Poincaré series for filtrations defined by discrete valuations with arbitrary center

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Dedicated to S.M. Gusein-Zade on the occasion of his sixtieth birthday

Abstract.— *To study singularities on complex varieties we study Poincaré series of filtrations that are defined by discrete valuations on the local ring at the singularity. In all previous papers on this topic one poses restrictions on the centers of these valuations and often one uses several definitions for Poincaré series. In this article we show that these definitions can differ when the centers of the valuations are not zero-dimensional, i.e. do not have the maximal ideal as center. We give a unifying definition for Poincaré series which also allows filtrations defined by valuations that are all nonzero-dimensional. We then show that this definition satisfies a nice relation between Poincaré series for embedded filtrations and Poincaré series for the ambient space and we give some application for singularities which are nondegenerate with respect to their Newton polyhedron.*

0. Introduction

In [C,D,K] one introduced a Poincaré series induced by a filtration on the ring of germs of a complex variety. This Poincaré series has been studied for several kinds of singularities, see for example [C,D,G-Z1], [C,D,G-Z2], [C,D,G-Z3], [Eb,G-Z], [CHR], [L1], [GP-H] and [N]. In some cases this Poincaré series determines the topology of the singularity and is related to its zeta function of monodromy.

In some of these papers one uses several definitions for Poincaré series. In Section 1 we will see that these definitions become not necessarily equivalent when not all considered valuations are centered at the maximal ideal. In [GP-H], one studies the Poincaré series for quasi-ordinary and for toric singularities and, in particular, a new definition for Poincaré series was introduced for this context which made it possible to treat also sets of monomial valuations where at least one valuation was centered at the maximal ideal. In Section 2 we give a unifying definition for the Poincaré series, i.e. a definition that coincides with the former definitions when all valuations are centered at the maximal ideal, that also coincides with the one in [GP-H] when the valuations are monomial and at least one of them is centered at the maximal ideal, and that even makes sense in some cases where none of the valuations is centered at the maximal ideal. This Poincaré series is defined in homological terms.

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In [L2] the second author introduced a Poincaré series for embedded varieties in an ambient space by taking multi-index filtrations coming from valuations on the ambient space. When at least one such valuation is centered at the maximal ideal and when the subspace corresponds to a principal ideal, then a nice formula relating the Poincaré series of the embedded space and the ambient space showed up. It was shown that this relating formula gives rise to interesting topological and geometrical information in the case of plane curve singularities and singularities that are nondegenerate with respect to their Newton polyhedron. We can now also define Poincaré series for embedded filtrations in homological terms, see Section 3. We show that the relating formula from [L2] still holds when none of the valuations is centered at the maximal ideal, under the condition that both Poincaré series are well defined. We then extend the topological and geometrical results from [L2] to the context of sets of valuations where none of them is centered at the maximal ideal.

1. Poincaré series - different definitions

In this section we would like to call attention to the fact that Poincaré series have been defined in several ways, at first sight maybe equivalent up to notation, but in fact different in the sense of less or more general.

Description 1.– Let (X, o) be a germ of a complex algebraic variety and let $\mathcal{O}_{X,o}$ be the local ring of germs of functions on (X, o). Let $\underline{\nu} = \{\nu_1, \cdots, \nu_r\}$ be a set of order functions from $\mathcal{O}_{X,o}$ to $\mathbb{Z}\cup\{\infty\}$, i.e. functions ν_j that satisfy $\nu_j(f+g) \geq \min\{\nu_j(f), \nu_j(g)\}$, $\nu_j(0) = \infty$ and $\nu_j(\lambda f) \geq \nu_j(f)$, for all $f, g \in \mathcal{O}_{X,o}$ and $\lambda \in \mathbb{C}$, $1 \leq j \leq r$. The set $\underline{\nu}$ defines a multi-index filtration on $\mathcal{O}_{X,o}$ by the vector spaces

$$M(\underline{v}) := \{g \in \mathcal{O}_{X,o} \mid \nu_j(g) \geq v_j, 1 \leq j \leq r\}, \qquad \underline{v} \in \mathbb{Z}^r.$$

If the dimensions of the complex vector spaces $M(\underline{v})/M(\underline{v}+\underline{1})$ are finite for all $\underline{v} \in \mathbb{Z}^r$, then originally (see [C,D,K] and [C,D,G-Z2]) the *Poincaré series* associated to this multi-index filtration was defined as

$$P_X^{\underline{\nu}}(t_1, \ldots, t_r) := \frac{\prod_{j=1}^r (t_j - 1)}{(t_1 \cdot \ldots \cdot t_r - 1)} \sum_{\underline{v} \in \mathbb{Z}^r} \dim(M(\underline{v})/M(\underline{v}+\underline{1}))\underline{t}^{\underline{v}}. \tag{1}$$

If the order functions ν_j, $1 \leq j \leq r$, satisfy the stronger property $\nu_j(gf) \geq \nu_j(f)$ for all $f, g \in \mathcal{O}_{X,o}$, then the multi-index filtration is by ideals. In particular, one may have the valuative case, i.e. when $\underline{\nu}$ is a set of discrete valuations of the function field $\mathbb{C}(X)$ whose valuation rings contain $\mathcal{O}_{X,o}$. Recall that an order function ν_j on $\mathcal{O}_{X,o}$ is called a discrete valuation if moreover it satisfies $\nu_j(fg) = \nu_j(f) + \nu_j(g)$, for all $f, g \in \mathcal{O}_{X,o}$. Then $\{f \in \mathcal{O}_{X,o} | \nu_j(f) > 0\}$ is a prime ideal which is called the center of the valuation ν_j, $1 \leq j \leq r$. In the valuative case, the Poincaré series (1) is well defined if the center of each valuation is the maximal ideal m of $\mathcal{O}_{X,o}$ (see [L1]).

One interesting valuative case is the toric one, i.e. when the data are an affine toric variety and the ν_j are monomial valuations. Let $N = \mathbb{Z}^n, n > 1$, and let M be the dual space to N, then there is a natural bilinear map $M \times N \to \mathbb{Z} : (m, n) \mapsto \langle m, n\rangle$. The dual cone $\check{\sigma}$ to a cone $\sigma \subset N \otimes_{\mathbb{Z}} \mathbb{R}$ is defined as the set $\{m \in \mathbb{R}^n \mid \langle m, x\rangle \geq 0, \forall x \in \sigma\}$. Now one considers a semigroup $S \subset M$ generating $\check{\sigma}$ as a cone, and the affine toric variety $X = \mathrm{Spec}\ \mathbb{C}[S]$. For each $\nu \in \sigma$, the duality allows to define a discrete valuation on $\mathbb{C}[S]$, also denoted by ν, given by $\nu(x) = \langle \nu, x\rangle$ on the monomials x of $\mathbb{C}[S]$. Such valuations, which are centered on a prime ideal of $\mathbb{C}[S]$, are called monomial, since their value on a function $f \in \mathbb{C}[S]$ is the minimum of the values $v(x)$ for x in the support of f.

Description 2.– For I a subset in $\{1, \ldots, r\}$, let $\underline{e}_I$ be the r-tuple with j-th component equal to 1 if $j \in I$ and equal to 0 otherwise. When I is a singleton, say $I = \{j\}$, we also denote $\underline{e}_I$ by $\underline{e}_j$. Notice that the coefficient of $\underline{t}^{\underline{v}}$ in the Poincaré series (1) can also be written as

$$\begin{aligned} &\sum_{I \subset \{1,\ldots,r\}} (-1)^{\#I} \dim \frac{M(\underline{v}+\underline{e}_I)}{M(\underline{v}+\underline{1})} \\ = &\sum_{K \subset \{1,\ldots,r-1\}} (-1)^{\#K} \dim \frac{M(\underline{v}+\underline{e}_K)}{M(\underline{v}+\underline{e}_K+\underline{e}_r)}. \end{aligned}$$

However, this last expression is well defined when the vector spaces $\frac{M(\underline{v}+\underline{e}_K)}{M(\underline{v}+\underline{e}_K+\underline{e}_r)}$ are of finite dimension, what in the valuative case means that only ν_r needs to be centered at m.

Description 3.– When only considering two valuations, the coefficient of $\underline{t}^{\underline{v}}$ in the Poincaré series (1) can also be written as

$$\dim \frac{M(v_1, v_2)}{M(v_1+1, v_2) + M(v_1, v_2+1)}.$$

This term can be well defined although none of the valuations is centered at m, as the following example shows. We will see in Section 2 that this description can be given in a much more general context.

Example 1. We take σ to be the cone $\mathbb{R}^2_{\geq 0}$. Let $S \subset \check{\sigma} \cap M$ be the semigroup generated by the vectors $(1, 0)$ and $(0, 1)$ and let X be the affine toric variety given by $\mathrm{Spec}\ \mathbb{C}[S]$. We consider the monomial valuations ν_1 and ν_2 corresponding to the vectors $(1, 0)$ and $(0, 1)$ in σ, i.e. $\nu_1(x^a y^b) = a$ and $\nu_2(x^a y^b) = b$ for $(a, b) \in \mathbb{Z}^2_{\geq 0}$. These valuations are obviously not centered at the maximal ideal. We get that the Poincaré series defined in this way is equal to $\frac{1}{(1-t_1)(1-t_2)}$. $\square$

Description 4.– In the paper [C,D,G-Z2] on Poincaré series for plane curve singularities, an alternative description was given for this Poincaré series: for $1 \leq j \leq r$, denote by $D_j(\underline{v})$ the complex vector space $M(\underline{v})/M(\underline{v}+\underline{e}_j)$. Let us consider the map

$$\begin{aligned} j_{\underline{v}} : M(\underline{v}) &\longrightarrow D_1(\underline{v}) \times \cdots \times D_r(\underline{v}) \\ g &\longmapsto (a_1(g), \ldots, a_r(g)) =: \underline{a}(g), \end{aligned}$$

where $a_j(g)$ is the projection of g on $D_j(\underline{v})$. Let $D(\underline{v})$ be the image of the map $j_{\underline{v}}$, then $D(\underline{v}) \simeq M(\underline{v})/M(\underline{v}+\underline{1})$. One defines the fibre $F_{\underline{v}}$ as the space $D(\underline{v}) \cap (D_1^*(\underline{v}) \times$

$\cdots \times D_r^*(\underline{v}))$, where $D_j^*(\underline{v})$ denotes $D_j(\underline{v}) \setminus \{0\}$. Then $F_{\underline{v}}$ is invariant with respect to multiplication by nonzero constants; let $\mathbb{P}F_{\underline{v}} := F_{\underline{v}}/\mathbb{C}^*$ be the projectivisation of $F_{\underline{v}}$. Let χ denote the topological Euler characteristic.

Theorem 1. *([C,D,G-Z2, Theorem 3]) If the valuations $\nu_1, \dots, \nu_r$ are centered at the maximal ideal, then $P_X^{\underline{\nu}}(\underline{t}) = \sum_{\underline{v}\in\mathbb{Z}^r} \chi(\mathbb{P}F_{\underline{v}})\underline{t}^{\underline{v}}$.*

This equality holds for all singularities (X, o) (not only plane curve singularities), but the proof only works if all valuations are centered at the maximal ideal, and more general, if each ν_j, $1 \leq j \leq r$, is a finite order function, i.e. an order function such that the quotient vector space $\{f \in \mathcal{O}_{X,o} \mid \nu_j(f) > v\}/\{f \in \mathcal{O}_{X,o} \mid \nu_j(f) > v+1\}$ is finitely dimensional for every integer v. Notice that discrete valuations centered in the maximal ideal are finite order functions. When this is not the case, it can happen that $\chi(\mathbb{P}F_{\underline{v}})$ is infinite but it can also happen that $\chi(\mathbb{P}F_{\underline{v}})$ is finite for all $\underline{v} \in \mathbb{Z}^r$ and thus that the expression $\sum_{\underline{v}\in\mathbb{Z}^r} \chi(\mathbb{P}F_{\underline{v}})\underline{t}^{\underline{v}}$ is well defined. In the toric case this expression is easy to compute, as shown in the following example.

Example 1 Continued.
In the toric monomial case one can decompose $F_{\underline{v}}$ as a (here infinite) disjoint union according to the support of the functions in $F_{\underline{v}}$. We see that only the monomials contribute to $\chi(\mathbb{P}F_{(v_1,v_2)})$. In particular, when $\chi(\mathbb{P}F_{\underline{v}})$ is finite, then it equals the number of monomials with value equal to $\underline{v}$ (see also [L1], Proof of Prop. 1). We get $\sum_{\underline{v}\in\mathbb{Z}^r} \chi(\mathbb{P}F_{\underline{v}})\underline{t}^{\underline{v}} = \frac{1}{(1-t_1)(1-t_2)}$. □

In the next section we will give a unifying definition for the above described Poincaré series.

2. Poincaré series defined in homological terms

Fix some $\underline{v} \in \mathbb{Z}^r$. We will denote $V_I := M(\underline{v} + \underline{e}_I)/M(\underline{v} + \underline{1})$ for $I \subseteq \{1, \dots, r\}$. Let

$$C_i := \begin{cases} \{0\} & \text{if } i = -1, i > r, \\ \bigoplus_{\substack{I\subset\{1,\dots,r\},\\ \#I=i}} V_I & \text{if } 0 \leq i \leq r. \end{cases}$$

For $-1 \leq i \leq r-1$, we define a map $\partial_{i+1} : C_{i+1} \to C_i$ by defining it on each component V_I ($\#I = i+1$). Suppose that $I = \{a_1, a_2, \dots, a_{i+1}\} \subseteq \{1, \dots, r\}$ with $a_1 < a_2 < \dots < a_{i+1}$, then we set

$$\begin{array}{ccc} V_I & \longrightarrow & C_i = \bigoplus_{\substack{J\subset\{1,\dots,r\},\\ \#J=i}} V_J \\ x & \longmapsto & (\underline{y})_J, \end{array}$$

where $y_J = 0$ if $J \not\subseteq I$ and $y_J = (-1)^k x$ if $J = I \setminus \{a_k\}$. For instance, if $r = 4$, then

$$\begin{array}{rcl} \partial_3 : V_{\{1,2,3\}} \oplus V_{\{1,2,4\}} \oplus V_{\{1,3,4\}} \oplus V_{\{2,3,4\}} & \longrightarrow & V_{\{1,2\}} \oplus V_{\{1,3\}} \oplus V_{\{1,4\}} \oplus V_{\{2,3\}} \oplus V_{\{2,4\}} \oplus V_{\{3,4\}} \\ (x, y, z, u) & \longmapsto & (-x-y, x-z, y+z, -x-u, -y+u, -z-u). \end{array}$$

For $i \leq r$, we define ∂_i to be the zero map. Notice that $\partial_i \circ \partial_{i+1} \equiv 0$ $(0 \leq i \leq r-1)$, so $\mathcal{C}_\bullet = (C_i, \partial_i)_{i \in \mathbb{Z}_{\geq -1}}$ defines a complex of vector spaces. Let $H_i(\mathcal{C}) =$ Ker $\partial_i /$ Im ∂_{i+1} and $h_i =$ dim $H_i(\mathcal{C})$. If confusion about the considered vector $\underline{v}$ in $\mathbb{Z}^r$ is possible, we will write $h_i^{\underline{v}}$ and $\mathcal{C}_\bullet^{\underline{v}} = (C_i^{\underline{v}}, \partial_i^{\underline{v}})$.

Theorem 2. *If $\nu_1, \ldots, \nu_r$ are order functions on $\mathcal{O}_{X,o}$ and if ν_r is finite, then the Poincaré series defined in* Description 1 *or* Description 2 *coincides with*

$$\sum_{\underline{v} \in \mathbb{Z}^r} \left(\sum_{i=0}^{r-1} (-1)^i h_i^{\underline{v}} \right) \underline{t}^{\underline{v}}.$$

Proof. As ν_r has its center at m, the Poincaré series of *Description 1* can be rewritten as in *Description 2*, and so it is sufficient to study the Poincaré series of *Description 2*. We first define a complex of finite dimensional vector spaces. Fix again $\underline{v} \in \mathbb{Z}^r$. For $I \subseteq \{1, \ldots, r-1\}$, let $\tilde{V}_I := M(\underline{v} + \underline{e}_I)/M(\underline{v} + \underline{e}_I + \underline{e}_r)$ and let

$$\tilde{C}_i := \begin{cases} \{0\} & \text{if } i = -1, i > r-1, \\ \bigoplus_{\substack{I \subset \{1,\ldots,r-1\}, \\ \#I = i}} \tilde{V}_I & \text{if } 0 \leq i \leq r-1. \end{cases}$$

For $-1 \leq i \leq r-2$, we define a map $\tilde{\partial}_{i+1} : \tilde{C}_{i+1} \to \tilde{C}_i$ by defining it on each component $\tilde{V}_I$ $(\#I = i+1)$, analogously as in the construction of the complex (C_i, ∂_i), and so we get a new complex of vector spaces $\tilde{\mathcal{C}}_\bullet = (\tilde{C}_i, \tilde{\partial}_i)_{i \in \mathbb{Z}_{\geq -1}}$. For $i \leq r-1$, we define ∂_i to be the zero map. Let $\tilde{H}_i(\mathcal{C}) =$ Ker $\tilde{\partial}_i /$ Im $\tilde{\partial}_{i+1}$ and $\tilde{h}_i =$ dim $\tilde{H}_i(\mathcal{C})$. If confusion about the considered vector $\underline{v}$ in $\mathbb{Z}^r$ is possible, we will write $\tilde{h}_i^{\underline{v}}$.

For $i \in \{0, \ldots, r-1\}$, consider now the map $\phi_i : C_i \to \tilde{C}_i$ that sends $x \in V_I$ to $(\overline{0}, \ldots, \overline{0}) \in \tilde{C}_i$ if $r \in I$, and to $(\overline{0}, \ldots, \overline{0}, \overline{x}, \overline{0}, \ldots, \overline{0}) \in \tilde{C}_i$ if $r \notin I$, with $\overline{x}$ on the component of index $\tilde{V}_I$. We set

$$\begin{aligned} L_i &:= \text{Ker } (\phi_i) \\ &= \bigoplus_{\substack{I \in \{1,\ldots,r\} \\ r \in I, \#I = i}} V_I \oplus \bigoplus_{\substack{I \in \{1,\ldots,r-1\} \\ \#I = i}} \frac{M(\underline{v} + \underline{e}_I + \underline{e}_r)}{M(\underline{v} + \underline{1})} \\ &= \bigoplus_{\substack{K \in \{1,\ldots,r-1\} \\ \#K = i}} \frac{M(\underline{v} + \underline{e}_K + \underline{e}_r)}{M(\underline{v} + \underline{1})} \oplus \bigoplus_{\substack{J \in \{1,\ldots,r-1\} \\ \#J = i-1}} \frac{M(\underline{v} + \underline{e}_J + \underline{e}_r)}{M(\underline{v} + \underline{1})}, \end{aligned}$$

and we denote the induced maps $\partial_i^L : L_i \to L_{i-1}$ giving rise to the complex $\mathcal{L}_\bullet = (L_i, \partial_i^L)_{i \in \mathbb{Z}_{\geq -1}}$, with induced homologies h_i^L. If $\chi(\mathcal{C}_\bullet)$ is finite, we have $\chi(\mathcal{C}_\bullet) = \chi(\tilde{\mathcal{C}}_\bullet) + \chi(\mathcal{L}_\bullet)$. We will now prove that $\chi(\mathcal{L}_\bullet) = 0$. Asking that the following diagram commutes

$$\begin{array}{ccccccc} 0 \longrightarrow & L_i & \longrightarrow & C_i & \longrightarrow & \tilde{C}_i & \longrightarrow 0 \\ & \downarrow \partial_i^L & & \downarrow \partial_i & & \downarrow \tilde{\partial}_i & \\ 0 \longrightarrow & L_{i-1} & \longrightarrow & C_{i-1} & \longrightarrow & \tilde{C}_{i-1} & \longrightarrow 0 \end{array}$$

means that for $f \in \bigoplus_{\substack{K\in\{1,\dots,r-1\}\\ \#K=i}} \frac{M(\underline{v}+\underline{e}_K+\underline{e}_r)}{M(\underline{v}+\underline{1})}$ and $g \in \bigoplus_{\substack{J\in\{1,\dots,r-1\}\\ \#J=i-1}} \frac{M(\underline{v}+\underline{e}_J+\underline{e}_r)}{M(\underline{v}+\underline{1})}$, the differential ∂_i^L acts as $\partial_i^L(f,g) = (\partial_i(f) + (-1)^i g, \partial_{i-1}(g))$. Now we can deduce that $h_i^L = 0$ for all i. Indeed, take $(f,g) \in$ Ker ∂_i^L. Then in particular $\partial_i(f) = (-1)^{i+1}g$ and $(f,g) = \partial_{i+1}^L(0, (-1)^{i+1}f)$ what implies that $\chi(\mathcal{L}_\bullet) = 0$ and so $\chi(\mathcal{C}_\bullet) = \chi(\tilde{\mathcal{C}}_\bullet)$.
As $\tilde{\mathcal{C}}_\bullet$ is a complex of vector spaces of finite dimension, we can compute $\chi(\tilde{\mathcal{C}}_\bullet)$ as $\sum(-1)^i \dim\tilde{C}_i$ what also equals $\sum_{K\subset\{1,\dots,r-1\}}(-1)^{\#K}\dim\frac{M(\underline{v}+\underline{e}_K)}{M(\underline{v}+\underline{e}_K+\underline{e}_r)}$. We can now close the proof. ■

We now compare $\sum_{\underline{v}\in\mathbb{Z}^r}\left(\sum_{i=0}^{r-1}(-1)^i h_i^{\underline{v}}\right)\underline{t}^{\underline{v}}$ with *Description 3* and *Description 4* to motivate a new definition for Poincaré series.

Description 3 goes beyond the domain of sets of valuations of which at least one is centered at the maximal ideal, or in general, sets of order functions of which one is finite. Example 1 is illustrating this. Indeed, there one has $h_0^{\underline{v}} = \dim \frac{M(v_1,v_2)}{M(v_1+1,v_2)+M(v_1,v_2+1)}$ and $h_1^{\underline{v}} = 0$ and so again the Poincaré series equals $\sum_{\underline{v}\in\mathbb{Z}^2}(h_0^{\underline{v}} - h_1^{\underline{v}})\underline{t}^{\underline{v}}$.

Description 4 was only defined in a context where all valuations were centered at the maximal ideal (see [C,D,G-Z2]), but in Section 1 we have seen how to extend and compute it in the toric monomial case.

With the following proposition, we get a homological description in Corollary 5 in the case of affine toric varieties and monomial valuations.

Proposition 3. *Suppose that there exist bases $\mathcal{B}_i$ for the vector spaces $V_{\{i\}}$, $1 \le i \le r$, such that $\mathcal{B} = \cup\mathcal{B}_i$ is a set of linearly independent vectors in $V_\emptyset = M(\underline{v})/M(\underline{v}+\underline{1})$. Then the complex $\mathcal{C}_\bullet = (C_i, \partial_i)$ is exact in C_i for $i \ge 1$, i.e. $h_i^{\underline{v}} = 0$ for $1 \le i \le r-1$.*

Proof. We will prove that dim $(\mathrm{Im}\partial_i) = \sum_{m\ge0}(-1)^m \dim C_{i+m}$. The statement can then be deduced from this equation. For $I, J \subset \{1,\dots,r\}$, first notice that $V_I \cap V_J = V_{I\cup J}$ and that $\cap_{i\in I}\mathcal{B}_i$ is a basis of V_I, as $\mathcal{B} = \cup\mathcal{B}_i$ is a set of linearly independent vectors in $M(\underline{v})/M(\underline{v}+\underline{1})$. For $x \in V_{\{1\}\cup K}$, where $K = \{a_1, \cdots, a_k\} \subset \{2, \cdots, r\}$ and $k \ge 1$, one sees easily that $\{\partial_k|_{V_{\{1\}\cup K\setminus\{a_1\}}}(x), \dots, \partial_k|_{V_{\{1\}\cup K\setminus\{a_k\}}}(x)\}$ is a linearly independent set. Let $i \le k$ and $(I_l)_{l\in L} \subset \{1\} \cup K$, with $|I_l| = i$ for $l \in L$. We denote $\mathcal{I} = \cup_{l\in L} I_l$. We have

$$\partial_k|_{V_K}(x) = \sum_{j=1}^{k}(-1)^{j+1}\partial_k|_{V_{\{1\}\cup K\setminus\{a_j\}}}(x) \tag{2}$$

from which it follows that

$$\dim \left(\bigoplus_{l \in L} < \partial_i|_{V_{I_l}}(x) > \right) = \sum_{I_l, 1 \in I_l} 1 = \sum_{I_l, 1 \in I_l} 1 + \left(\sum_{I_l, 1 \notin I_l} 1 - \sum_{\substack{I \subset \mathcal{I}, |I| = i+1 \\ 1 \in I}} 1 \right) + \left(- \sum_{\substack{I \subset \mathcal{I}, |I| = i+1 \\ 1 \notin I}} 1 + \sum_{\substack{I \subset \mathcal{I}, |I| = i+2 \\ 1 \in I}} 1 \right) + \left(\sum_{\substack{I \subset \mathcal{I}, |I| = i+2 \\ 1 \notin I}} 1 - \sum_{\substack{I \subset \mathcal{I}, |I| = i+3 \\ 1 \in I}} 1 \right) + \dots$$

We can now deduce that

$$\dim\ (\mathrm{Im}\partial_i) = \ \dim C_i - \ \dim C_{i+1} + \ \dim C_{i+2} - \cdots \qquad \blacksquare$$

To prove Corollary 5 below we need the following lemma.

Lemma 4. *[Ma, p. 62] Let* (A, m) *be a local Noetherian ring. If* M *is a finite* A*-module and* $N \subset M$ *a submodule, then*

$$\bigcap_{n>0} (N + \mathsf{m}^n M) = N.$$

The following corollary reflects well the nature of affine toric varieties in these homological terms.

Corollary 5. *If* X *is an affine toric variety and* $\nu_1, \dots, \nu_r$ *are monomial valuations, then* $h_i^{\underline{v}} = 0$ *for* $i \geq 1$ *and in particular,* $P_X^{\underline{\nu}}(\underline{t}) = \sum_{\underline{v} \in \mathbb{Z}^r} h_0^{\underline{v}} \underline{t}^{\underline{v}}$ *is well-defined if one of the following two equivalent conditions is satisfied:*

1. *dim* $\frac{M(\underline{v})}{M(\underline{v}+\underline{e}_1)+\cdots+M(\underline{v}+\underline{e}_r)} < \infty$*;*

2. *there exists* $T \in \mathbb{Z}_{>0}$ *such that* $M(\underline{v}) \cap \mathsf{m}^T \subset M(\underline{v} + \underline{e}_1) + \cdots + M(\underline{v} + \underline{e}_r)$*.*

Proof. It follows immediately from Proposition 3 that $h_i^{\underline{v}} = 0$ for $i \geq 1$ and obviously $h_0^{\underline{v}} = \dim \frac{M(\underline{v})}{M(\underline{v}+\underline{e}_1)+\cdots+M(\underline{v}+\underline{e}_r)}$. We now show the equivalence between Condition *1* and *2*. If Condition *2* holds, then

$$\dim \frac{M(\underline{v})}{M(\underline{v} + \underline{e}_1) + \cdots + M(\underline{v} + \underline{e}_r)} \leq \dim \frac{M(\underline{v})}{M(\underline{v}) \cap \mathsf{m}^T} \leq \ \dim \mathcal{O}_{X,o}/\mathsf{m}^T$$

and hence is finite.

Suppose now that Condition *1* holds. Say $\frac{M(\underline{v})}{M(\underline{v}+\underline{e}_1)+\cdots+M(\underline{v}+\underline{e}_r)}$ is m-dimensional with basis $\{\overline{f_1}, \dots, \overline{f_m}\}$. For $T \in \mathbb{Z}_{>0}$, we consider the vector space

$$L_T := \{(\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m \mid \sum_{i=1}^m \lambda_i f_i \in M(\underline{v} + \underline{e}_1) + \cdots + M(\underline{v} + \underline{e}_r) + (M(\underline{v}) \cap \mathsf{m}^T)\}.$$

We have $\mathbb{C}^m \supseteq L_T \supseteq L_{T+1} \supseteq \dots$ and hence this chain stabilizes from some T_0 on. Then for $(\lambda_1, \dots, \lambda_m) \in L_{T_0}$, we have

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i f_i &\in \bigcap_{T \geq T_0} L_T \\
&\subseteq \bigcap_{T \geq T_0} \left(M(\underline{v}+\underline{e}_1) + \cdots + M(\underline{v}+\underline{e}_r) + \mathsf{m}^T \right) \\
&\subseteq \bigcap_{T \geq T_0} \left(M(\underline{v}+\underline{e}_1) + \cdots + M(\underline{v}+\underline{e}_r) \right) \qquad \text{(Lemma 4)}.
\end{aligned}$$

This implies that $L_T = 0$ for $T \geq T_0$. It now follows easily for all $T \geq T_0$ that $M(\underline{v}) \cap \mathsf{m}^T \subset M(\underline{v}+\underline{e}_1) + \cdots + M(\underline{v}+\underline{e}_r)$. ■

Remark 1. If $X = \mathbb{C}^d$ equipped with monomial valuations $\nu_j = (\nu_{j1}, \dots, \nu_{jn})$, $1 \leq j \leq r$, then $P_X^{\underline{\nu}}(\underline{t})$ is well-defined if for all $i \in \{1, \dots, n\}$, there exists some $j \in \{1, \dots, r\}$ such that $\nu_{ji} \neq 0$.

The above observations make it natural to propose the following definition for Poincaré series, which we will denote by $\boldsymbol{P}_X^{\underline{\nu}}(\underline{t})$.

Definition 1. *Let (X, o) be a germ of a complex algebraic variety and let $\underline{\nu} = \{\nu_1, \cdots, \nu_r\}$ be a set of order functions on $\mathcal{O}_{X,o}$. If all $h_i^{\underline{v}}$, defined as above, are finite, then the Poincaré series $\boldsymbol{P}_X^{\underline{\nu}}(\underline{t})$ is defined as*

$$\sum_{\underline{v} \in \mathbb{Z}^r} \left(\sum_{i=0}^{r-1} (-1)^i h_i^{\underline{v}} \right) \underline{t}^{\underline{v}}.$$

3. Poincaré series of embedded filtrations

In [L2], the second author introduced a Poincaré series for an embedded subspace V defined by an ideal $\mathcal{I}$ in $\mathcal{O}_{X,o}$ and order functions $\underline{\nu} = \{\nu_1, \cdots, \nu_r\}$ on $\mathcal{O}_{X,o}$. Keeping the notation from Section 1, this Poincaré series was defined as

$$\mathcal{P}_V^{\underline{\nu}}(t_1, \cdots, t_r) := \frac{\prod_{j=1}^{r}(t_j - 1)}{(t_1 \cdots t_r - 1)} \sum_{\underline{v} \in \mathbb{Z}^r} \dim(M(\underline{v}) + \mathcal{I}/M(\underline{v}+\underline{1}) + \mathcal{I}) \underline{t}^{\underline{v}},$$

if the dimension of $(M(\underline{v})+\mathcal{I}/M(\underline{v}+\underline{1})+\mathcal{I})$ is finite. Similarly as above, this description can be generalized to a homological description.

Let $V_I^{\mathcal{I}} := (M(\underline{v}+\underline{e}_I)+\mathcal{I})/(M(\underline{v}+\underline{1})+\mathcal{I})$ for $I \subseteq \{1, \dots, r\}$. Let

$$C_i^{\mathcal{I}} := \begin{cases} \{0\} & \text{if } i = -1, i > r, \\ \bigoplus_{\substack{I \subset \{1,\dots,r\}, \\ \#I = i}} V_I^{\mathcal{I}} & \text{if } 0 \leq i \leq r. \end{cases}$$

For $-1 \leq i \leq r-1$, we define a map $\partial_{i+1}^{\mathcal{I}} : C_{i+1}^{\mathcal{I}} \to C_i^{\mathcal{I}}$ by defining it on each component $V_I^{\mathcal{I}}$ ($\#I = i+1$). Suppose that $I = \{a_1, a_2, \ldots, a_{i+1}\} \subseteq \{1, \ldots, r\}$, with $a_1 < a_2 < \ldots < a_{i+1}$, then we set

$$\begin{array}{rcl} V_I^{\mathcal{I}} & \longrightarrow & C_i^{\mathcal{I}} = \bigoplus_{\substack{J \subset \{1,\ldots,r\}, \\ \#J = i}} V_J^{\mathcal{I}} \\ x & \longmapsto & (\underline{y})_J, \end{array}$$

where $y_J = 0$ if $J \nsubseteq I$ and $y_J = (-1)^k x$ if $J = I \setminus \{a_k\}$. We get a complex $\mathcal{C}_{\bullet}^{\mathcal{I},\underline{v}} = \mathcal{C}_{\bullet}^{\mathcal{I}} = (C_i^{\mathcal{I}}, \partial_i^{\mathcal{I}})_{i \in \mathbb{Z}_{\geq -1}}$ of vector spaces and we will denote the arising homologies here by $h_i^{\mathcal{I},\underline{v}}$. We give the following definition.

Definition 2. *Let (X,o) be a germ of a complex algebraic variety and let $\underline{\nu} = \{\nu_1, \ldots, \nu_r\}$ be a set of order functions on $\mathcal{O}_{X,o}$. Let $\mathcal{I}$ be an ideal in $\mathcal{O}_{X,o}$ defining the subspace V. If all $h_i^{\mathcal{I},\underline{v}}$ are finite, then the Poincaré series $\boldsymbol{\mathcal{P}}_V^{\underline{\nu}}(\underline{t})$ is defined as*

$$\sum_{\underline{v} \in \mathbb{Z}^r} \left(\sum_{i=0}^{r-1} (-1)^i h_i^{\mathcal{I},\underline{v}} \right) \underline{t}^{\underline{v}}.$$

When the order functions are all valuations and when the ideal $\mathcal{I} = (h)$ is principal, then it was shown in [L2, Theorem 1] that

$$\mathcal{P}_V^{\underline{\nu}}(\underline{t}) = (1 - \underline{t}^{\underline{q}}) P_X^{\underline{\nu}}(\underline{t})$$

where $\underline{q} = \underline{\nu}(h)$. This proof worked whenever at least one valuation was centered in the maximal ideal. We now want to extend this formula for the Poincaré series defined in homological terms, i.e. including sets of valuations where none of them is centered at the maximal ideal.

Theorem 6. *Let (X,o) be irreducible, $\mathcal{I} = (h)$ a principal ideal in $\mathcal{O}_{X,o}$ and V the analytic subspace of (X,o) determined by the ideal $\mathcal{I}$. Let $\underline{\nu} = \{\nu_1, \ldots, \nu_r\}$ be a set of discrete valuations of $\mathbb{C}(X)$ such that $\boldsymbol{P}_X^{\underline{\nu}}(\underline{t})$ and $\boldsymbol{\mathcal{P}}_V^{\underline{\nu}}(\underline{t})$ are well-defined. We assume that $\nu_j(h) < \infty$ for all $j \in \{1, \ldots, r\}$ and we write $\underline{q} = \underline{\nu}(h)$. Then*

$$\boldsymbol{\mathcal{P}}_V^{\underline{\nu}}(\underline{t}) = (1 - \underline{t}^{\underline{q}}) \boldsymbol{P}_X^{\underline{\nu}}(\underline{t}). \tag{3}$$

Proof. Let us study the coefficient of $\underline{t}^{\underline{v}}$ in both members. With the notation of Section 2, the coefficient of $\underline{t}^{\underline{v}}$ in the right hand side member is $\sum_{i=0}^{r-1} (-1)^i h_i^{\underline{v}} - \sum_{i=0}^{r-1} (-1)^i h_i^{\underline{v}-\underline{q}}$. With the notation of Section 3, the coefficient of $\underline{t}^{\underline{v}}$ in the left hand side member can be written as $\sum_{i=0}^{r-1} (-1)^i h_i^{\mathcal{I},\underline{v}}$. Now we will make the link between the three complexes that arise. Notice that

$$0 \to M(\underline{v} - \underline{q}) \xrightarrow{\alpha} M(\underline{v}) \xrightarrow{\pi} M(\underline{v})/(h)M(\underline{v} - \underline{q}) \to 0,$$

is an exact sequence, where α is the multiplication map with h and π is the projection map. As $\nu_1, \ldots, \nu_r$ are valuations, one has $(h)M(\underline{v} - \underline{q}) = (h) \cap M(\underline{v})$, and so

$M(\underline{v})/(h)M(\underline{v}-\underline{q}) \cong M(\underline{v})+(h)/(h)$. This makes that we have a short exact sequence of the complexes

$$0 \to \mathcal{C}_{\bullet}^{\underline{v}-\underline{q}} \to \mathcal{C}_{\bullet}^{\underline{v}} \to \mathcal{C}_{\bullet}^{\mathcal{I},\underline{v}} \to 0.$$

This sequence induces a long exact sequence of the homology groups (see for example [H, p. 203]) from which one then easily deduces that $\sum_{i=0}^{r-1}(-1)^i h_i^{\mathcal{I},\underline{v}} = \sum_{i=0}^{r-1}(-1)^i h_i^{\underline{v}} - \sum_{i=0}^{r-1}(-1)^i h_i^{\underline{v}-\underline{q}}$. ■

Remark 2. *Notice that it follows from the proof of Theorem 6 that the existence of* $\boldsymbol{P}_X^{\underline{\nu}}(\underline{t})$ *implies the existence of* $\boldsymbol{\mathcal{P}}_V^{\underline{\nu}}(\underline{t})$ *for every analytic subspace* V *of* (X, o) *determined by a principal ideal* $\mathcal{I}$.

As this relation between the embedded Poincaré series and the ambient Poincaré series holds in this broader context, one can extend properties such as [L2, Thm. 6] about functions $h \in \mathbb{C}[x_1, \ldots, x_n]$ that are nondegenerate with respect to their Newton polyhedron $\mathcal{N}$. The condition that $\mathcal{N}$ has at least one compact facet in [L2, Thm. 6] (notice that a compact facet induces a valuation centered at the maximal ideal) can now be weakened to the condition that the variable x_i does not divide the polynomial expression of h if the monomial x_i is already contained in the support of h, $1 \le i \le d$. The property becomes then:

Theorem 7. *Suppose that* $h \in \mathbb{C}[x_1, \ldots, x_n]$ *and that no monomial* x_i *is dividing the polynomial expression of* h *for which* x_i *is already contained in the support of* h, $1 \le i \le d$. *Let* $\underline{\nu} = \{\nu_1, \cdots, \nu_r\}$ *be the monomial valuations on* $\mathbb{C}^d$ *induced by the facets of* $\mathcal{N}$. *Then the Poincaré series* $\boldsymbol{\mathcal{P}}_V^{\underline{\nu}}(\underline{t})$ *contains the same information as the Newton polyhedron. In particular, when* h *is nondegenerate with respect to* $\mathcal{N}$, *then the Poincaré series determines the zeta function of monodromy of* h.

Proof. It follows from Remark 1 that $\mathbf{P}_{\mathbb{C}^d}^{\underline{\nu}}(\underline{t})$ is well defined because $\{\nu_1, \cdots, \nu_r\}$ contains the valuations $\nu_1 := (1, 0, \ldots, 0), \ldots, \nu_d := (0, \ldots, 0, 1)$, and hence by Remark 2 that also $\boldsymbol{\mathcal{P}}_V^{\underline{\nu}}(\underline{t})$ exists. With $\underline{q} := \underline{\nu}(h)$, we know from Theorem 6 that

$$\boldsymbol{\mathcal{P}}_V^{\underline{\nu}}(\underline{t}) = (1 - \underline{t}^{\underline{q}})\mathbf{P}_{\mathbb{C}^d}^{\underline{\nu}}(\underline{t}). \tag{4}$$

No factors cancel in Equation (4). Indeed, suppose that $\underline{\nu}$ contains only the valuations $\nu_1, \ldots, \nu_d$, then the form of the Newton polyhedron implies that h is a constant and so $\underline{q} = \underline{0}$ contradicting the cancelation, or that h is a multiple of some monomial m. Then $\underline{\nu}(h) = \underline{\nu}(m)$ but a cancelation would imply that $m = x_i$ for some $i \in \{1, \ldots, d\}$ what contradicts the hypothesis on h. Suppose now that $\underline{\nu}$ contains more valuations, say we have $\nu_{d+1} = (\nu_{d+1,1}, \ldots, \nu_{d+1,d})$ with at least two entries different from 0. This means that $\mathcal{N}$ contains a facet such that the affine space generated by this facet has as equation

$$\nu_{d+1,1}x_1 + \cdots + \nu_{d+1,d}x_d = N,$$

with $N \ge \nu_{d+1,1} + \cdots + \nu_{d+1,d}$. Notice that N is also equal to $\nu_{d+1}(h) = q_{d+1}$. If there would be some cancelation, then $\underline{q} = (\nu_{1,i}, \ldots, \nu_{r,i})$ for some $i \in \{1, \ldots, d\}$. Thus $N = q_{d+1} = \nu_{d+1,i}$ which contradicts the fact that ν_{d+1} contains at least two entries different from 0. ■

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