi(\overset{\circ}{E}_i) < 0$. We bewijzen:

$\chi(\overset{\circ}{E}_j) < 0$ *als en slechts als de configuratie in* $E_j \cong \mathbb{P}^2$ *bestaat uit tenminste drie rechten - eventueel exceptioneel - die allen door een zelfde punt gaan.*

Als toepassing kunnen we dan het volgende resultaat aantonen.

**Stelling 4** *Als* $\chi(\overset{\circ}{E}_j) > 0$*, dan is* $e^{-2\pi i \frac{\nu_j}{N_j}}$ *een eigenwaarde van monodromie van* $f$*.*

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