The monodromy conjecture for plane meromorphic germs

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Abstract.— *A notion of Milnor fibration for meromorphic functions and the corresponding concepts of monodromy and monodromy zeta function have been introduced in* [GZLM1]. *In this article we define the topological zeta function for meromorphic germs and we study its poles in the plane case. We show that the poles do not behave as in the holomorphic case but still do satisfy a generalization of the monodromy conjecture.*

0. INTRODUCTION

In the last decades there has been an increasing interest in zeta functions associated to a germ of holomorphic function, such as the Igusa zeta function, the topological zeta function and the motivic zeta function. These zeta functions are rational and can be described in terms of an embedded resolution of the germ. Each irreducible component of the total transform of an embedded resolution of the germ yields a 'candidate pole' of these zeta functions. However, many of these candidate poles are canceled. In general no geometric criterion is known to sort out the poles from a set of candidate poles. The poles are completely characterised in dimension two. In [Ve1, Theorem 4.3] Veys gives a criterion to deduce easily the poles from the embedded resolution graph.

The poles of these zeta functions also show up in the monodromy conjecture. The monodromy conjecture predicts that if s_0 is a pole of the topological zeta function (or the Igusa or the motivic zeta function) associated to a holomorphic germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, then $e^{2\pi i s_0}$ is an eigenvalue of monodromy at some point of $f^{-1}\{0\}$ in a neighbourhood of the origin. This conjecture has been proven for plane curves by Loeser in [L1]. An alternative proof is given by Rodrigues in [Ro]. The conjecture is still open in arbitrary dimension. Particular cases are proven in [L2], [ACLM1], [ACLM2], [LV] and [LVa].

These poles are rational numbers of the form $-\nu_i/N_i, i \in S$, where the numbers $\nu_i - 1$ and N_i are given by the multiplicities of the divisors $\pi^* dx_1 \wedge \cdots \wedge dx_{n+1}$ and $\pi^* f$ along E_i for an embedded resolution $\pi : X \to \mathbb{C}^{n+1}$ of f (see Section 1). Némethi and Veys propose an original approach to the conjecture (at least for dimension 2) in [Ve4], [NV1] and [NV2]. By considering another differential form than the standard form a different set of numbers $(\nu_i)_{i \in S}$ is generated. They investigate under which conditions the corresponding new set of poles still gives rise to eigenvalues of monodromy and which eigenvalues can be found in this way.

In this article we take a complementary approach. In our case the values $(\nu_i)_{i \in S}$ are the standard ones but we vary the set $(N_i)_{i \in S}$. In concrete, we consider the case

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of plane meromorphic germs $f = \frac{P}{Q}$. The series of numbers $(N_i = N_i^P - N_i^Q)_{i \in S}$ are now more general than in the case of plane curves. The notions of Milnor fibration and monodromy for germs of meromorphic functions have been defined in [GZLM1]. These definitions were motivated by the classification of germs of fractions due to Arnold in [A]. We define a topological zeta function for meromorphic germs and then, the context for the monodromy conjecture being prepared, we show that the monodromy conjecture also holds for plane meromorphic germs. Very recently, Raibaut defined a motivic Milnor fibre and motivic zeta function for rational functions (see [R1, R2]).

The outline of this paper is as follows. In the first section we recall the notions of Milnor fibre for meromorphic germs and zeta function of monodromy, as they were introduced by Gusein-Zade, Luengo and Melle-Hernández in [GZLM1]. In the second section we define the topological zeta function in the meromorphic context and we study its poles in the plane case. We show that now just only one implication in the criterion of Veys holds. In Section 3 we propose the monodromy conjecture for meromorphic germs and we prove the conjecture in the plane case.

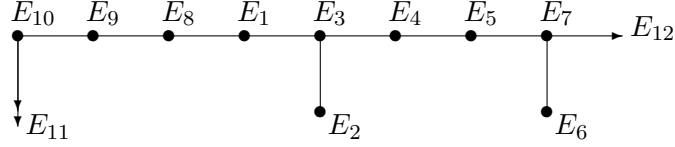## 1. Monodromy for meromorphic germs

Definition 1 *A germ of a meromorphic function on $(\mathbb{C}^{n+1}, 0)$ is a fraction $f = \frac{P}{Q}$, where P and Q are germs of holomorphic functions $P, Q : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Two meromorphic functions $f = P/Q$ and $f' = P'/Q'$ are said to be equal if there exists a germ of a holomorphic function $u : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with $u(0) \neq 0$ such that $P' = Pu$ and $Q' = Qu$.*

Definition 2 *Let X be an $(n+1)$-dimensional smooth analytic manifold, $\mathcal{U}$ a neighbourhood of $0 \in \mathbb{C}^{n+1}$ and $\pi : X \to \mathcal{U}$ a proper analytic map which is an isomorphism outside a proper analytic subspace in $\mathcal{U}$. We say that π is an* embedded resolution of *the germ of the meromorphic function $f = \frac{P}{Q}$ if*

1. *the total transform $\pi^{-1}(H)$ of the hypersurface $H = \{P_{|\mathcal{U}} = 0\} \cup \{Q_{|\mathcal{U}} = 0\}$ is a normal crossing divisor at each point of X;*

2. *the lifting $\tilde{f} = f \circ \pi = \frac{P \circ \pi}{Q \circ \pi}$ defines a holomorphic map $\tilde{f} : X \to \mathbb{P}^1$.*

Remark 1 One can obtain an embedded resolution π for $\frac{P}{Q}$ from an embedded resolution π' for PQ by blowing up along the intersections of irreducible components of $\pi^{-1}\{PQ = 0\}$ until the irreducible components of the strict transform in $\pi^{-1}\{P = 0\}$ and $\pi^{-1}\{Q = 0\}$ are separated by a *dicritical component*, i.e. an exceptional divisor E of π for which $\tilde{f}_{|E} : E \to \mathbb{P}^1$ is a surjective map.

Example 1 We compute an embedded resolution π for the germ $f = P/Q = ((y^2 - x^3)^2 - xy^5)/(x - y)$ and represent it by its *dual resolution graph*, i.e. to each irreducible component of $\pi^{-1}(H)$ we associate a vertex and we join two vertices if the corresponding components have a non-empty intersection. The vertices corresponding to the components of the strict transform of H are represented by arrowheads. An arrowhead $\to$ (resp. $\twoheadrightarrow$) corresponds to an irreducible component of the strict transform of P (resp. Q). The following figure shows the dual resolution graph of f.

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$$E_{10} \quad E_9 \quad E_8 \quad E_1 \quad E_3 \quad E_4 \quad E_5 \quad E_7 \longrightarrow E_{12}$$

$$E_{11} \qquad\qquad\qquad E_2 \qquad\qquad E_6$$

The exceptional components $E_1, \cdots, E_7$ are created to resolve $\{PQ = 0\}$. We then perform some additional blow ups until we get the dicritical component E_{10} that separates the components according to the value they take when applying $f \circ \pi$. In this example E_{11} is send to $\infty \in \mathbb{P}^1$ and the other components different from the dicritical component E_{10} are mapped to $0 \in \mathbb{P}^1$. □

In [GZLM2], Gusein-Zade, Luengo and Melle-Hernández introduced the notion of a-Milnor fibre for meromorphic germs. Let $f = P/Q$ be a meromorphic germ on $(\mathbb{C}^{n+1}, 0)$. Let B_ϵ be the closed ball of radius ϵ centered at $0 \in \mathbb{C}^{n+1}$ with ϵ sufficiently small such that P and Q are defined on B_ϵ and the sphere ∂B_ϵ intersects transversally $\{P = 0\}$ and $\{Q = 0\}$. Then $f : B_\epsilon \setminus \{P = Q = 0\} \to \mathbb{P}^1$ is a locally trivial C^∞ fibration over a small enough punctured disc with center $a \in \mathbb{P}^1$.

Definition 3 *The a-Milnor fibre of the meromorphic germ f at $0 \in \mathbb{C}^{n+1}$ is*

$$\mathcal{M}_f^a = \{z \in B_\epsilon \mid f(z) = \frac{P(z)}{Q(z)} = c\},$$

with $c \in \mathbb{C}$ and $||c - a||$ small enough and different from 0.

They also introduced the corresponding a-monodromy zeta function. Consider a counterclockwise going loop around the value a and let h_f^a be the induced diffeomorphism of the a-Milnor fibre $\mathcal{M}_f^a$. The *a-monodromy transformation h_{f*}^a* at the origin is then the induced automorphism of the homology groups of the a-Milnor fibre and its zeta function is defined as

$$\zeta_f^a(t) \;\; = \;\; \prod_{q \geq 0} \left(\det(Id - t h_{f*|H_q(\mathcal{M}_f^a, \mathbb{C})}^a) \right)^{(-1)^q}.$$

Notice that ∞-Milnor fibre of f is the 0-Milnor fibre of $1/f$ (and $\zeta_f^\infty = \zeta_{1/f}^0$) and that the a-Milnor fibre of f, for $a \in \mathbb{P}^1 \setminus \{0, \infty\}$, is the 0-Milnor fibre of $f - a$ (and then $\zeta_f^a = \zeta_{f-a}^0$). Therefore in the rest of the paper we permit us to think at $a = 0$ and we will not explicitly denote the value $a = 0$.

In [GZLM1], the monodromy zeta function is expressed in terms of an embedded resolution of f. We first introduce some useful notation. Let $\pi : X \to \mathcal{U}$ be an embedded resolution of the germ of meromorphic function $f = \frac{P}{Q}$ and let $(E_i)_{i \in S}$ be the irreducible components of the total transform $\pi^{-1}(H)$. We will denote $E_i^\circ := E_i \setminus (\cup_{j \in S \setminus \{i\}} E_j)$, for $i \in S$. For $i \in S$, let N_i^P (resp. N_i^Q) be the multiplicity of $P \circ \pi$ (resp. $Q \circ \pi$) along E_i and $\nu_i - 1$ the multiplicity of $\pi^*(dx_1 \wedge \cdots \wedge dx_{n+1})$ along E_i. The data (N_i^P, N_i^Q, ν_i) are called the *numerical data* corresponding to the component E_i. Sometimes we just write (N_i, ν_i), with $N_i := N_i^P - N_i^Q$. Finally, denote by S_0 the set $\{i \in S \mid N_i^P > N_i^Q\}$.

Remark that the irreducible components of the total transform that are send to 0 (resp. to ∞) under $f \circ \pi$ are then exactly those E_i for which $N_i^P > N_i^Q$ (resp. $N_i^P < N_i^Q$). The dicritical components verify that $N_i^P = N_i^Q$. However $N_i^P = N_i^Q$ does not imply that E is dicritical.

We denote the topological Euler characteristic by $\chi(\cdot)$.

Theorem 1 [GZLM1, Theorem 1] *Let $\pi : X \to \mathcal{U}$ be an isomorphism outside the hypersurface $\{P_{|\mathcal{U}} = 0\} \cup \{Q_{|\mathcal{U}} = 0\}$. Then*

$$\zeta_f(t) \;=\; \prod_{i \in S_0} \left(1 - t^{N_i^P - N_i^Q}\right)^{\chi(E_i^\circ \cap \pi^{-1}\{0\})},$$

Example 1 Continued The numerical data corresponding to the embedded resolution π for the germ $f = P/Q = ((y^2 - x^3)^2 - xy^5)/(x - y)$ of Example 1 are written in the following table.

$i =$	1	2	3	4	5	6	7	8	9	10	11	12
N_i^P	4	6	12	14	16	17	34	4	4	4	0	1
N_i^Q	1	1	2	2	2	2	4	2	3	4	1	0
ν_i	2	3	5	6	7	8	15	3	4	5	1	1

We thus find

$$\zeta_f(t) = \frac{(1 - t^5)(1 - t^{15})}{(1 - t^{10})(1 - t^{30})}.$$

$\square$

2. Topological zeta function for meromorphic functions

In this section we extend the definition of the local topological zeta function for holomorphic functions to meromorphic functions. Let $f = \frac{P}{Q}$ be a meromorphic function on $(\mathbb{C}^{n+1}, 0)$ and let $\pi : X \to \mathcal{U}$ be an embedded resolution of f as in Section 1. We denote $E_I := \cap_{i \in I} E_i$ and $E_I^\circ := E_I \setminus (\cup_{j \notin I} E_j)$.

Definition 4 *The 0-local topological zeta function associated to f at $0 \in \mathbb{C}^{n+1}$ is the rational function*

$$Z_{top,f}(s) = \sum_{\substack{I \subseteq S \\ I \cap S_0 \neq \emptyset}} \chi(E_I^\circ \cap \pi^{-1}\{0\}) \prod_{i \in I} \frac{1}{(N_i^P - N_i^Q)s + \nu_i}.$$

As in the holomorphic case, the global version is defined by taking $\chi(E_I^\circ)$ instead of $\chi(E_I^\circ \cap \pi^{-1}\{0\})$. Checking that this definition does not depend on the chosen embedded resolution for f goes analogously as in the holomorphic case.

The definition of the topological zeta function yields a set of candidate poles

$$\left\{ -\frac{\nu_i}{N_i^P - N_i^Q} \mid i \in S_0 \right\}$$

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that are negative rational numbers, depending on the chosen embedded resolution. Notice that in particular the dicritical components do not give rise to candidate poles and neither components of the strict transform of Q do.

Example 1 Continued Continued. The local topological zeta function of $f = \frac{(y^2-x^3)^2-xy^5}{x-y}$ at 0 is

$$
\begin{aligned}
Z_{top,f}(s) &= -\frac{1}{10s+5} - \frac{1}{30s+15} + \frac{1}{5(s+4)} + \frac{1}{(s+4)(2s+3)} + \frac{1}{(2s+3)(3s+2)} \\
&+ \frac{1}{(10s+5)(3s+2)} + \frac{1}{(10s+5)(5s+3)} + \frac{1}{(10s+5)(12s+6)} \\
&+ \frac{1}{(12s+6)(14s+7)} + \frac{1}{(14s+7)(30s+15)} + \frac{1}{(30s+15)(15s+8)} \\
&+ \frac{1}{(30s+15)(s+1)} + \frac{1}{15s+8} + \frac{1}{5s+3} \\
&= \frac{20s^2+33s+12.}{15(s+1)(2s+1)^2}
\end{aligned}
$$

$\square$

In the above example we see that most candidate poles are canceled. This is a general phenomenon and it appears already in the holomorphic case. For dimension 2 there is a complete geometric criterion to determine the poles of the local topological zeta function (or the other local zeta functions) associated to a holomorphic function. The general case is however poorly understood. We recall this criterion.

Theorem 2 [Ve1] *Let $f \in \mathbb{C}[x,y]$ and let π be the minimal embedded resolution of f. We have that s_0 is a pole of $Z_{top,f}(s)$ if and only if $s_0 = -\nu_i/N_i$ for some exceptional curve E_i intersecting at least three times other components, or $s_0 = -1/N_j$ for some irreducible component E_j of the strict transform of $f^{-1}\{0\}$.*

We will now study the poles in the case of plane meromorphic germs. The next example shows that Theorem 2 does not hold anymore for plane meromorphic functions.

Example 2 Let $f = P/Q$ with

$$
P(x,y) = \prod_{i=1}^{3} \prod_{j=1}^{5} ((y+a_ix) - b_jx^2)P'(x,y) \quad \text{and}
$$

$$
P'(x,y) = \prod_{k=1}^{4} \left(\prod_{l=1}^{2} ((y+c_kx) - x^2 - c_kd_lx^2 - d_lxy) \prod_{m=1}^{3} ((y+c_kx) - 2x^2 - c_kf_mx^2 - f_mxy) \right),
$$

$$
Q(x,y) = \prod_{n=1}^{35} (y - g_nx),
$$

with $a_i, c_k, g_n \in \mathbb{C}$ all different, all the $b_j \in \mathbb{C}$ different, the $d_l \in \mathbb{C}$ different and the $f_m \in \mathbb{C}$ different. The dual resolution graph is as follows
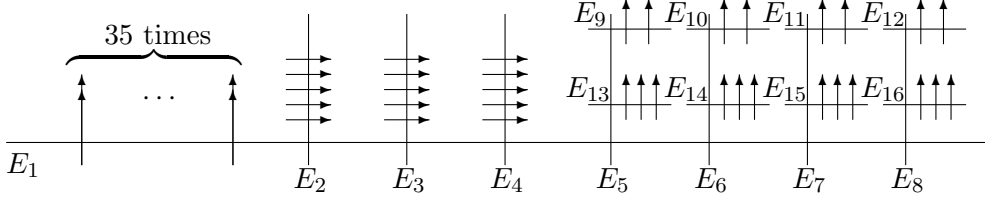
We compute the numerical data.

$i=$	1	$2, 3, 4$	$5, 6, 7, 8$	$9, 10, 11, 12$	$13, 14, 15, 16$
N_i^P	35	40	40	42	43
N_i^Q	35	35	35	35	35
N_i	0	5	5	7	8
ν_i	2	3	3	4	4

We find that $Z_{top,f}(s) = -\frac{560s^3 + 1274s^2 + 767s + 132}{2(7s+4)(s+1)(2s+1)}$ and so the candidate pole $-3/5$ is not a pole. $\qquad\square$

To prove the criterion for poles of plane curves, Veys points out in [Ve1] that there is an 'ordered tree structure', i.e. the candidate poles are ordered increasingly along well-described paths in the tree. This structure fails in the meromorphic case. The other implication in Theorem 2 still holds, i.e. if s_0 is a pole of $Z_{top,f}(s)$, then $s_0 = -\nu_i/N_i$ for some exceptional curve E_i intersecting at least three times other components, or $s_0 = -1/N_j$ for some irreducible component E_j of the strict transform of $f^{-1}\{0\}$. We prove this result now in the plane meromorphic case.

Let $\pi : X \to \mathcal{U}$ be an embedded resolution for $f = P/Q$ as in Section 1 and let $(E_i)_{i \in S}$ be the irreducible components of the total transform $\pi^{-1}(H)$. Let $E(N, \nu)$ be an exceptional component of π for which $N = N^P - N^Q \neq 0$ and suppose that E intersects exactly k other irreducible components of $\pi^{-1}(H)$, say $E_1(N_1, \nu_1), \ldots, E_k(N_k, \nu_k)$. For $i \in \{1, \cdots, k\}$, we set $\alpha_i = \nu_i - \frac{\nu}{N}N_i$.

Lemma 3 *The following relations hold:*

$$\sum_{i=1}^{k} N_i = (-E \cdot E)N; \tag{1}$$

$$\sum_{i=1}^{k} \alpha_i = k - 2. \tag{2}$$

Proof. We can follow the conceptual proof of Veys in [Ve3] for holomorphic functions. In $Pic(X)$ we have $\sum_{i \in S} N_i^P E_i = 0$ and $\sum_{i \in S} N_i^Q E_i = 0$. Hence $NE = -\sum_l (N_l^P - N_l^Q)E_l$, where l runs over all components E_l, $l \in S$, except E. Intersecting with E yields $N(E \cdot E) = -\sum_{i=1}^{k} N_i$. This gives us Relation (1).

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For Relation (2) we have that $K_X = \sum_{i \in S}(\nu_i - 1)E_i$. Hence in $Pic(X) \otimes \mathbb{Q}$ we have

$$
\begin{aligned}
K_X &= \sum_{i \in S}(\nu_i - 1)E_i - \frac{\nu}{N}(\sum_{i \in S} N_i E_i) \\
&= (\nu - 1)E + \sum_{l}(\nu_l - 1)E_l - \frac{\nu}{N}(\sum_l N_l E_l) - \nu E \\
&= -E + \sum_l (\alpha_l - 1)E_l
\end{aligned}
$$

where l runs over all components E_l, $l \in S$, except E.
Intersecting with E gives $K_X \cdot E = -E \cdot E + \sum_{i=1}^{k}(\alpha_i - 1)$ and by the adjunction formula we get $-2 = \deg K_E = \sum_{i=1}^{k}(\alpha_i - 1)$. ∎

If $\nu/N \neq \nu_i/N_i$ for $i \in \{1, \cdots, k\}$, then the contribution $\mathcal{R}$ of the exceptional divisor E to the residue of $Z_{top,f}(s)$ at $s = -\nu/N$ is given by

$$
\mathcal{R} = \frac{1}{N}(2 - k + \sum_{i=1}^{k} \frac{1}{\alpha_i}).
$$

Completely analogous to the holomorphic case, one can now prove the following result.

Proposition 4 *Let $f = \frac{P}{Q}$ be a germ of a plane meromorphic function. If s_0 is a pole of $Z_{top,f}(s)$, then $s_0 = -\nu_i/N_i$ for some exceptional component intersecting at least three other components or $s_0 = -1/N_i$ for some irreducible component of the strict transform of P.*

Proof. If s_0 is not a candidate pole of order two, then one can deduce from Relation (2) that the exceptional components giving rise to the candidate pole s_0 and that have at most two intersections do not contribute to the residue. If s_0 is a candidate pole of order two, then suppose E_i intersects E_j and $\nu_i/N_i = \nu_j/N_j$. If E_i or E_j is a component of the strict transform of P, then we are done. Suppose now that E_i and E_j are exceptional components. If E_i or E_j has no other intersections, then we get a contradiction with Relation (2). If they intersect exactly one other component, then Relation (2) implies that this component also yields the same candidate pole. By iterating the previous arguments, we can conclude that there is a component of the strict transform or an exceptional component intersecting at least three other components that also gives rise to the candidate pole $-\nu_i/N_i$. ∎

3. Some facts about resolution for plane curves

In this section we formulate some properties of resolution of singularities of plane curves. These properties should be well known but we recall them here because they are important to understand well the resolution of plane meromorphic germs. This section will serve as a preparation for Section 4 where we will treat the monodromy conjecture for plane meromorphic germs.

(1) Notation. Let us fix some terminology about the dual resolution graph $\mathcal{G}$ (of plane curves or of plane meromorphic germs). For a vertex of $\mathcal{G}$, let its *valence* be the number of irreducible components that meet the irreducible component corresponding to that vertex. A *bamboo* leaving out of a vertex of $\mathcal{G}$ is a connected path in $\mathcal{G}$ starting in that vertex, having an end point with valence 1 and in which the vertices in between have all valence 2. A bamboo is said to be a *primitive bamboo* if its valence one vertex does not correspond to an irreducible component of the strict transform (i.e. it is not an arrowhead). The *branches* leaving out of a vertex of $\mathcal{G}$ are the connected components of $\mathcal{G}$ minus that vertex. Such a branch is called primitive if none of its valence one vertices corresponds to an irreducible component of the strict transform (i.e. the branch does not contain arrowheads).

(2) Dual resolution graphs. The following properties of dual resolution graphs of plane curve singularities will be useful. See [BK, III. Section 8.4, Prop. 16] or [dJP, Section 5.4, Theorem 5.4.5] for a proof.

Proposition 5 *For the dual resolution graph $\mathcal{G}$ of the minimal embedded resolution of a plane curve the following statements hold:*

1. *The graph $\mathcal{G}$ is finite and does not contain cycles;*

2. *For every vertex of $\mathcal{G}$ there is at most one primitive bamboo leaving out of that vertex, except for possibly one vertex from which at most two primitive bamboos leave;*

3. *For every vertex of $\mathcal{G}$ there is at most one primitive branch leaving out of that vertex which is not a bamboo.*

Remark 2 To create the minimal embedded resolution of a meromorphic germ P/Q, notice that one can take the minimal embedded resolution of PQ to which one adds the necessary blow ups to create the dicritical components. Notice that the blow ups (the intersections of irreducible components of $\pi^{-1}\{PQ = 0\}$) necessary to create the dicritical components only create vertices with valence 2. This implies that the statements in Proposition 5 still hold for the dual minimal embedded resolution graph of a plane meromorphic germ.

(3) Properties of the resolution data. Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic germ. Let $E(N, \nu)$ be an exceptional component with $N \neq 0$ and suppose that E intersects exactly k other irreducible components of $(f \circ \pi)^{-1}\{0\}$, say $E_1(N_1, \nu_1), \ldots, E_k(N_k, \nu_k)$. For $i \in \{1, \cdots, k\}$, we set $\alpha_i = \nu_i - \frac{\nu}{N} N_i$. The bounds on the numbers α_i play an essential role in the proof by Rodrigues of the monodromy conjecture for plane curves (see [Ro]). In the holomorphic case Loeser stated the following bounds:

Proposition 6 [L1, Proposition II.3.1] *Suppose that E is an exceptional component in the minimal embedded resolution of a plane curve which intersects at least two irreducible components in the total transform, one of them being E_j. Then $-1 < \alpha_j < 1$. If the embedded resolution of the plane curve is not minimal, then $-1 \leq \alpha_j \leq 1$.*

Remark 3 If the exceptional component E intersects exactly one component, say E_j, then it follows from Relation (2) for holomorphic germs (see [Ve2, Lemma 2.3]) that $\alpha_j = -1$.

In our proof of the monodromy conjecture for plane meromorphic germs we will follow the same strategy as Rodrigues but we will have to study carefully the numbers α_i. Indeed, in the case of meromorphic germs in general the numbers α_i are not bounded even in the minimal embedded resolution (see Example 1 with $E = E_8$ and $E_j = E_1$, then $\alpha_1 = -5/2$). In what follows we will look at subgraphs of the dual resolution graph of a plane meromorphic function in which the numbers α_i are contained in the interval $(-1, 1)$.

Let $\pi : X \to \mathcal{U}$ be an embedded resolution of the germ of the plane meromorphic function $f = \frac{P}{Q}$ at 0 and let $(E_i)_{i \in S}$ be the irreducible components of the total transform of f. Denote by $\mathcal{E}$ the exceptional divisor of π; i.e. $\mathcal{E} = \cup_{i \in S} E_i \cap \pi^{-1}(0)$, and by $\mathcal{D}$ the union of the dicritical components of π.

In order to prove the monodromy conjecture for plane meromorphic germs we need to take into account some features of the connected components of the closure $(\mathcal{E} - \mathcal{D})^c$ of $\mathcal{E} - \mathcal{D}$. These components where studied by Delgado and Maugendre in relation to the geometry of pencils of plane curve singularities (see [DM]). Here we recall some properties needed in our proof.

Let $E(N, \nu)$ be an exceptional component of π with $N = N^P - N^Q \neq 0$ and suppose that E intersects exactly k other irreducible components of $(f \circ \pi)^{-1}\{0\}$, say $E_1(N_1, \nu_1), \ldots, E_k(N_k, \nu_k)$. Recall that π is also a resolution of the curves P and Q and that the corresponding resolution data are related by $N_i = N_i^P - N_i^Q$. From the identity [Ve3, 2.2 (ii)], which is the version of (2) for resolutions of plane curve singularities, we have that $\sum_{i=1}^k \alpha_i^P = k - 2 = \sum_{i=1}^k \alpha_k^Q$. Equivalently, we have the identity

$$\sum_{i=1}^{k} N_i^P / N^P = \sum_{i=1}^{k} N_i^Q / N^Q, \tag{3}$$

which implies the following result.

Proposition 7 [DM, Proposition 1] *Assume that none of the components $E_1, \ldots, E_k$ belongs to the strict transform of $\{P_{|\mathcal{U}}\} \cup \{Q_{|\mathcal{U}}\}$. Then, there exists an index $r \in \{1, \ldots, k\}$ such that $N_r^P N^Q > N_r^Q N^P$ if and only if there exists an index $s \in \{1, \ldots, k\}$ such that $N_s^P N^Q < N_s^Q N^P$.*

Corollary 8 [DM, Corollary 1]Proposition 7 implies that the quotient N^P/N^Q is constant on a primitive bamboo.

Example 3 Let us consider two irreducible plane curves $P(x, y) = x^{103} - y^{24} = 0$ and $Q(x, y) = x^{30} - y^7 = 0$. The dual resolution graph of P/Q is as follows
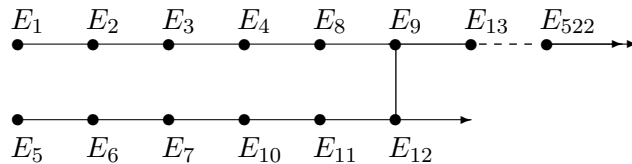
where $E_{13}, \dots, E_{522}$ are exceptional components created to get a dicritical component. The numerical data for the components $E_1, \dots, E_{12}$ are given in the following table

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}	E_{11}	E_{12}
N_i^P	24	48	72	96	103	206	309	408	720	1030	1751	2472
N_i^Q	7	14	21	28	30	60	90	119	210	300	510	720
ν_i	2	3	4	5	6	11	16	21	37	53	90	127

Corollary 8 tells us that

$$\frac{24}{7} = \frac{N_1^P}{N_1^Q} = \frac{N_2^P}{N_2^Q} = \frac{N_3^P}{N_3^Q} = \frac{N_4^P}{N_4^Q} = \frac{N_8^P}{N_8^Q} = \frac{N_9^P}{N_9^Q} \qquad \text{and}$$

$$\frac{103}{30} = \frac{N_5^P}{N_5^Q} = \frac{N_6^P}{N_6^Q} = \frac{N_7^P}{N_7^Q} = \frac{N_{10}^P}{N_{10}^Q} = \frac{N_{11}^P}{N_{11}^Q} = \frac{N_{12}^P}{N_{12}^Q}.$$

$\square$

(4) Some special subgraphs. If $\mathcal{K}$ is a connected component of $(\mathcal{E} - \mathcal{D})^c$, then $f \circ \pi$ is constant along $\mathcal{K}$. Let us assume that $(f \circ \pi)_{|\mathcal{K}}$ is constantly 0.

Proposition 9 [DM, Theorem 1] $\mathcal{K}$ *intersects the strict transform of the hypersurface* $\{P_{|\mathcal{U}} = 0\}$.

Proof. If there are no dicritical components, we are in the case of plane curves and obviously some exceptional component intersects a component of the strict transform of $\{P_{|\mathcal{U}} = 0\}$. So suppose that the connected component $\mathcal{K}$ meets a dicritical component. Since $\mathcal{K}$ is finite we can take an element $E \in \mathcal{K}$ such that $\frac{N^P}{N^Q}$ is maximal in $\mathcal{K}$. If E does not intersect the strict transform of the hypersurface $\{P_{|\mathcal{U}} = 0\}$ and if $E_1, \dots, E_k$ are the components that intersect E, then Proposition 7 implies that $\frac{N^P}{N^Q} = \frac{N_1^P}{N_1^Q} = \cdots = \frac{N_r^P}{N_r^Q}$.

In this way, one gets that the quotient $\frac{N^P}{N^Q}$ is constant in all $\mathcal{K}$, provided that $\mathcal{K}$ does not intersect the strict transform of the hypersurface $\{P_{|\mathcal{U}} = 0\}$. However, again Proposition 7 prevents that any of the components in $\mathcal{K}$ intersects a dicritical component because $\frac{N^P}{N^Q} > 1$ for all components in $\mathcal{K}$ as $(f \circ \pi)_{|\mathcal{K}}$ is constantly 0 and $\frac{N^P}{N^Q} = 1$ for all dicritical components. Thus we reach a contradiction. $\blacksquare$

From now on we take $\pi : X \to \mathcal{U}$ the minimal embedded resolution of the plane meromorphic function $f = \frac{P}{Q}$.

Proposition 10 *Let* $P, Q : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ *and let* $\mathcal{G}$ *be the dual resolution graph corresponding to the minimal embedded resolution* π *of* P/Q. *Let* $\mathcal{P}$ *be a connected subgraph of* $\mathcal{G}$ *that contains exactly one vertex which is connected to vertices not belonging to* $\mathcal{P}$, *that does not contain vertices corresponding to components of the strict transform and on which* $(f \circ \pi)_{|\mathcal{P}}$ *is constantly 0. Let* E_i *and* E_j *be two exceptional components which are represented by adjacent vertices in* $\mathcal{P}$. *If* E_i *is intersected by at least one more component, then* $-1 < \alpha_{i,j} := \nu_j - \frac{\nu_i}{N_i} N_j < 1$.

Proof. Let us first argue on how the subgraph $\mathcal{P}$ then looks like. Proposition 5 and Remark 2 imply that $\mathcal{P}$ has the following shape, where we only draw the vertices with valence different from two.
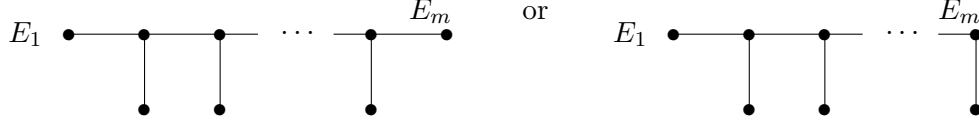
with particular cases
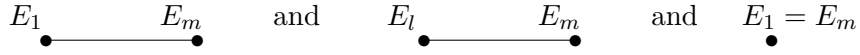
Here E_1 denotes the exceptional component corresponding to the first blow up of π, the component E_m is the component that intersects some components outside of $\mathcal{P}$ and $E_l \neq E_1$. In the last case where $\mathcal{P}$ only consists of one exceptional divisor, $E_m = E_1$.

Next notice that $\mathcal{P}$ was already a subgraph of the minimal resolution of PQ. Indeed, suppose that any vertex E in $\mathcal{P}$ is created to pass from the minimal embedded resolution of PQ to the embedded resolution of $f = \frac{P}{Q}$. According to Remark 2, E has valence 2 and erasing E divides the resolution graph into two disjoint parts, one of then containing at least irreducible components of the strict transform of $\pi^{-1}\{P = 0\}$ and the other containing at least irreducible components of the strict transform $\pi^{-1}\{Q = 0\}$. This is a contradiction as we supposed that $\mathcal{P}$ contains no irreducible components of the strict transform of $\pi^{-1}\{P = 0\}$ neither of $\pi^{-1}\{Q = 0\}$.

Therefore we can assume that E_i and E_j appeared in the minimal resolution graph of PQ and Proposition 6 tells us then that

$$-1 < \nu_j - \frac{\nu_i}{N_i^P + N_i^Q}(N_j^P + N_j^Q) < 1. \tag{4}$$

Taking into account the form of the subgraph $\mathcal{P}$, we can now prove that N^P/N^Q is constant on the whole graph $\mathcal{P}$. Corollary 8 deals with the case that $\mathcal{P}$ is a primitive bamboo. Otherwise let E be an exceptional component in $\mathcal{P}$ corresponding to a vertex of valence equal to 3 from which two bamboos are leaving. Then it follows from Corollary 8 that the quotient N_i^P/N_i^Q is constant on these bamboos and equal to N^P/N^Q. Denote by E_1, E_2 and E_3 the exceptional components intersecting E. Since $N^P/N^Q = N_1^P/N_1^Q = N_2^P/N_2^Q$, we can deduce that $N^P/N^Q = N_3^P/N_3^Q$. Indeed, Relation (2) in the holomorphic case gives

$$\nu_1 - \frac{\nu}{N^P}N_1^P + \nu_2 - \frac{\nu}{N^P}N_2^P + \nu_3 - \frac{\nu}{N^P}N_3^P = 1 = \nu_1 - \frac{\nu}{N^Q}N_1^Q + \nu_2 - \frac{\nu}{N^Q}N_2^Q + \nu_3 - \frac{\nu}{N^Q}N_3^Q,$$

from which we can deduce $N^P/N^Q = N_3^P/N_3^Q$. Next we can propagate this equality along a chain of vertices of valence 2. Let E_j be a vertex of has valence 2 intersecting E_i and $E_k = E$. Then we have that

$$\nu_i - \frac{\nu_j}{N_j^P}N_i^P + \nu_k - \frac{\nu_j}{N_j^P}N_k^P = 0 = \nu_i - \frac{\nu_j}{N_j^Q}N_i^Q + \nu_k - \frac{\nu_j}{N_j^Q}N_k^Q.$$

Now we can deduce that $N_i^P/N_i^Q = N_j^P/N_j^Q$. We can iterate this until we arrive at another vertex of valence equal to 3, from which a primitive bamboo is leaving. Using again the previous argument and taking into account the shape of $\mathcal{P}$, we deduce that the quotients N_i^P/N_i^Q are equal for all E_i in $\mathcal{P}$.

Finally, since $\frac{N_i^P}{N_j^P} = \frac{N_i^Q}{N_j^Q}$ and hence $\frac{N_i^P+N_i^Q}{N_j^P+N_j^Q} = \frac{N_i^P}{N_j^P} = \frac{N_i}{N_j}$. We substitute in (4) and get $-1 < \nu_j - \frac{\nu_i}{N_i}N_j < 1$. ∎

(5) Topology of trees of nonsingular rational curves The following lemma was shown by Rodrigues to give his own proof of the monodromy conjecture for plane curves.

Lemma 11 [Ro, Lemma(2.2)] *Let $\cup_{i=1}^r E_i$ be a tree of nonsingular rational curves on a nonsingular surface, then*

$$\sum_{i=1}^r \chi(E_i^\circ) = 2.$$

4. The monodromy conjecture for plane meromorphic germs

By analogy with the holomorphic case, we propose the monodromy conjecture for meromorphic germs as follows.

Conjecture 1 *(Monodromy Conjecture for meromorphic germs) Let $f = \frac{P}{Q}$ be a germ of a meromorphic function on $(\mathbb{C}^{n+1}, 0)$ and let s_0 be a pole of $Z_{top,f}$, then $e^{2\pi i s_0}$ is an eigenvalue of the monodromy transformation h_f at some point of $P^{-1}\{0\}$ in a neighbourhood of 0.*

Theorem 12 *Let $f = \frac{P}{Q}$ be a germ of a meromorphic function on $(\mathbb{C}^2, 0)$ and let s_0 be a pole of $Z_{top,f}(s)$, then $e^{2\pi i s_0}$ is an eigenvalue of the monodromy transformation h_f at some point of $P^{-1}\{0\}$ in a neighbourhood of 0.*

Proof. Let $\pi : X \to \mathcal{U}$ be the minimal embedded resolution of f. By Proposition 4 it follows that either $s_0 = -1/N$ for some irreducible component $E(N, 1)$ of the strict transform of P or $s_0 = -\nu_i/N_i$ for some exceptional divisor E_i of π with $\chi(E_i^\circ) < 0$. In the first case we find the eigenvalue $e^{2\pi i s_0}$ at any point of $\{P = 0\}$ in a punctured neighbourhood of 0. So suppose now that $s_0 = -\nu_i/N_i$ for some exceptional divisor E_i of π with $\chi(E_i^\circ) < 0$. We write $\nu_i/N_i = a/d$ with $\gcd(a, d) = 1$. We can suppose that there is no component $E_S(N_S, \nu_S)$ of the strict transform of P for which $d \mid N_S$. Otherwise $e^{2\pi i s_0}$ is an eigenvalue of monodromy at any point of $\{P = 0\}$ in a punctured neighbourhood of 0.

Let us define

$$C_d := \{E_k \text{ exceptional divisor of } \pi \text{ such that } N_k > 0 \text{ and } d \mid N_k\}.$$

Now we study $\sum_{E_k \in \mathcal{K}} \chi(E_k^\circ)$ for each connected component $\mathcal{K}$ of C_d. Firstly, Proposition 9 shows that there is no such component $\mathcal{K}$ intersecting only dicritical divisors.

Notice that an exceptional component E with $N = 0$ intersecting such a connected component $\mathcal{K}$ is also dicritical because $(f \circ \pi)_{|E}$ then reaches at least two different values. Hence, each connected component $\mathcal{K}$ in C_d does contain a component E_l that intersects some irreducible component E of the total transform $\pi^{-1}(H)$ outside $\mathcal{K}$ with $N \neq 0$. Relation (1) in Lemma 3 implies that there is at least another irreducible component of $\pi^{-1}(H)$ outside $\mathcal{K}$ satisfying these properties. Lemma 11 then implies that $\sum_{E_k \in \mathcal{K}} \chi(E_k^\circ) \leq 0$ for each connected component $\mathcal{K}$ of C_d. Now the result follows if for at least one such component $\sum_{E_k \in \mathcal{K}} \chi(E_k^\circ) < 0$ holds. We show that this is the case for the component $\mathcal{K}$ of C_d that contains the exceptional component E_i. We denote this component by $\tilde{\mathcal{N}}$. Let us argue by contradiction. We suppose that $\sum_{E_k \in \tilde{\mathcal{N}}} \chi(E_k^\circ)$ is equal to 0. This assumption implies that $\tilde{\mathcal{N}}$ contains only one exceptional component intersecting irreducible components of $\pi^{-1}(H)$ not belonging to $\tilde{\mathcal{N}}$ and with $N \neq 0$. Furthermore the number of the intersections must be exactly two. Thus $\tilde{\mathcal{N}}$ satisfies the conditions of the component $\mathcal{P}$ in Proposition 10. In particular, we can disregard the case of $\tilde{\mathcal{N}}$ being a singleton (i.e. $\tilde{\mathcal{N}}$ only consists of E_i) because $\chi(E_i^\circ) < 0$. Remark that in this case $e^{-2\pi i a/d}$ is an eigenvalue of monodromy at the origin. If $\tilde{\mathcal{N}}$ is not a singleton, then let E_j be an exceptional component in $\tilde{\mathcal{N}}$ intersecting E_i. As $\alpha_{i,j} = \nu_j - \frac{\nu_i}{N_i} N_j = \nu_j - \frac{a}{d} N_j \in \mathbb{Z}$, it follows by Proposition 10 that $\alpha_{i,j} = 0$. Hence $\nu_j/N_j = a/d$. On the other hand, as $\alpha_{i,j} = 0$, it follows from Relation (2) that E_j intersects exactly one other component E_k. If $E_k \in \tilde{\mathcal{N}}$, then analogously we find $\nu_k/N_k = a/d$ and using Relation (2) we see that E_k intersects exactly one other component. As the graph $\mathcal{G}$ is finite, we must end with an exceptional component not in $\tilde{\mathcal{N}}$. However, as $\tilde{\mathcal{N}}$ only contains one exceptional component which intersects exceptional components not in $\tilde{\mathcal{N}}$ and as E_i is intersecting at least three components, we can do the same reasoning for the other exceptional components intersecting E_i and hence get a contradiction. ∎

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