THE HOLOMORPHY CONJECTURE FOR IDEALS IN DIMENSION TWO

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Abstract. The holomorphy conjecture predicts that the topological zeta function associated to a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ and an integer $d > 0$ is holomorphic unless d divides the order of an eigenvalue of local monodromy of f. In this note, we generalise the holomorphy conjecture to the setting of arbitrary ideals in $\mathbb{C}[x_1, \ldots, x_n]$, and we prove it when $n = 2$.

0. Introduction

Let f be a complex polynomial. The topological zeta function associated to f and an integer $d > 0$ is a rational function on the complex line. It can be computed explicitly on an embedded resolution of singularities of f. This expression yields a complete set of candidate poles for the topological zeta function, but many of these will not be actual poles, due to cancelations in the formula. This phenomenon would partially be explained by the *monodromy conjecture* and the *holomorphy conjecture*. The monodromy conjecture states that poles of the topological zeta function should give rise to eigenvalues of local monodromy of f (see [DL]). The conjecture we study in this note, the holomorphy conjecture, predicts that the topological zeta function is holomorphic unless d devides the order of an eigenvalue of local monodromy of f (see [V]). Both conjectures were motivated by similar conjectures about Igusa's p-adic zeta function, due to Igusa (see [Ig]) and Denef (see [D2]), respectively.

In this article we introduce the holomorphy conjecture for ideals in $\mathbb{C}[x_1, \ldots, x_n]$. The notion of embedded resolution is here replaced by the notion of log-principalisation of the ideal. In Section 2 we go on by providing some preliminary results in dimension 2 which we will use in Section 3 to prove the holomorphy conjectures for ideals in $\mathbb{C}[x, y]$.

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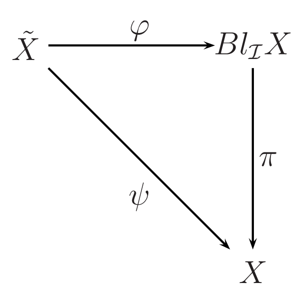
1. The holomorphy conjecture

Verdier introduced a notion of eigenvalues of monodromy for ideals, coinciding with the classical notion for principal ideals (see [Ver]). Based on this notion of Verdier, the second author and Veys gave a criterion à la A'Campo for being an eigenvalue of monodromy of a given ideal.

To recall this criterion, fix an ideal $\mathcal{I} \subset \mathbb{C}[x_1, \ldots, x_n]$. Let Y be the zero locus of $\mathcal{I}$ in $X := \mathbb{C}^n$, containing the origin 0. We construct the blowing-up $\pi : Bl_{\mathcal{I}}X \to X$ of X in Y and we denote by E the inverse image $\pi^{-1}(Y)$. Now consider a log-principalisation $\psi : \tilde{X} \to X$ of $\mathcal{I}$ (the existence of that is guaranteed by Hironaka in [H]). This means that ψ is a proper birational map from a nonsingular variety $\tilde{X}$ such that the total transform $\mathcal{I}\mathcal{O}_{\tilde{X}}$ is locally principal and moreover is the ideal of a simple normal crossings divisor. Let $\sum_{i\in S} N_i E_i$ denote this divisor, written in such a way that the $E_i, i \in S$, are the irreducible components occurring with multiplicity N_i. Let $\nu_i - 1$ be the multiplicity of E_i in the divisor $\psi^*(dx_1 \wedge \cdots \wedge dx_n)$. The couples $(N_i, \nu_i), i \in S$, are called the numerical data of the log-principalisation ψ. For $I \subset S$, denote $E_I := \cap_{i\in I} E_i$ and $E_I^\circ := E_I \setminus (\cup_{j\in S, j\notin I} E_j)$. We denote furthermore the topological Euler-Poincaré characteristic by $\chi(\cdot)$. By the Universal Property of Blowing Up, there exists a unique morphism φ that makes the following diagram commutative.



Theorem 1. [VV2, Theorem 4.2] *The number α is an eigenvalue of monodromy of $\mathcal{I}$ if and only if there exists a point $e \in E$ such that α is a zero or pole of the function*

$$Z_{\mathcal{I},e}(t) = \prod_{j\in S} (1 - t^{N_j})^{\chi(E_j^\circ \cap \varphi^{-1}(e))}.$$

Definition 2. *Let ψ be a log-principalisation of $\mathcal{I}$. The* local topological zeta function at the point 0 associated to the ideal $\mathcal{I}$ and the positive integer *d is the rational function in one complex variable*

$$Z^{(d)}_{top,\mathcal{I},\psi}(s) := \sum_{\substack{I\subset S\\ d\mid N_i}} \chi(E_I^\circ \cap \psi^{-1}\{0\}) \prod_{i\in I} \frac{1}{N_i s + \nu_i}.$$

Proposition 3. *The function $Z_{top,\mathcal{I},\psi}^{(d)}$ does not depend on the choice of ψ.*

Proof. The Weak Factorization Theorem (see [AKMW, §4]) assures that it is sufficient to check whether $Z_{top,\mathcal{I},\psi}^{(d)}$ remains invariant when composing ψ with a blowing-up with smooth centre having normal crossings with $\mathcal{I}\mathcal{O}_{\tilde{X}}$. We leave this as an easy excercice to the reader. □

From now on we will write $Z_{top,\mathcal{I}}^{(d)}$ for the local topological zeta function associated to $\mathcal{I}$ and d at the origin. When $\mathcal{I}$ is principal, it coincides with the zeta function defined in [DL].

Conjecture 4. *(Holomorphy Conjecture)*
Let d be a positive integer. If d does not divide the order of any eigenvalue of monodromy associated to the ideal $\mathcal{I}$ in points of $\pi^{-1}\{0\}$, then $Z_{top,\mathcal{I}}^{(d)}$ is holomorphic on the complex plane.

Note that $Z_{top,\mathcal{I}}^{(d)}$ is holomorphic if and only if it is identically zero. The formulation we use is motivated by the analogy with Denef's original p-adic version of the conjecture. When $\mathcal{I}$ is principal, this conjecture was formulated in [V, Remark 3.4.]. For principal ideals in $\mathbb{C}[x,y]$, this conjecture has been shown by Veys in [V, Theorem 3.1]. Veys and the first author confirmed the conjecture for principal ideals in $\mathbb{C}[x,y,z]$ defining a surface that is general for a toric idealistic cluster (see [LV, Theorem 24]). In this article we will prove the holomorphy conjecture in the special case that $\mathcal{I}$ is an ideal in $\mathbb{C}[x,y]$. The structure of our proof is inspired by the structure of the proof in [V].

2. Preliminary results

From now on, we put $X = \mathbb{A}^2_{\mathbb{C}}$ and we consider a finitely generated ideal $\mathcal{I}$ in $\mathbb{C}[x,y]$. We fix once and for all a set of generators for $\mathcal{I}$, say $f_1,\ldots,f_r$, and a log-principalisation $\psi : \tilde{X} \to X$ of $\mathcal{I}$. Notice that ψ also gives an (non-minimal) embedded resolution for all elements of some Zariski open subset of the linear system $\{\lambda_1 f_1 + \cdots + \lambda_r f_r = 0 \mid \lambda_1,\ldots,\lambda_r \in \mathbb{C}\}$. We will call these elements *totally general for* $(f_1,\ldots,f_r)$. Moreover, the numerical data associated to the principalisation and to the embedded resolution are the same. A proof of this statement can be found in [VV1, §2]. Let us write $\mathcal{I}$ as $\mathcal{I} = (h)(f'_1,\ldots,f'_r)$ with $(f'_1,\ldots,f'_r)$ finitely supported. We fix a principalisation for $(f'_1,\ldots,f'_r)$ and we will say that a totally general element for $(f'_1,\ldots,f'_r)$ with respect to the chosen principalisation is *general for* $\mathcal{I}$. We will use the notation introduced in Section 1. In particular the $E_i, i \in S$, will be the irreducible components of $\mathcal{I}\mathcal{O}_{\tilde{X}}$. We choose a totally general element f for $\mathcal{I}$ and we can write

$\psi^{-1}(f^{-1}\{0\}) = \sum_{i\in T} N_i E_i$, with T a set containing S. Let $k_i, i \in S$, be the number of intersection points of E_i with other components of $\psi^{-1}\mathcal{I}$. Analogously, for $i \in T$, let k'_i be the number of intersection points of E_i with other components of $\psi^{-1}(f^{-1}\{0\})$. So $k_i \leq k'_i$ for $i \in S$, with equality if and only if E_i is not intersected by the strict transform of a general element for $\mathcal{I}$.

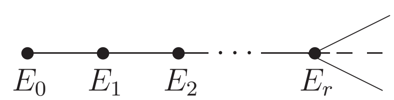
If $\mathcal{I}$ has components of codimension one, we can write the total transform as a product of two principal ideals: the support of the first one is the exceptional locus, where the support of the second one is formed by the irreducible components of the total transform that are not contained in the exceptional locus. This second ideal is the *weak transform* of $\mathcal{I}$.

We will use the following congruence.

Lemma 5. [L, Lemme II.2] *If we fix one exceptional curve E_i, intersecting k'_i times other components $E_1, \ldots, E_{k'_i}$ of $\psi^{-1}(f^{-1}\{0\})$, then $\sum_{j=1}^{k'_i} N_j \equiv 0$ mod N_i.*

Veys shows the following result in his proof for the holomorphy conjecture for plane curves. He proved this for the minimal embedded resolution, but the proof remains valid for non-minimal resolutions induced by log-principalisations.

Lemma 6. [V, Lemma 2.3] *Let E_0 be an exceptional curve with $k'_0 = 1$. Then for some $r \geq 1$ there exists a unique path*



in the resolution graph consisting entirely of exceptional curves, such that

1. $k'_j = k_j = 2$ *for* $j = 1, \ldots, r-1$;
2. $k'_r \geq 3$;
3. $N_0 | N_j$ *for all* $j = 1, \ldots, r$;
4. $N_0 < N_1 < \cdots < N_r$.

We will now provide a set of eigenvalues of monodromy. Let $n : \overline{Bl_{\mathcal{I}}X} \to Bl_{\mathcal{I}}X$ be the normalization map. Recall that the Rees components of an ideal $\mathcal{I}$ are the irreducible components of the exceptional divisor on $\overline{Bl_{\mathcal{I}}X}$. Let $\sigma : \tilde{X} \to \overline{Bl_{\mathcal{I}}X}$ be such that $\varphi = n \circ \sigma$. We will also call the corresponding exceptional components in $\tilde{X}$ Rees components, so an exceptional component E in $\tilde{X}$ is Rees if and only if $\dim(\sigma(E)) = \dim(E)$. As the normalization map is a finite map, being contracted by φ is equivalent to being contracted by σ. Theorem 1 gives us:

Corollary 7. *If the exceptional component E_i in $\tilde{X}$ is Rees for $\mathcal{I}$, then all N_ith roots of unity are eigenvalues of monodromy.*

We can recognize these Rees components in the resolution graph in a very easy way.

Lemma 8. *An exceptional component E on $\tilde{X}$ is contracted by the map $\varphi : \tilde{X} \to Bl_{\mathcal{I}}\,\mathbb{C}^2$ if and only if the strict transform of a general element for $\mathcal{I}$ does not intersect E.*

Proof. Let D be the Cartier divisor on $\tilde{X}$ such that $\mathcal{I}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-D)$ and let F be the Cartier divisor on $Bl_{\mathcal{I}}\,\mathbb{C}^2$ such that $\mathcal{I}\mathcal{O}_{Bl_{\mathcal{I}}\,\mathbb{C}^2} = \mathcal{O}_{Bl_{\mathcal{I}}\,\mathbb{C}^2}(-F)$. Then by the projection formula one has $(-D)\cdot E = -\varphi^*(F)\cdot E = (-F)\cdot \varphi_* E$. Suppose E is contracted by φ, then $\varphi_* E = 0$ and $(-D)\cdot E = 0$. If E is not contracted by φ, then $\varphi_* E = k\varphi(E)$ for some strictly positive integer k. Since $-F$ is very ample relative to X, we have $-F \cdot \varphi(E) > 0$ and thus $(-D)\cdot E > 0$.

We now write $\mathcal{I} = h\mathcal{I}'$ with $\mathcal{I}'$ an ideal of finite support. For a totally general element $f = hf'$ for $\mathcal{I}$, we can write its total transform $\psi^{-1}(f^{-1}\{0\}) = D + S$, where S is the strict transform of f'. By the projection formula, one always has that $(D+S)\cdot E = 0$.

Combining these formulas, one gets the statement of Lemma 8. □

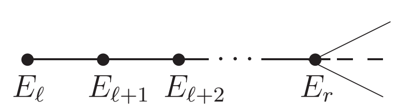
Proposition 9. *Let E_j be an exceptional curve with $k_j \geq 3$. Then N_j divides the order of an eigenvalue of monodromy of $\mathcal{I}$.*

Proof. If E_j is Rees for $\mathcal{I}$, then Corollary 7 yields exactly this result. Suppose now that E_j is not Rees for $\mathcal{I}$ and let a be the point on the exceptional locus of $Bl_{\mathcal{I}}\,\mathbb{C}^2$ such that $a = \varphi(E_j)$, where $\varphi : \tilde{X} \to Bl_{\mathcal{I}}\,\mathbb{C}^2$. We define S_a as the set of indices $i \in S$ which satisfy $\varphi(E_i) = a$. By Theorem 1 it is enough to prove that

$$\sum_{i\in S_a, N_j|N_i} \chi(E_i^\circ) \neq 0.$$

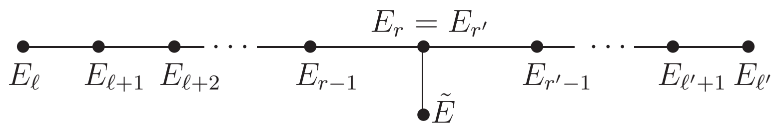
It is given that $\chi(E_j^\circ) < 0$. We will now prove that every positive contribution to this sum is canceled by another negative contribution.

Suppose $\ell \in S_a, N_j|N_\ell$ and $\chi(E_\ell^\circ) > 0$. This means that $\chi(E_\ell^\circ) = 1$ and k_ℓ is equal to 1. If $k'_\ell \neq 1$, then by Lemma 8 E_ℓ is Rees for $\mathcal{I}$ and N_j is a divisor of the order of an eigenvalue of monodromy (Corollary 7). If $k'_\ell = 1$, then by Lemma 6 there exists a path with $k'_r \geq 3$.



If E_r is Rees for $\mathcal{I}$, Corollary 7 tells us that $e^{\frac{2\pi i}{N_r}}$ is an eigenvalue of monodromy and as $N_j|N_r$, also N_j divides the order of it. Suppose now that E_r is not Rees for $\mathcal{I}$. By Lemma 8, $E_{\ell+1}, \cdots, E_r$ are not Rees and as $E_\ell, \cdots, E_r$ are connected, it follows that also $E_{\ell+1}, \cdots, E_r$ are contracted to the point a. As E_r is not Rees, it follows by Lemma 8 that $k_r = k'_r$ and thus $\chi(E_r^\circ) < 0$. Now $N_j|N_\ell$ and by Lemma 6 $N_\ell|N_r$, so we have found a negative contribution canceling $\chi(E_\ell^\circ)$.

We now check whether there can exist two exceptional curves E_ℓ and $E_{\ell'}$ with $\varphi(E_\ell) = \varphi(E_{\ell'}) = a, \chi(E_\ell^\circ) = \chi(E_{\ell'}^\circ) = 1, N_j|N_\ell$ and $N_j|N_{\ell'}$, for which the respectively associated E_r and $E_{r'}$ yielded by Lemma 6 are equal, such as illustrated in the figure below. By Property 4 of Lemma 6, we know that E_r is created later in the principalisation process than $E_\ell, \ldots, E_{r-1}, E_{\ell'}, \ldots, E_{r'-1}$. So at the stage where E_r is created, the resolution graph looks as follows.



Note that by the principalisation process it is impossible to have more than two exceptional curves intersecting E_r. We denote by $\tilde{E}$ the components of the strict transform of the curves that belong to the support of $\mathcal{I}$. These components might be singular and are only present in the principalisation graph if $\mathcal{I}$ is not finitely supported. Since the principalisation graph is connected, there are no other components at that moment. As E_j is intersected at least three times, it follows that E_j is equal to E_r and if not, then by the general form of a resolution graph of a plane curve, it follows that E_j is created later than E_r, what means $N_j \geq N_r$. By Lemma 6, $N_\ell < N_r$. This contradicts the assumption that $N_j|N_\ell$. □

3. Holomorphy conjecture for ideals in $\mathbb{C}[x, y]$

Now we prove the holomorphy conjecture for the local topological zeta function associated to an ideal in dimension two. Actually we are going to show that $Z_{top,\mathcal{I}}^{(d)}$ is identically zero. The terminology 'holomorphic' has its origins in the context of p-adic Igusa zeta functions.

Theorem 10. *Let $\mathcal{I}$ be an ideal in $\mathbb{C}[x, y]$ and $\pi : Bl_{\mathcal{I}}\mathbb{C}^2 \to \mathbb{C}^2$ be the blowing-up of $\mathbb{C}^2$ in the ideal $\mathcal{I}$. Suppose d is a positive integer that does not divide the order of any eigenvalue of monodromy associated to the ideal $\mathcal{I}$ in points of $\pi^{-1}\{0\}$. Then $Z_{top,\mathcal{I}}^{(d)}$ is identically 0 on the complex plane.*

Proof. We search for components that contribute to the local topological zeta function. If $\mathcal{I}$ is a principal ideal, then we refer to [V, Theorem 3.1].

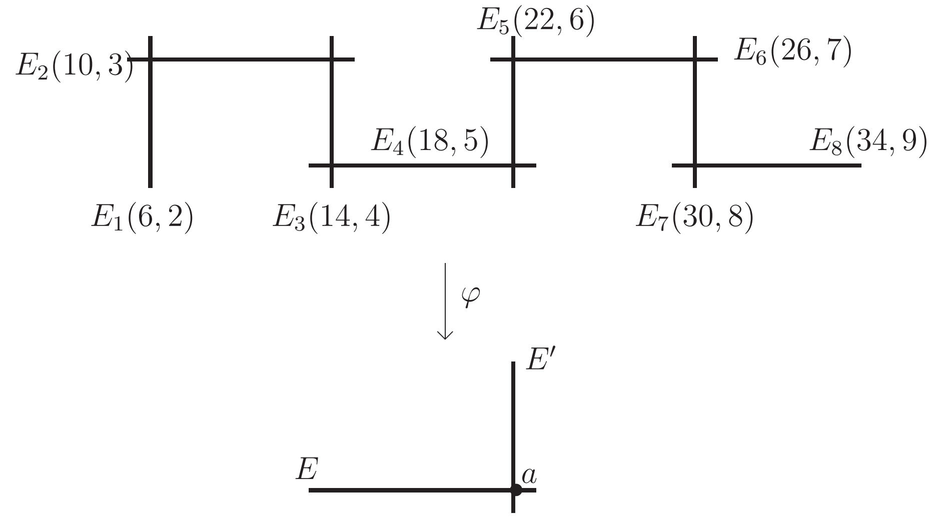
Suppose that $E_i(N_i, \nu_i)$ is an exceptional component of the principalisation satisfying $d|N_i$. By Corollary 7 it follows that E_i is not Rees for $\mathcal{I}$ and thus $k_i = k_i'$. If $k_i \geq 3$, we use Proposition 9 to see that d would be a divisor of the order of a monodromy eigenvalue. If $k_i' = 1$, we use Lemma 6 to find an exceptional curve E_r with $k'_r \geq 3$. If $k_r = k'_r$, we are again in the situation of Proposition 9. Since $d|N_i$ and $N_i|N_r$, this leads to a contradiction. If $k_r \neq k'_r$, the component E_r is Rees for $\mathcal{I}$ and Corollary 7 brings the same conclusion. Hence, we obtain that having $d|N_i$ for an exceptional component $E_i(N_i, \nu_i)$ implies that $k_i = 2$.

Suppose now that $E_i(N_i, \nu_i)$ is a component of the support of the weak transform satisfying $d|N_i$. The only possible contribution of E_i comes from an intersection point of E_i with an exceptional component $E_j(N_j, \nu_j)$ for which $d|N_j$. By Corollary 7 it follows that E_j is not Rees for $\mathcal{I}$. Then we showed that there exists exactly one other component E_k that intersects E_j. From Lemma 5 it follows that $d|N_k$. If E_k is Rees for $\mathcal{I}$, then we have a contradiction. If E_k is a component of the support of the weak transform, then there is no Rees component in the principalisation graph. This implies that $\mathcal{I}$ is a principal ideal. If E_k is exceptional and not Rees for $\mathcal{I}$, we can iterate this argument. By finiteness of the resolution graph we should once meet a component that is Rees for $\mathcal{I}$ or that is a component of the support of the weak transform. This has been discussed before.

The only contribution to the topological zeta function can come from an exceptional component E_i with $\chi(E_i^\circ) = 0$. In particular, the contribution has to come from intersections with other exceptional components. Suppose that E_j is a component that intersects E_i and that $d|N_j$. Then E_j must be exceptional. We do the same reasoning for E_j and we find that k_j must be two. Suppose E_k is the other component that intersects E_j. By Lemma 5 we know that d must divide N_k. We iterate this argument and get the existence of a component $E(N, \nu)$ that is Rees for $\mathcal{I}$ and for which $d|N$. This contradicts the choice of d (Corollary 7) and so d does not divide N_j.

We conclude that $Z_{top,\mathcal{I}}^{(d)} = 0$. □

Example. We consider the ideal $\mathcal{I} = (x^2y^4, x^{34}, y^6) \subset \mathbb{C}[x, y]$. A log-principalisation of $\mathcal{I}$ consists of eight successive blowing-ups. The intersection diagram with the numerical data can be found in the following figure. We use Theorem 1 to find the eigenvalues of monodromy. The exceptional curves $E_2, \ldots, E_7$ are contracted by the map φ to the intersection point a of the exceptional components E and E' in $Bl_{\mathcal{I}}\,\mathbb{C}^2$. The exceptional curves E_1 and E_8 are respectively mapped surjectively to E and E'. As eigenvalues of monodromy we get the 6th roots of unity and the 34th roots of unity. For instance $d = 5$ is no divisor of the order of an eigenvalue of monodromy. The components E_2 and E_7 satisfy $\chi(E_2^\circ) = \chi(E_7^\circ) = 0$ and have an empty intersection. This implies that $Z_{top,\mathcal{I}}^{(5)}(s)$ is equal to zero. □



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References

[AKMW] D. Abramovich, K. Karu, K. Matsuki, J. Włodarczyk, *Torification and factorization of birational maps,* J. Amer. Math. Soc. **15** (2002), 531-572.

[D] J. Denef, *Report on Igusa's local zeta function,* Sém. Bourbaki 741, Astérisque **201/202/203** (1991), 359-386.

[D2] J. Denef, *Degree of local zeta functions and monodromy*, Comp. Math. **89** (1994), 207-216.

[DL] J. Denef and F. Loeser, *Caractéristique d'Euler-Poincaré, fonctions zêta locales et modifications analytiques*, J. Amer. Math. Soc. **5, 4** (1992), 705-720.

[H] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. Math. **79** (1964), 109-326.

[Ig] J. Igusa, *B-functions and p-adic integrals*, Algebraic Analysis, Academic Press (1988), 231-241.

[LV] A. Lemahieu and W. Veys, *Zeta functions and monodromy for surfaces that are general for a toric idealistic cluster*, Int. Math. Res. Notices, ID rnn 122 (2009), 52 pages.

[L] F. Loeser, *Fonctions d' Igusa p-adiques et polynômes de Bernstein,* Amer. J. Math. **110** (1988), 1-22.

[VV1] L. Van Proeyen and W. Veys, *Poles of the topological zeta function associated to an ideal in dimension two,* Math. Z. **260** (2008), 615-627.

[VV2] L. Van Proeyen and W. Veys, *The monodromy conjecture for zeta functions associated to ideals in dimension two,* preprint (2007), 17p.

[Ver] J.L. Verdier, *Spécialisation de faisceaux et monodromie modérée,* Astérisque **101-102** (1983), 332-364.

[V] W. Veys, *Holomorphy of local zeta functions for curves,* Math. Annalen **295** (1993), 635-641.