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# Zeta Functions and Monodromy for Surfaces that are General for a Toric Idealistic Cluster

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In this article, we consider surfaces that are general with respect to a three-dimensional toric idealistic cluster. In particular, this means that blowing up a toric constellation provides an embedded resolution of singularities for these surfaces. First we give a formula for the topological zeta function directly in terms of the cluster. Then we study the eigenvalues of monodromy. In particular, we derive a useful criterion to be an eigenvalue. In a third part, we prove the monodromy and the holomorphy conjecture for these surfaces.

## 1 Introduction

Weil [33] introduced some zeta functions $\mathcal{Z}(K, f)$ that are integrals over a $p$-adic field $K$ and that are associated with a polynomial $f(\underline{x}) \in K[\underline{x}]$. Using an embedded resolution of singularities, Igusa showed that these zeta functions are rational and he studied their poles (see [14] and [15]). One can define the analogous integrals over $K = \mathbb{R}$ or $\mathbb{C}$. Also these zeta functions are rational (see for example [4] and [5]) and it is known that their poles are contained in the set of roots—and roots shifted by a negative integer—of the Bernstein polynomial $b_f$. According to Malgrange [24], if $\alpha$ is a root of $b_f$, then $e^{2\pi i\alpha}$ is an eigenvalue of the local monodromy of $f$ at some point of $f^{-1}(0)$. So when $K = \mathbb{R}$ or

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$\mathbb{C}$, then the poles of the zeta function induce eigenvalues of the local monodromy. This result was a motivation to study this relation at the $p$-adic side. The study of concrete examples made it natural to propose the following conjecture.

### 1.1 Monodromy conjecture [16]

Let $F \subset \mathbb{C}$ be a number field and $f \in F[\underline{x}]$. For almost all $p$-adic completions $K$ of $F$, if $s_0$ is a pole of $\mathcal{Z}(K, f)$, then $e^{2\pi i Re(s_0)}$ is an eigenvalue of the local monodromy of $f$ at some point of the hypersurface $f = 0$.

Loeser verified this conjecture for plane curves (see [22]). He also gave a proof for a class of polynomials in higher dimensions; the polynomial should be nondegenerate with respect to its Newton polyhedron and should satisfy some numerical conditions ([23] and Section 3).

When Denef and Loeser introduced the topological zeta function in 1992 in [11], an analogous version of the monodromy conjecture arose. This monodromy conjecture relates the poles of the topological zeta function $Z_{top,f}$ associated with a polynomial function or a germ of a holomorphic function $f$ with the eigenvalues of monodromy of the hypersurface $f = 0$.

### 1.2 Monodromy conjecture

If $s_0$ is a pole of $Z_{top,f}$, then $e^{2\pi i s_0}$ is an eigenvalue of the local monodromy of $f$ at some point of the hypersurface $f = 0$.

By the original definition of the topological zeta function, it follows that the monodromy conjecture for the Igusa zeta function implies the monodromy conjecture for the topological zeta function. Artal Bartolo, Cassou-Noguès, Luengo, and Melle Hernández proved the monodromy conjecture for some surface singularities, such as the superisolated ones (see [2]), and for quasiordinary polynomials in [3]. The second author provided results in [30–32], and together with Rodrigues in [28]. In [18], the authors consider the same context as in this paper but they had to impose a restricting condition on the surfaces. Through geometrical arguments, they showed that the monodromy conjecture holds for candidate poles of the topological zeta function of order 1 that are poles.

There are more conjectures relating the poles of the topological zeta function (Igusa zeta function) and the eigenvalues of monodromy. There exist the rational functions $Z^{(r)}_{top,f}$ ($r \in \mathbb{Z}_{>0}$) that are variants of the topological zeta function and that play a role in the holomorphy conjecture, as stated by Denef.

### 1.3 **Holomorphy conjecture [9]**

If $r \in \mathbb{Z}_{>0}$ does not divide the order of any eigenvalue of the local monodromy of $f$ at any point of $f^{-1}\{0\}$, then $Z^{(r)}_{top,f}$ is holomorphic on $\mathbb{C}$.

Originally, the holomorphy conjecture was formulated for the Igusa zeta function. We refer to [9] for the inspiration. Denef showed that the conjecture is true for the relative invariants of a few prehomogeneous vector spaces. The second author proved the conjecture for plane curves (see [29]) and together with Rodrigues for homogeneous polynomials (see [27]).

Although the monodromy conjecture and/or holomorphy conjecture has been proven for these kinds of singularities, one did not get a better understanding of the deep reason why the conjectures hold for them. Until now, the attempts are thus restricted to prove the conjecture for classes of singularities.

This article deals with the class of surfaces that are general with respect to a three-dimensional toric idealistic cluster. This implies that we work with surfaces for which there exists an embedded resolution of singularities by blowing up in points that are orbits for the action of the torus, i.e., in a toric constellation. We refer to Section 2 for a recap about clusters and in Section 3 we explain the objects that play the main role in the conjecture. In Section 4, we show how the topological zeta function can be computed directly in terms of the toric cluster for the surfaces that we consider. We use the embedded resolution provided by the blowing up of the constellation. Let $\pi : Z \to \mathbb{C}^3$ be that resolution of such a surface $f = 0$ and let $E_j$, $j \in S$, be the irreducible components obtained by this resolution of which $E_1, \ldots, E_r$ are the exceptional ones. We will denote $E_j^\circ := E_j \setminus (\cup_{i \in S\setminus\{j\}} E_i)$, for $j \in S$. We write $N_j$ and $\nu_j - 1$ for the multiplicities of $E_j$ in the divisor on $Z$ of $f \circ \pi$ and $\pi^*(dx \wedge dy \wedge dz)$, respectively. The numbers $-\nu_j/N_j$, $j \in S$, form a complete list of candidate poles of $Z_{top,f}$.

We compute in particular the Euler characteristic of the spaces $E_j^\circ, 1 \leq j \leq r$, in terms of the cluster. They show up in A'Campo's formula for the eigenvalues of monodromy and they are very relevant for the monodromy conjecture. In Section 5, we analyse these Euler characteristics. Our goal is to determine when these numbers are less than or equal to 0. A geometric argument will show that we can reduce this job to the investigation of a finite number of families of constellations. We complete Section 5 with combinatorial preparations. These make it possible to determine the sign of the Euler characteristics that we are looking for. We carry this out in Section 6. We then prove the following result.

**Theorem.** If $\chi(E_j^\circ) > 0$, then $e^{-2\pi i \frac{\nu_j}{N_j}}$ is an eigenvalue of monodromy of $f$. □

Using this result, we prove in Section 7 the monodromy conjecture for candidate poles of order 1 that are poles, and in Section 8 the monodromy conjecture for candidate poles of order 2 or 3 that are poles. Hence, we obtain.

**Theorem.** Let $f$ be a germ of a polynomial map that is general with respect to a three-dimensional toric idealistic cluster. If $s_0$ is a pole of $Z_{top,f}$, then $e^{2\pi i s_0}$ is an eigenvalue of monodromy of $f$ at some point of the hypersurface $f = 0$. □

In Section 9, we prove the holomorphy conjecture for these surfaces.

**Theorem.** Let $f$ be a germ of a polynomial map that is general with respect to a three-dimensional toric idealistic cluster. If $r \in \mathbb{Z}_{>0}$ does not divide the order of any eigenvalue of the local monodromy of $f$ at any point of $f = 0$, then $Z^{(r)}_{top,f}$ is holomorphic on $\mathbb{C}$. □

## 2 Toric Clusters

In this section, we introduce the terminology of infinitely near points, (toric) clusters etc. according to [6]. We would like to refer to [6] for some historical notes on clusters. See also [7, 8, 13, 19–21], and [34] for more details on the theory of clusters.

### 2.1 Clusters

Let $X$ be a nonsingular variety of dimension $d \geq 2$ and let $Z$ be a variety obtained from $X$ by a finite succession of point blowing-ups. A point $Q \in Z$ is said to be *infinitely near* to a point $P \in X$ if $P$ is in the image of $Q$; we write $Q \geq P$. A *constellation* is a finite sequence $\mathcal{C} := \{Q_1, Q_2, \dots, Q_r\}$ of infinitely near points of $X$ with $Q_1 \in X =: X_0$ and each $Q_{j+1}$ is a point on the variety $X_j$ obtained by blowing up $Q_j$ in $X_{j-1}$, $j \in \{1, \dots, r-1\}$. The variety $X(\mathcal{C}) := X_r$ obtained by blowing up $Q_r$ in $X_{r-1}$ is called the *sky*. The relation '$\geq$' gives rise to a partial ordering on the points of a constellation. In the case that they are totally ordered, so $Q_r \geq \cdots \geq Q_1$, the constellation $\mathcal{C}$ is called a *chain*. For every $Q_j$ in $\mathcal{C}$, the subsequence $\mathcal{C}^j := \{Q_i \mid Q_j \geq Q_i\}$ of $\mathcal{C}$ is a chain. The integer $l(Q_j) := \#\mathcal{C}^j - 1$ is called the *level* of $Q_j$. In particular $Q_1$ has level 0. If no other point of $\mathcal{C}$ has level 0, then $Q_1$ is called the *origin* of

$\mathcal{C}$. We will always work with constellations that have an origin and we will also denote the origin of the constellation by $o$. If $Q_j \geq Q_i$ and $l(Q_j) = l(Q_i) + 1$, we will write $Q_j \succ Q_i$ or $j \succ i$. For each $Q_i \in \mathcal{C}$, denote the exceptional divisor of the blowing up in $Q_i$ by $E_i$, as well as its strict transform at some intermediate stage (including the final stage) $X_j$, $i \leq j \leq r$. The total transform at some intermediate stage (including the final stage) will be denoted by $E_i^*$. If $Q_j \in E_i$, then one says that $Q_j$ is *proximate* to $Q_i$. This will be denoted as $Q_j \to Q_i$ or $j \to i$. As $E_i = E_i^* - \sum_{j \to i} E_j^*$, it follows that also $\{E_1^*, \dots, E_r^*\}$ is a basis of the group of divisors with exceptional support $\oplus_{j=1}^r \mathbb{Z} E_j$. A pair $\mathcal{A} := (\mathcal{C}, \underline{m})$ consisting of a constellation $\mathcal{C} := \{Q_1, \dots, Q_r\}$ and a sequence $\underline{m} := (m_1, \dots, m_r)$ of nonnegative integers is called a *cluster*. One calls $m_j$ the *weight* or *multiplicity* of $Q_j$ in the cluster and we write $D(\mathcal{A}) := \sum_{j=1}^r m_j E_j^*$. Introducing the numbers $v_j$, $1 \leq j \leq r$, by setting $m_j := v_j - \sum_{j \to i} v_i$, allows us to write also $D(\mathcal{A}) = \sum_{j=1}^r v_j E_j$.

The idea of clusters is to express that a system of hypersurfaces is passing through the points of the constellation with (at least) the given multiplicities. Blowing up a point $Q_i \in \mathcal{C}$ induces a discrete valuation $\nu_i$ on $\mathbb{C}(X) \setminus \{0\}$: for $g \in \mathbb{C}(X) \setminus \{0\}$, the value $\nu_i(g)$ is the order of the pullback of $g$ (at the stage $X_i$) along $E_i$. To a cluster we can then associate the (complete) ideal

$$I(v_1, \dots, v_r) = \{g \in \mathcal{O}_{X,o} \mid \nu_j(g) \geq v_j, 1 \leq j \leq r\} \cup \{0\}.$$

If we want that these ideals principalise by blowing up the points of the constellation, we require the ideals to be finitely supported. Formally, an ideal $I$ in $\mathcal{O}_{X,o}$ is called *finitely supported* if $I$ is primary for the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{X,o}$ —so supported at the closed point—and if there exists a constellation $\mathcal{C}$ of infinitely near points of $X$ such that $I\mathcal{O}_{X(\mathcal{C})}$ is an invertible sheaf.

However, given a finitely supported ideal $I$, one can associate a cluster to it. Let $\mathcal{C}_I =: \{Q_1, \dots, Q_r\}$ be the constellation of base points of $I$, i.e., the minimal constellation $\mathcal{C}$ such that $I\mathcal{O}_{X(\mathcal{C})}$ is an invertible sheaf. Let $m_j$ be the order of the point $Q_j$, $1 \leq j \leq r$ in the strict transform of the ideal $I$ in $\mathcal{O}_{X_j, Q_j}$. Then the ideal sheaf $I\mathcal{O}_{X(\mathcal{C}_I)}$ is associated with $-D(\mathcal{A}_I) := -\sum_{j=1}^r m_j E_j^*$.

If $\mathcal{C}$ is a constellation with origin at $Q_1$, the cluster $\mathcal{A} := (\mathcal{C}, \underline{m})$ is called *idealistic* if there exists a finitely supported ideal $I$ in $\mathcal{O}_{X,Q_1}$ such that $I\mathcal{O}_{X(\mathcal{C})}$ is the ideal sheaf associated with $-D(\mathcal{A})$. For an idealistic cluster $\mathcal{A}$, Lipman proved that there exists a unique finitely supported complete ideal $I_{\mathcal{A}}$ such that $I_{\mathcal{A}}\mathcal{O}_{X(\mathcal{C})} = \mathcal{O}_{X(\mathcal{C})}(-D(\mathcal{A}))$, namely that given by the direct image of $\mathcal{O}_{X(\mathcal{C})}(-D(\mathcal{A}))$ in $X$, see [20].

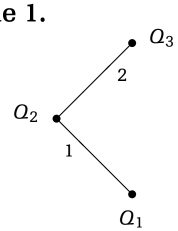
In the next subsection we will illustrate these notions in the context of our results.

### 2.2 Toric clusters in $\mathbb{C}^3$

From now on suppose that $X$ is the affine toric variety $\mathbb{C}^3$. Let $Q_1$ be the origin of $\mathbb{C}^3 = X_0$. A three-dimensional *toric constellation* of infinitely near points with origin $Q_1$ is a constellation $\mathcal{C} := \{Q_1, Q_2, \ldots, Q_r\}$ such that each $Q_{j+1}$ is a zero-dimensional orbit in the toric variety $X_j$ obtained by blowing up $Q_j$ in $X_{j-1}$, $1 \leq j \leq r-1$. Blowing up in orbits of smooth varieties corresponds to making star subdivisions of the fan corresponding to the variety (see for example [26]). In this way each blowing up in a zero-dimensional orbit induces the creation of three cones of dimension 3 and thus of three new zero-dimensional orbits. Hence, the choice of a point $Q_i$ in a toric chain is equivalent to the choice of an integer $a_i \in \{1, 2, 3\}$, which determines a three-dimensional cone in the fan. A tree with a root such that each vertex has at most three following adjacent vertices is called a 3-*nary tree*. The above observation shows that there is a natural bijection between the set of three-dimensional toric constellations with origin and the set of finite 3-nary trees with a root, with the edges labeled with positive integers not greater than 3, such that two edges with the same source have different labels.

A cluster $\mathcal{A} := (\mathcal{C}, \underline{m})$ is called *toric* if the constellation $\mathcal{C}$ is toric. The blowing-ups now induce monomial valuations (i.e., valuations determined by their values on monomials) and the ideal $I(v_1, \ldots, v_r)$ associated with a toric cluster is thus monomial.

**Example 1.**



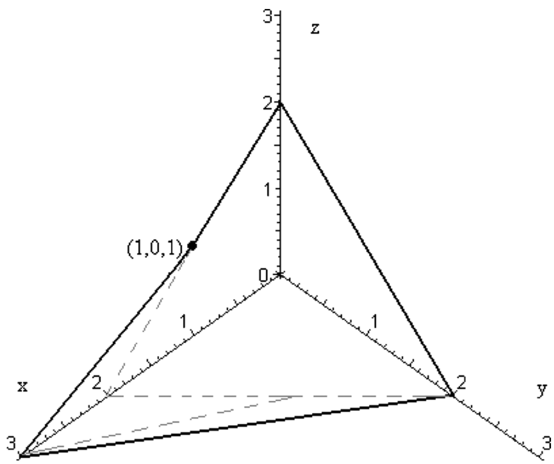

Suppose $d = 3$ and $\mathcal{C}$ is the constellation pictured at the left. It represents the following resolution process: by blowing up in the origin $Q_1$ we get an exceptional variety $E_1 \cong \mathbb{P}^2$. In $E_1$ there is one point in which we blow up, namely $Q_2$. The labels indicate in which affine chart the points of the constellation are created. $\square$

We call the affine chart with label 1 that one in which the equation of $E_1$ is $x = 0$. In the chart with label 2 one has $E_1 \leftrightarrow y = 0$ and in the chart with label 3 one has $E_1 \leftrightarrow z = 0$. The point $Q_2$ is the origin of the affine chart with label 1. After blowing up in $Q_2$ we get an exceptional variety $E_2 \cong \mathbb{P}^2$, where we blow up in the point $Q_3$ that is the origin of chart 1.2. There one has $E_2 \leftrightarrow y = 0$ and (the transform of) $E_1 \leftrightarrow x = 0$.

We now point out how the induced valuations $\nu_1, \nu_2$, and $\nu_3$ act. For $a, b, c \in \mathbb{Z}_{\geq 0}$, $\nu_1(x^a y^b z^c) = a + b + c$ because the pullback of $x^a y^b z^c$ in chart 1 is $x^{a+b+c} y^b z^c$. The pullback in chart 1.2 becomes $x^{a+b+c}y^{a+2b+2c}z^c$ and thus $\nu_2(x^a y^b z^c) = a + 2b + 2c$. Analogously, we find $\nu_3(x^a y^b z^c) = 2a + 3b + 4c$. We can represent these valuations by the following vectors in the lattice $\mathbb{N}^3$:

$$\nu_1 \leftrightarrow (1,1,1) \quad \nu_2 \leftrightarrow (1,2,2) \quad \nu_3 \leftrightarrow (2,3,4).$$

We have $I(v_1, v_2, v_3) = (x^a y^b z^c \mid a + b + c \geq v_1, a + 2b + 2c \geq v_2, 2a + 3b + 4c \geq v_3)$. To compute such an ideal, one can picture the hyperplanes induced by the valuations. We compute this ideal for the cluster $\mathcal{A} := (\mathcal{C}, \underline{m})$ with $(m_1, m_2, m_3) = (2,1,1)$. Then $(v_1, v_2, v_3) = (2,3,6)$ and one can see from

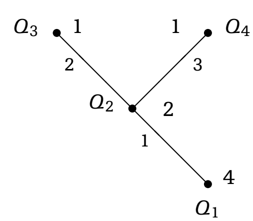



that $I(2,3,6) = (x^3, y^2, z^2, xz, x^2y, yz)$. One can verify that this ideal is finitely supported and that the cluster associated with this ideal is exactly the cluster $\mathcal{A}$. Hence this cluster is idealistic.

Let us now consider the cluster $\mathcal{B}$ consisting of the above constellation and for which $(m_1, m_2, m_3) = (4,1,2)$ or $(v_1, v_2, v_3) = (4,5,11)$. Analogously, one finds that $I := I(4,5,11)$ is equal to

$$(x^6, y^4, z^4, xy^3, x^3y^2, x^4y, x^3z^2, x^4z, xz^3, yz^3, y^2z^2, y^3z, xyz^2, xy^2z, x^2yz).$$

One finds that this ideal is finitely supported but the cluster associated to this ideal is



From this we can deduce that $\mathcal{B}$ is not an idealistic cluster. Indeed, if $\mathcal{B}$ was idealistic, then there would exist a finitely supported complete ideal $J$ such that $J\mathcal{O}_{X(\mathcal{C})} = \mathcal{O}_{X(\mathcal{C})}(-D(\mathcal{B}))$ and we also know that $J$ would be included in $I$. As $I$ and $J$ are both complete ideals, they should be equal but we mentioned already that $I\mathcal{O}_{X(\mathcal{C})} \neq \mathcal{O}_{X(\mathcal{C})}(-D(\mathcal{B}))$.

Finally, let us consider the cluster consisting of the constellation $\mathcal{C}$ with $(m_1, m_2, m_3) = (2, 2, 2)$ or $(v_1, v_2, v_3) = (2, 4, 8)$. One can check that

$$I(2, 4, 8) = (x^4, y^3, z^2, xy^2, x^3y, x^2z, y^2z, xyz)$$

and that this ideal is not even finitely supported. In the same way we can conclude that this cluster is not idealistic. □

### 2.3 Properties

In this subsection, we recall some properties of clusters, in particular of toric clusters. (1) In the case of toric clusters, there exists a combinatorial characterisation for the idealistic clusters. Fix a point $Q_i$ in a toric three-dimensional constellation $\mathcal{C}$ and some integers $a, b$ such that $a, b \in \{1, 2, 3\}$ and $a \neq b$. For $s, t \in \mathbb{Z}_{\geq 0}$, let $Q_i(a^s, b^t)$ be the terminal point of the chain with origin $Q_i$ coded by $(a, \ldots, a, b, \ldots, b)$ where $a$ appears $s$ times and $b$ appears $t$ times. If $t = 0$, it is denoted by $Q_i(a^s)$. The point $Q_i(a^s, b^t)$ may not belong to $\mathcal{C}$. A point $Q_j \in \mathcal{C}$ that is infinitely near to $Q_i$ is said to be *linearly proximate* to $Q_i$, if $Q_j = Q_i(a, b^t)$, with $a, b$ and $t$ as above. We denote this relation by $Q_j \twoheadrightarrow Q_i$ or $j \twoheadrightarrow i$. Then we have that $Q_j$ is linearly proximate to $Q_i$ if and only if there exists a one-dimensional orbit $\mathrm{l}$ in $B_i$ such that $Q_j$ belongs to the strict transform of the closure of $\mathrm{l}$ in $E_i$. This explains the terminology. Denote $M_{Q_i}(a, b) := \sum_{t \geq 0} m_{Q_i(a, b^t)}$. Campillo, Gonzalez-Sprinberg, and Lejeune-Jalabert show the following:

1. A toric cluster $\mathcal{A} = (\mathcal{C}, \underline{m})$ is idealistic if and only if for each point $Q_i$ of the constellation $\mathcal{C}$ and for each pair of integers $a$ and $b$ such that $a, b \in \{1, 2, 3\}$ and $a \neq b$, the following inequality is satisfied:

$$M_{Q_i}(a, b) + M_{Q_i}(b, a) \leq m_{Q_i}.$$

These inequalities are called the *linear proximity inequalities*.

2. Let $\mathcal{A} = (\mathcal{C}, \underline{m})$ be a three-dimensional toric idealistic cluster with associated divisor $D(\mathcal{A}) = \sum_{j=1}^{r} m_j E_j^* = \sum_{j=1}^{r} v_j E_j$ and let $\nu_1, \ldots, \nu_r$ be the induced discrete valuations. Such a valuation is called Rees for the ideal $I(\underline{v}) := I(v_1, \ldots, v_r)$ if it is a valuation induced by an irreducible component of the

exceptional divisor of the normalised blowing up $\overline{Bl_{I(\underline{v})}X}$ of $I(\underline{v})$. Then

$$\forall Q_i \in \mathcal{C} : m_i^2 \geq \sum_{j \to i} m_j^2 \tag{1}$$

and

$$\nu_i \text{ is Rees for } I(\underline{v}) \text{ if and only if } m_i^2 > \sum_{j \to i} m_j^2. \tag{2}$$

(2) To a monomial ideal $I$ one can associate a Newton polyhedron $\mathcal{N}_I$. It is the convex hull of $m + \mathbb{R}^3_{\geq 0}$ as $m$ runs through the set of exponents of monomials in $I$. We refer to [17] for the proofs of the following properties:

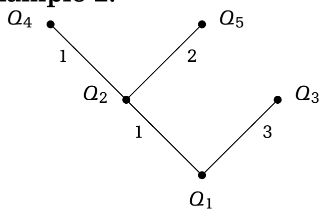
1. The compact facets of $\mathcal{N}_I$ correspond with the Rees valuations of $I$.
2. A monomial ideal is complete if and only if it contains every monomial whose exponent is a point of $\mathcal{N}_I \cap \mathbb{Z}^3_{\geq 0}$.

(3) Campillo, Gonzalez-Sprinberg, and Lejeune-Jalabert generate a very interesting set of 'general' hypersurfaces in [6].

**Theorem 1.** The canonical map from the sky of the constellation of base points of a finitely supported ideal $I$ to $X$ is an embedded resolution of the subvariety of $(X, o)$ defined by a general enough element in $I$. □

We will call these 'general enough' elements *general for* $I$ or *general for* $\mathcal{C}_I$. We will prove the monodromy and holomorphy conjectures for the class of surfaces that are general for a finitely supported monomial ideal. In particular, this means that our results apply to all surfaces for which there exists an embedded resolution consisting of toric point blowing-ups and for which the corresponding toric cluster is idealistic. According to Theorem 1, such surfaces in a finitely supported ideal form an open dense set.

**Example 2.**




Suppose $d = 3$ and $\mathcal{C}$ is the constellation pictured at the left. It represents the following resolution process: by blowing up in the origin $Q_1$ we get an exceptional variety $E_1 \cong \mathbb{P}^2$. In $E_1$ there are two points in which we blow up, namely $Q_2$ and $Q_3$. After blowing up in $Q_2$ we get an exceptional variety $E_2 \cong \mathbb{P}^2$, where we again blow up in two points.

The induced valuations are represented by the following vectors in the lattice $\mathbb{N}^3$:

$$\nu_1 \leftrightarrow (1,1,1) \quad \nu_2 \leftrightarrow (1,2,2) \quad \nu_3 \leftrightarrow (2,2,1) \quad \nu_4 \leftrightarrow (1,3,3) \quad \nu_5 \leftrightarrow (2,3,4).$$

Consider the multiplicities $(m_1, m_2, m_3, m_4, m_5) = (3,2,1,1,1)$ for the points of this constellation, equivalently $(v_1, v_2, v_3, v_4, v_5) = (3,5,4,6,9)$. By the linear proximity inequalities it follows that the cluster $\mathcal{A} := (\mathcal{C}, \underline{m})$ is idealistic. Now let $I_{\mathcal{A}}$ be the ideal generated by the monomials whose exponents are in the associated Newton polyhedron. We find

$$I_{\mathcal{A}} = (x^6, y^3, z^4, x^3y, x^2y^2, yz^2, y^2z, x^3z, xz^2, xyz).$$

The blowing up of the constellation gives an embedded resolution for a general element of $I_{\mathcal{A}}$, such as $h(x,y,z) := x^6 + y^3 + z^4 + x^3y + x^2y^2 + yz^2 + y^2z + x^3z + xz^2 - xyz$. □

**Example 3.** Let us consider the noncomplete ideal $I = (x^3, y^2, z^2, xz, x^2y)$. This ideal is finitely supported and the associated cluster is the idealistic cluster $\mathcal{A}$ from Example 1. Theorem 1 says then that the blowing up of that constellation gives an embedded resolution for a general element of $I$, such as $x^3 + y^2 + z^2 + xz + x^2y$. □

(4) We now first recall the notion for a polynomial to be *nondegenerate with respect to its Newton polyhedron*. Let $f \in \mathbb{C}[x_1, \ldots, x_d]$ be a nonconstant polynomial vanishing in the origin. Write $\underline{x}^{\underline{k}} := x_1^{k_1} \cdots x_d^{k_d}$ and $f := \sum_{\underline{k} \in \mathbb{N}^d} c_{\underline{k}} \underline{x}^{\underline{k}}$. The *support of* $f$ is $\operatorname{supp}(f) := \{\underline{k} \in \mathbb{N}^d \mid c_{\underline{k}} \neq 0\}$. The *Newton polyhedron* $\Gamma$ of $f$ is the convex hull of $\operatorname{supp}(f) + \mathbb{R}^d_{\geq 0}$. For a face $\tau$ of $\Gamma$ we write $f_\tau := \sum_{\underline{k} \in \tau} c_{\underline{k}} \underline{x}^{\underline{k}}$. A polynomial $f$ is called *nondegenerate with respect to* $\Gamma$ if for every compact face $\tau$ of $\Gamma$, the polynomials $f_\tau$ and $\partial f_\tau / \partial x_i$ have no common zeroes in $(\mathbb{C}^*)^d$, $1 \leq i \leq d$.

**Proposition 2.** Every hypersurface that is general with respect to some three-dimensional toric idealistic cluster is nondegenerate with respect to its Newton polyhedron.

□

**Proof.** Let $\mathcal{A} = (\mathcal{C}, \underline{m})$ be a toric idealistic cluster such that $f$ is general with respect to $\mathcal{A}$. Suppose that $f$ is degenerate with respect to $\mathcal{N}(f)$.

Let $\tau$ be a compact face of $\mathcal{N}(f)$ for which there exists a point $p \in (\mathbb{C}^*)^3$ such that $f_\tau(p) = \partial f_\tau / \partial x(p) = \partial f_\tau / \partial y(p) = \partial f_\tau / \partial z(p) = 0$.

If $\tau$ is a facet, then $\tau$ corresponds to some exceptional irreducible component created by the blowing up of the constellation, say to $E_i$. More specifically, the strict transform of $f_\tau$ is equal to $E_0 \cap E_i$. As $p$ is not an orbit, it follows that there exists a point in which $E_0 \cap E_i$ does not have normal crossings and that is not an orbit. If the dimension of $\tau$ is one and if $\tau$ is the intersection of two compact facets, then analogously we have that there exist two irreducible exceptional components $E_i$ and $E_j$ such that $E_0 \cap E_i \cap E_j$ does not have normal crossings in a point that is not an orbit. Remains the case that $\tau$ is the intersection of a compact facet and a coordinate plane. Suppose that compact facet corresponds to $E_i$ and that the coordinate plane is given by $\{x = 0\}$. Again we get that then $E_0 \cap E_i$ does not have normal crossings in a point that is not an orbit. Indeed, if $E_i$ has equation $y = 0$ in some affine chart, then there is a point $(0, 0, p_z)$ with $p_z \neq 0$ in which there are no normal crossings. ■

## 3 Conjectures

Let $f$ be a complex polynomial in $d$ variables and let $\pi : Z \to \mathbb{C}^d$ be an embedded resolution of singularities of $f^{-1}\{0\}$. We write $E_j$, $j \in S$, for the irreducible components of $\pi^{-1}(f^{-1}\{0\})$ and we denote by $N_j$ and by $\nu_j - 1$ the multiplicities of $E_j$ in the divisor on $Z$ of $f \circ \pi$ and $\pi^*(dx_1 \wedge \ldots \wedge dx_d)$, respectively. The couples $(\nu_j, N_j)$, $j \in S$, are called the numerical data of the embedded resolution $(Z, \pi)$. We denote also $E_j^\circ := E_j \setminus (\cup_{i \in S \setminus \{j\}} E_i)$, for $j \in S$. Let the $E_j$, $j \in J := \{1, \ldots, r\} \subset S$, be the exceptional irreducible components of $\pi^{-1}(\{0\})$.

### 3.1 Monodromy

We assume that $f(b) = 0$. Take $\epsilon > 0$ small enough such that the open ball $B_\epsilon$ with radius $\epsilon$ around $b$ in $\mathbb{C}^d$ intersects the fiber $f^{-1}(0)$ transversally. Then choose $\epsilon \gg \eta > 0$ such that for $t$ in the disc $D_\eta \subset \mathbb{C}$ around the origin, the fiber $f^{-1}(t)$ intersects $B_\epsilon$ transversally. Write $X := f^{-1}(D_\eta) \cap B_\epsilon$, $X_t := f^{-1}(t) \cap B_\epsilon$ for $t \in D_\eta$ and $D_\eta^* := D_\eta \setminus \{0\}$ for the pointed disc. Milnor showed that $f_{|X \setminus X_0} : X \setminus X_0 \to D_\eta^*$ is a locally trivial fibration, see [25]. A fiber $X_t$ of this bundle is called *Milnor fiber of $f$ at $b$*. We will denote it by $F_b$. Consider the loop $\gamma$ encircling the origin once counterclockwise. Since $f_{|X \setminus X_0}$ is a locally trivial fibration, the loop $\gamma$ lifts to a diffeomorphism $h$ of the Milnor fiber $F_b$, which is well determined up to homotopy. In this way $\gamma$ induces an automorphism $h^* : H^i(F_b, \mathbb{C}) \to H^i(F_b, \mathbb{C})$, $i \geq 0$, that is called the *monodromy transformation*.

The surfaces for which we will prove the monodromy conjecture have exactly one isolated singularity in the origin. A result of Milnor (see [25]) then says that $H^i(F_0, \mathbb{C}) = 0$, for $i \neq 0$ and $i \neq d - 1$, and $H^0(F_0, \mathbb{C}) = \mathbb{C}$ with trivial monodromy action. The formula of A'Campo ([1]) describes the characteristic polynomial of the monodromy action on $H^{d-1}(F_0, \mathbb{C})$ in terms of an embedded resolution of the hypersurface $f^{-1}(0)$.

We may suppose that $\pi$ is an isomorphism outside the inverse image of the origin.

**Theorem 3** (A'Campo) [1]. The characteristic polynomial of the monodromy action on $H^{d-1}(F_0, \mathbb{C})$ is equal to

$$\left[\frac{\prod_{j=1}^{r}(1-t^{N_j})^{\chi(E_j^\circ)}}{1-t}\right]^{(-1)^{d-1}}.$$ □

### 3.2 Topological zeta function

In 1992, Denef and Loeser created a new zeta function which they called the topological zeta function because of the topological Euler–Poincaré characteristic $\chi(\cdot)$ turning up in it. It is associated with a complex polynomial $f$ with $f(0) = 0$. If $E_I := \cap_{i \in I} E_i$ and $E_I^\circ := E_I \setminus (\cup_{j \notin I} E_j)$, then they introduced it in [11] in the following way.

**Definition 4.** The local *topological zeta function associated with* $f$ is the rational function in one complex variable

$$Z_{top,f}(s) := \sum_{I \subset S} \chi\left(E_I^\circ \cap \pi^{-1}\{0\}\right) \prod_{i \in I} \frac{1}{N_i s + \nu_i}.$$ □

Denef and Loeser proved that every embedded resolution gives rise to the same function, so the topological zeta function is a well-defined singularity invariant (see [11]). Once the motivic Igusa zeta function was introduced, they proved this result alternatively in [12] by showing that this more general zeta function specialises to the topological one. There exists a global version, replacing $E_I^\circ \cap \pi^{-1}\{0\}$ by $E_I^\circ$.

### 3.3 Monodromy conjecture

One calls $\alpha$ an *eigenvalue of monodromy of* $f$ *at* $b \in f^{-1}\{0\}$ if $\alpha$ is an eigenvalue for some $h^* : H^i(F_b, \mathbb{C}) \to H^i(F_b, \mathbb{C})$.

**Conjecture 5.** (Monodromy conjecture) If $s_0$ is a pole of $Z_{top,f}$, then $e^{2\pi i s_0}$ is an eigenvalue of monodromy of $f$ at some point of the germ at 0 of the hypersurface $f = 0$. □

Let $f$ be a polynomial that is general with respect to a three-dimensional toric idealistic cluster. Consider the embedded resolution $\pi : Z \to \mathbb{C}^3$ of $f^{-1}\{0\}$ that corresponds to the blowing up of the constellation. We fix a candidate pole $s_0 = -\nu_j/N_j$ of $Z_{top,f}$. If $E_j$ is not an exceptional component, then $\nu_1 = 1$ and $N_1 = 1$. As 1 is always an eigenvalue of the local monodromy of $f$, this candidate pole does not pose any difficulty. If $s_0 = -\nu_j/N_j$ is a candidate pole of $Z_{top,f}$ induced by an exceptional component $E_j$, then we write $\nu_j/N_j$ as $a/b$ such that $a$ and $b$ are coprime. We define the set $J_b := \{j \in J \mid b \text{ divides } N_j\}$. It follows from A'Campo's formula that

$$e^{2\pi i s_0} \text{ is an eigenvalue of monodromy of } f \text{ at the origin } 0$$
$$\Updownarrow$$
$$\sum_{j\in J_b} \chi(E_j^\circ) \neq 0.$$

In general, there can be a lot of cancelations which make that $\sum_{j\in J_b} \chi(E_j^\circ) = 0$. To control this, we will1 determine when $\chi(E_j^\circ)$ is positive, negative or zero. We will see that the cases where $\chi(E_j^\circ) \leq 0$ are very rare in this context.

### 3.4 Holomorphy conjecture

For every $r \in \mathbb{Z}_{>0}$, one can define a variant $Z^{(r)}_{top,f}$ of the topogical zeta function that is also a rational function in one complex variable.

**Definition 6.**

$$Z^{(r)}_{top,f} := \sum_{\substack{I\subset S\\ \forall i\in I: r|N_i}} \chi\left(E_I^\circ \cap \pi^{-1}\{0\}\right) \prod_{i\in I} \frac{1}{N_i s + \nu_i}. \qquad \square$$

The functions $Z^{(r)}_{top,f}$ are limits of more general Igusa zeta function associated with a polynomial and a character, see [9]. In particular $Z^{(1)}_{top,f} = Z_{top,f}$. Clearly, they are holomorphic on $\mathbb{C}$ if and only if they do not have a pole. The holomorphy conjecture stated by Denef predicts the following relation.

**Conjecture 7.** (Holomorphy conjecture) If $r \in \mathbb{Z}_{>0}$ does not divide the order of any eigenvalue of monodromy of $f$, then $Z^{(r)}_{top,f}$ is holomorphic on $\mathbb{C}$. $\square$

In Section 9, we provide a proof of the holomorphy conjecture for the surfaces we are studying. Again, the classification of $\chi(E_j^\circ)$ according to the sign will be the key to solve the conjecture.

## 4 Computation of the Topological Zeta Function

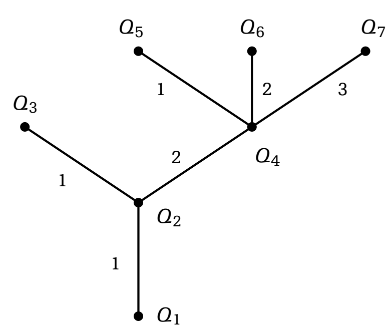
Given a germ of a polynomial function $f$ in $d$ variables over $\mathbb{C}$, its topological zeta function $Z_{top,f}$ can be calculated by computing an embedded resolution. If $f$ is nondegenerate with respect to its Newton polyhedron, then there exists also the formula for $Z_{top,f}$ in terms of its Newton polyhedron, see [11]. In our context, we show that, directly from the tree that represents the toric constellation, one can read all information needed to write down the topological zeta function.

Concretely, we consider a toric idealistic cluster in $\mathbb{C}^3$ and a complex polynomial $f$ in three variables in a finitely supported ideal such that the cluster gives an embedded resolution for the surface $S := V(f) \subset \mathbb{C}^3$. To determine the topological zeta function of $f$, we determine the numbers $\chi(E_I^\circ)$. We will denote the strict transform of $S$ by $\hat{S}$, whatever the stage is, and we will denote the curves $\hat{S} \cap E_i$ by $C_i$. We will write $p_a$ for the geometric genus.

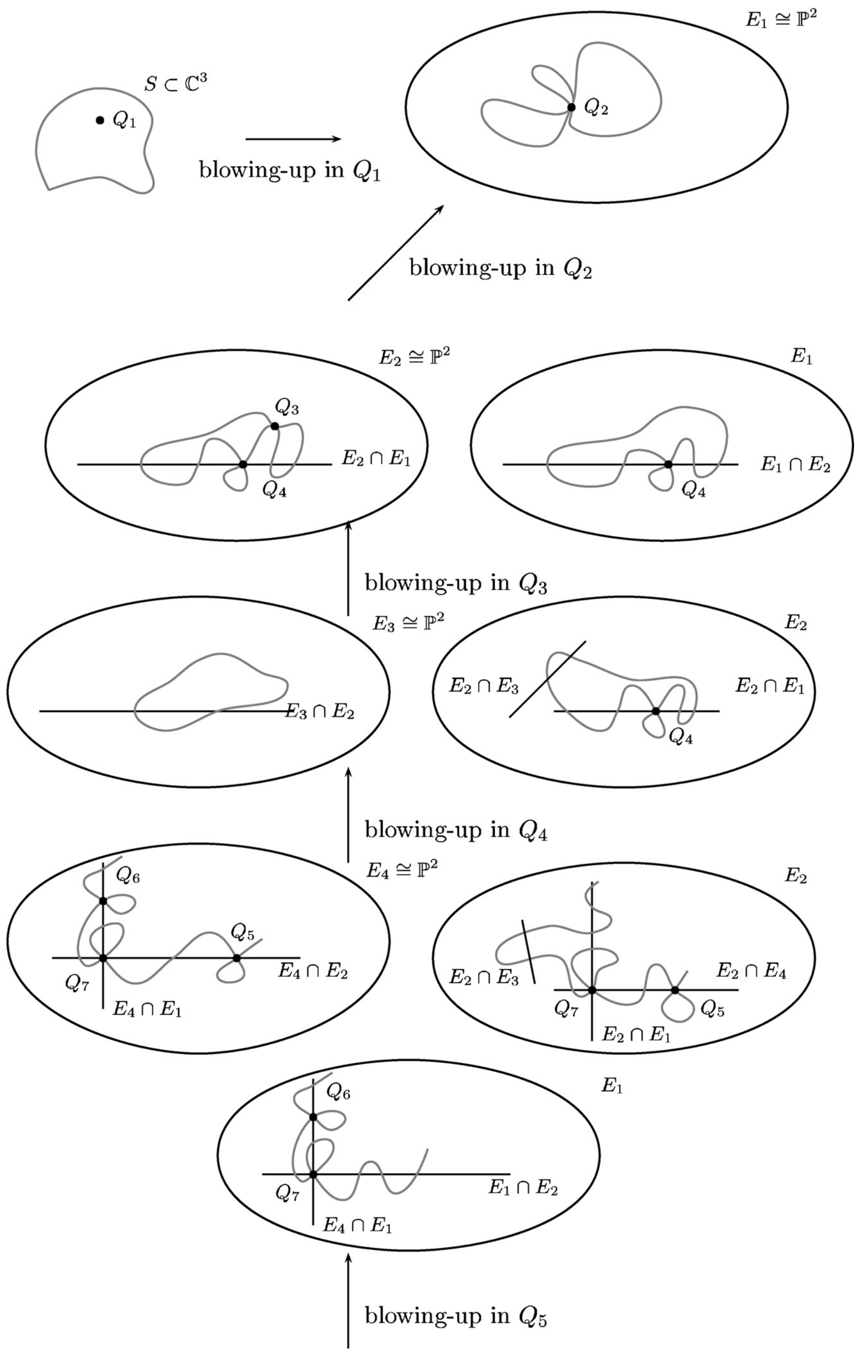
First of all, notice that when blowing up in a point of multiplicity $m$ on $S$, and $E$ being the created exceptional divisor, the curve $\hat{S} \cap E$ has degree $m$. Another important observation is that if $Q \in E$, then the multiplicity of $Q$ on $\hat{S} \cap E$ is equal to the multiplicity of $Q$ on $\hat{S}$.

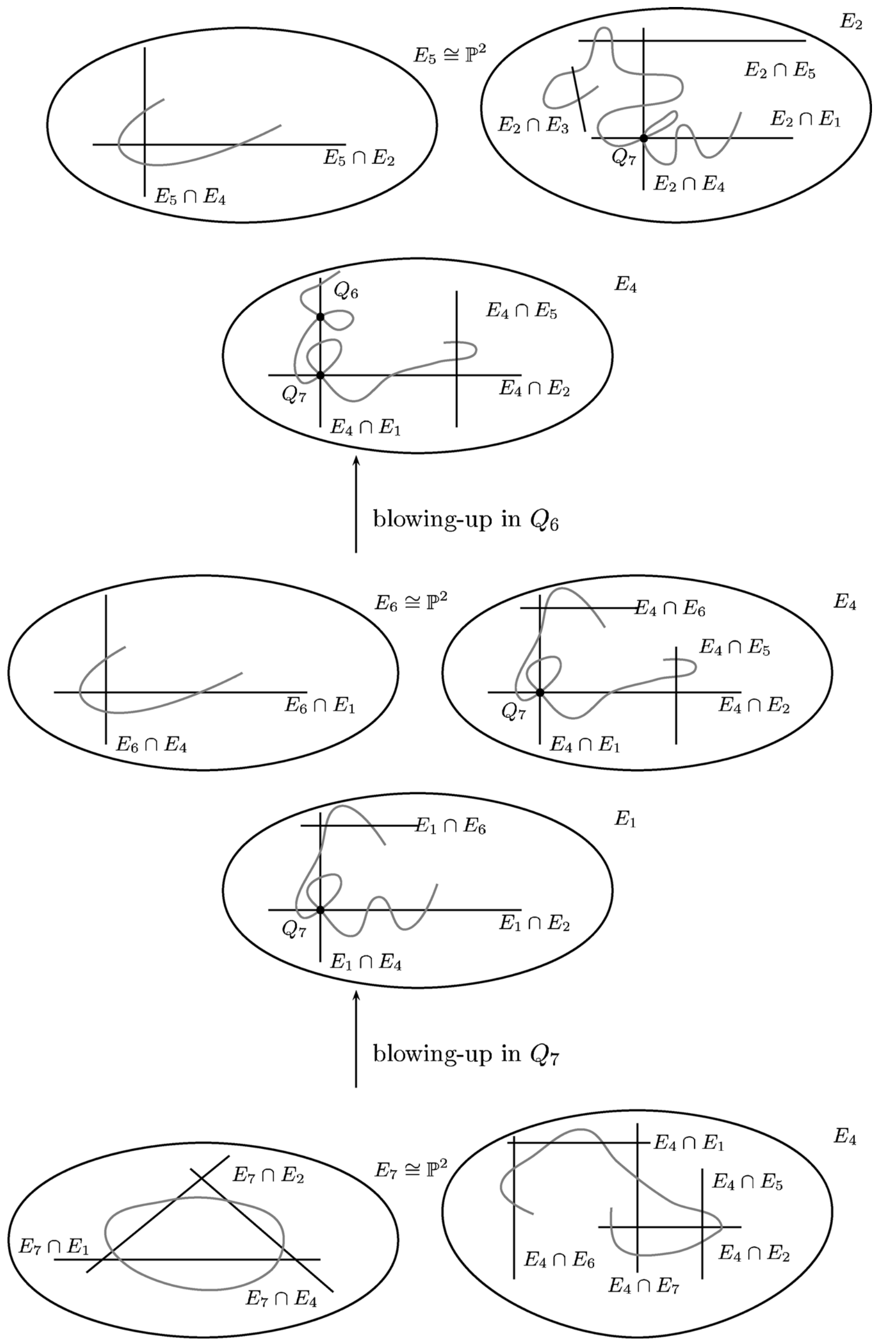
We give a formula for the topological zeta function but first we illustrate the computation by following the embedded resolution process of the following toric constellation. We think that such concrete pictures are very useful to understand the computation of the $\chi(E_I^\circ)$ in general.

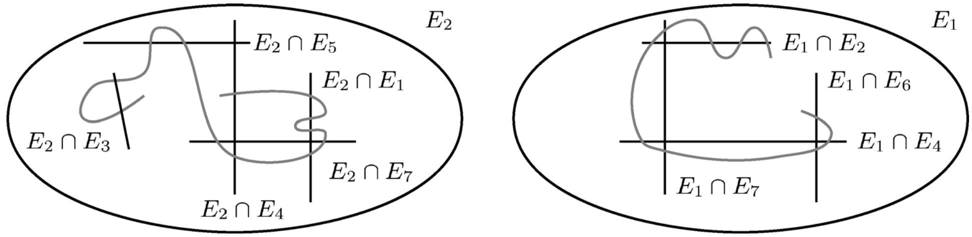
**Example 4.** Consider the toric constellation represented by the following tree:



Let $S$ be a surface in $\mathbb{C}^3$ that is general for the above toric constellation. We follow the resolution process and we picture the intersections that are relevant in the calculation of the numbers $\chi(E_I^\circ)$. The gray curve (that can be reducible) pictured in the ambient $E_j$ represents the curve $C_j$.




$E_5 \cong \mathbb{P}^2$
$E_5 \cap E_2$
$E_5 \cap E_4$
$E_2$
$E_2 \cap E_5$
$E_2 \cap E_3$
$E_2 \cap E_1$
$Q_7$
$E_2 \cap E_4$
$E_4$
$Q_6$
$E_4 \cap E_5$
$Q_7$
$E_4 \cap E_2$
$E_4 \cap E_1$
blowing-up in $Q_6$
$E_6 \cong \mathbb{P}^2$
$E_6 \cap E_1$
$E_6 \cap E_4$
$E_4$
$E_4 \cap E_6$
$E_4 \cap E_5$
$Q_7$
$E_4 \cap E_2$
$E_4 \cap E_1$
$E_1$
$E_1 \cap E_6$
$Q_7$
$E_1 \cap E_2$
$E_1 \cap E_4$
blowing-up in $Q_7$
$E_7 \cong \mathbb{P}^2$
$E_7 \cap E_2$
$E_7 \cap E_1$
$E_7 \cap E_4$
$E_4$
$E_4 \cap E_1$
$E_4 \cap E_5$
$E_4 \cap E_6$
$E_4 \cap E_2$
$E_4 \cap E_7$




□

We now proceed to the computation of the $\chi(E_I^\circ)$. We will write $m_j$ for the multiplicity of the point $Q_j$ on $\hat{S}$ and $E_0$ for the strict transform $\hat{S}$.

1. $I := \{0, i, j\}$ **with** $0 < i < j$ **and** $j \to i$.

   From the number of intersection points of $C_j$ and $E_i$ in $E_j \cong \mathbb{P}^2$, we subtract the number of points in which we will blow up. Then we get $\chi(E_I^\circ) = m_j - \sum_{\substack{k \succ j \\ k \to i}} (C_j(E_i \cap E_j))_{Q_k}$. We can conclude

$$\chi\left(E_I^\circ\right) = m_j - \sum_{\substack{k \to i \\ k \twoheadrightarrow j}} m_k.$$

2. $I := \{i, j, k\}$ **with** $0 \neq i < j < k, k \to i$ **and** $k \to j$.

   The contribution to $\chi(E_I^\circ)$ comes from the intersection point of $E_i \cap E_j \cap E_k$ unless it is a point in which we will blow up. We can express this as follows:

$$\chi\left(E_I^\circ\right) = 1 - \#\{l \mid l \to i, l \to j \text{ and } l \to k\}.$$

3. $I := \{0, i\}$ **with** $0 \neq i$.

   We look at $E_i$ in the final stage. There we have to subtract from $E_0 \cap E_i$ the intersection points with the other exceptional components.

$$\chi\left(E_I^\circ\right) = \chi(C_i) - \sum_{j \to i} \chi(E_0 \overset{\circ}{\widehat{\cap E_i \cap}} E_j) - \sum_{i \to j} \chi(E_0 \overset{\circ}{\widehat{\cap E_i \cap}} E_j).$$

We have $\chi(C_i) = 2 - 2p_a(C_i)$ for the nonsingular $C_i$ that can be irreducible or reducible. This leads to the formula

$$\chi\left(E_I^\circ\right) = m_i(3-m_i) + \sum_{j\to i} m_j(m_j-1) - \sum_{j\to i}\left(m_j - \sum_{\substack{k\to i,\\ k\twoheadrightarrow j}} m_k\right) - \sum_{i\to j}\left(m_i - \sum_{\substack{k\to j\\ k\twoheadrightarrow i}} m_k\right).$$

4. $I := \{i, j\}$ **with** $0 \neq i < j, j \to i$.

We compute the contribution from the configuration in $E_j \cong \mathbb{P}^2$.

$$\begin{aligned}\chi\left(E_I^\circ\right) &= 2 - \left(\chi(\overbrace{E_0 \cap E_i}^{\circ} \cap E_j) + \#A_{ij} + \#B_{ij} - \#C_{ij}\right)\\ &= 2 - \left(m_j - \sum_{\substack{k\to i,\\ k\twoheadrightarrow j}} m_k\right) - \#A_{ij} - \#B_{ij} + \#C_{ij},\end{aligned}$$

with

$$\begin{aligned}A_{ij} &:= \{k \mid k \succ j, k \to i\}\\ B_{ij} &:= \{k \mid k \neq i, j \to k\}\\ C_{ij} &:= \{k \mid k \succ j, k \to i \text{ and } \exists l : l \neq i, k \to l \text{ and } j \to l\}.\end{aligned}$$

5. $I := \{i\}$ **with** $i \neq 0$.

We look in $E_i \cong \mathbb{P}^2$ and find

$$\begin{aligned}\chi\left(E_I^\circ\right) &= 3 - \left(\chi\left(\overbrace{E_0 \cap E_i}^{\circ}\right) + \#A_i + 2\#B_i - \binom{\#B_i}{2}\right)\\ &= 3 + m_i(m_i-3) - \sum_{j\to i} m_j(m_j-1)\\ &\quad + \sum_{j\to i}\left(m_j - \sum_{\substack{k\to i,\\ k\twoheadrightarrow j}} m_k\right) + \sum_{i\to j}\left(m_i - \sum_{\substack{k\to j\\ k\twoheadrightarrow i}} m_k\right) - \#A_i - 2\#B_i + \binom{\#B_i}{2},\end{aligned}$$

with

$$\begin{aligned}A_i &:= \{k \mid k \succ i \text{ and } \nexists l : i \to l \text{ and } k \to l\}\\ B_i &:= \{k \mid i \to k\}.\end{aligned}$$

6. For $I$ not of the form of one of the sets described above, $\chi(E_I^\circ) = 0$.

Also the numerical data are completely determined by the tree. We obtain the numbers $N_i$ via the recursive formula $N_i = m_i + \sum_{i\to j} N_j$. For the $\nu_i$, we find $\nu_i = \sum_{i\to j}(\nu_j - 1) + 3$.

## 5 Analysis of $\chi(E_i^\circ)$

In order to investigate the conjectures, we study the expression for $\chi(E_i^\circ)$ that we obtained in the previous section:

$$\chi\left(E_i^\circ\right) = m_i(m_i-3) - \sum_{j\to i} m_j(m_j-1) + \sum_{j\to i}\left(m_j - \sum_{\substack{k\to i\\ k\twoheadrightarrow j}} m_k\right) + \sum_{i\to j}\left(m_i - \sum_{\substack{k\to j\\ k\twoheadrightarrow i}} m_k\right) + 3 - \#A_i - 2\#B_i + \binom{\#B_i}{2},$$

with $A_i = \{k \mid k \succ i \text{ and } \nexists l : i \to l \text{ and } k \to l\}$ and $B_i = \{k \mid i \to k\}$.

Notice that the linear proximity inequalities imply that $m_j - \sum_{\substack{k\to i\\ k\twoheadrightarrow j}} m_k \geq 0$, for all $j \to i$ and that $m_i - \sum_{\substack{k\to j\\ k\twoheadrightarrow i}} m_k \geq 0$ for all $i \to j$. Moreover, for a point $Q_j$ with maximal level in the set of the points that are proximate to $Q_i$, we have $m_j - \sum_{\substack{k\to i\\ k\twoheadrightarrow j}} m_k = m_j > 0$ and so $\sum_{j\to i}(m_j - \sum_{\substack{k\to i\\ k\twoheadrightarrow j}} m_k) > 0$.

Let $T := 3 - \#A_i - 2\#B_i + \binom{\#B_i}{2}$. Then $T$ takes the following values:

| $\#B_i$ | $\#A_i$ | $T$ |
|---|---|---|
| 0 | 0 | 3 |
| 0 | 1 | 2 |
| 0 | 2 | 1 |
| 0 | 3 | 0 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |
| 2 | 0 | 0 |
| 3 | 0 | 0 |

*Table 1*

We want to investigate when $\chi(E_i^\circ) \leq 0$. *A priori* there are infinitely many constellations to consider. The first result in this section will permit us to reduce our study to a finite number of cases. Secondly, we will rewrite $\chi(E_i^\circ)$ and via combinatorics we will analyse this new description.

**Lemma 8.** Let $\mathcal{A} = (\mathcal{C}, \underline{m})$ be a three-dimensional toric idealistic cluster and let $Q_i$ be a point of the constellation $\mathcal{C}$. If $\#\{t \in \mathbb{Z}_{\geq 0} \mid Q_i(a, b^t) \in \mathcal{C}\} + \#\{t \in \mathbb{Z}_{\geq 0} \mid Q_i(b, a^t) \in \mathcal{C}\} \geq 3$ for all $a, b \in \{1, 2, 3\}$ with $a \neq b$, then $\chi(E_i^\circ) > 0$. □

**Proof.** If $\#\{t \in \mathbb{Z}_{\geq 0} \mid Q_i(a, b^t) \in \mathcal{C}\} + \#\{t \in \mathbb{Z}_{\geq 0} \mid Q_i(b, a^t) \in \mathcal{C}\} \geq 3$ for all $a, b \in \{1, 2, 3\}$ with $a \neq b$, then it follows that $m_i > 3$ except when there are exactly 6 points—that have multiplicity 1—that are proximate to $Q_i$ and such that $\#\{t \in \mathbb{Z}_{\geq 0} \mid Q_i(1, 3^t) \in \mathcal{C}\} = \#\{t \in \mathbb{Z}_{\geq 0} \mid Q_i(2, 1^t) \in \mathcal{C}\} = \#\{t \in \mathbb{Z}_{\geq 0} \mid Q_i(3, 2^t) \in \mathcal{C}\} = 2$, up to permutation of the labels. In that case $m_i$ can be equal to 3 and then one finds that $\chi(E_i^\circ) > 0$.

When $m_i > 3$ we construct a new cluster. We define $m'_i := m_i - 3$, $m'_j := m_j - 1$ for all $j$ for which $j \to i$ and we do not change the weights of the other points in $\mathcal{C}$. Let $\mathcal{C}'$ be the subconstellation of $\mathcal{C}$ that contains exactly the points $Q_j$ of $\mathcal{C}$ for which $m_j > 1$ and let $\mathcal{A}'$ be the cluster $(\mathcal{C}', \underline{m'})$. Then also $\mathcal{A}'$ satisfies the linear proximity inequalities and thus $\mathcal{A}'$ is a toric idealistic cluster. Let us now consider a surface $\mathcal{S}$ that is general with respect to $\mathcal{A}$ and a surface $\mathcal{S}'$ that is general with respect to $\mathcal{A}'$. Blowing up the point $Q_i$ provides two curves $C_i = E_i \cap \hat{\mathcal{S}}$ and $C'_i = E_i \cap \hat{\mathcal{S}}'$ in the exceptional variety $E_i \cong \mathbb{P}^2$ of degree $m_i$ and $m'_i$, respectively. From Bezout's formula, it follows that $m_i m'_i \geq \sum_{\substack{j \to i \\ Q_j \in \mathcal{C}'}} m_j m'_j$. The latter sum is also equal to $\sum_{\substack{j \to i \\ Q_j \in \mathcal{C}}} m_j m'_j$. We can conclude that $\chi(E_i^\circ) > 0$. ■

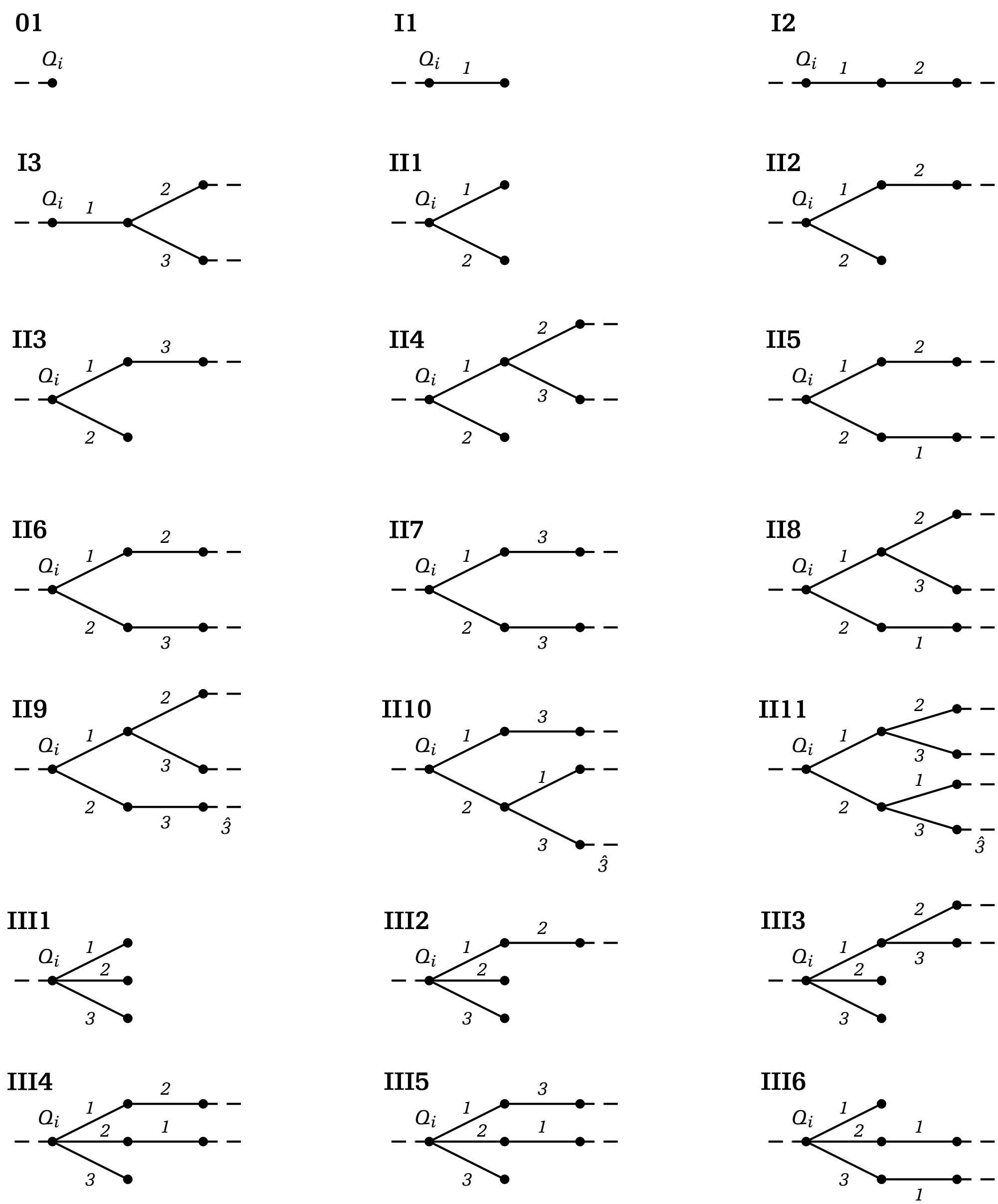
This lemma will allow us to work with a finite number of families of constellations. We represent these families in List 1. We first explain some notations.

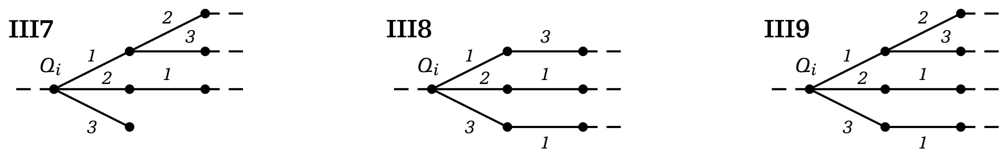
To save place, from now on we draw the clusters from left to right. So if there is an edge between $Q_i$ and $Q_j$ and if $Q_j$ is at the right from $Q_i$, then $Q_j > Q_i$. If there exists an edge with label $x$ between points of the chain $\mathcal{C}^i := \{Q_j \mid Q_i \geq Q_j\}$, then we will simply say that 'label $x$ appears below $Q_i$'.

The constellations are listed according to the number of points $Q_j$ for which $Q_j \succ Q_i$ (indicated by a roman number). We only draw the subconstellation that shows $Q_i$ and the points $Q_j$ that are proximate to $Q_i$ and for which holds that $j \succ i$ or for which there exists a point $Q_k$ such that $k \succ i$ and $j \succ k$. By drawing '$--$' going out of a point $Q_j$ for which $j \to i$, we mean that there can exist a point $Q_k$ for which $k > j$ and $k \to i$.

We also draw the symbol '——' arriving in the point $Q_i$ when $Q_i$ is not necessarily the origin. When $Q_j$ is a point of the constellation, we will denote its multiplicity by $m_{Q_j}$ or by $m_j$.

List 1 contains the constellations we should study, according to Lemma 8, up to permutation of the labels. In constellations II9, II10 and II11, we mean by $\hat{3}$ that label 3 should not occur at that place, so $\#\{t \in \mathbb{Z}_{\geq 0} \mid Q_i(2, 3^t) \in \mathcal{C}\} = 2$.






*List 1*

In the next step, we give an alternative description for $\chi(E_i^\circ)$. We first introduce some new notation.

**Notation 1.** We write $D := m_i^2 - \sum_{j\to i} m_j^2$ and $r_{ab} := m_i - M_{Q_i}(a,b) - M_{Q_i}(b,a)$ for $a,b \in \{1,2,3\} = \{a,b,c\}$, $a \neq b$. Let $R$ be equal to $\hat{r_{12}} + \hat{r_{13}} + \hat{r_{23}}$ where $\hat{r_{ab}} :=$

$$\begin{cases} r_{ab} & \text{if label } c \text{ does not appear under } Q_i; \\ 0 & \text{else.} \end{cases}$$

We refer to the beginning of Section 5 for the definition of $T$ and to Table 1 for the values that $T$ takes. □

**Lemma 9.**

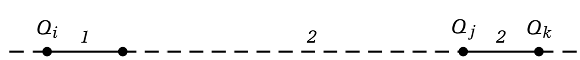
$$\chi\left(E_i^\circ\right) = D - R + T.$$ □

**Proof.** We will prove that

$$R = 3m_i - 2\sum_{j\to i} m_j + \sum_{j\to i}\left(\sum_{\substack{k\to i\\ k\twoheadrightarrow j}} m_k\right) + \sum_{i\to j}\left(\sum_{\substack{k\to j\\ k\twoheadrightarrow i}} m_k\right) - \sum_{i\to j} m_i. \tag{3}$$

Let $X$ be the right-hand side in (3), let $X_1 := \sum_{j\to i}(\sum_{\substack{k\to i\\ k\twoheadrightarrow j}} m_k)$ and $X_2 := \sum_{i\to j}(\sum_{\substack{k\to j\\ k\twoheadrightarrow i}} m_k)$. For $k \to i$, one has one of the following situations:

- There exist exactly two points $Q_{j_1}$ and $Q_{j_2}$ that are proximate to $Q_i$ and for which $k \twoheadrightarrow j_1$ and $k \twoheadrightarrow j_2$. Then $m_k$ appears twice as term in $X_1$ and $Q_k$ is not linearly proximate to $Q_i$; hence $m_k$ does not appear in $X_2$. This implies that $m_k$ does not show up in $X$.
- There exists exactly one point $Q_j$ that is proximate to $Q_i$ and for which $k \twoheadrightarrow j$. We are in the following situation:



Then $m_k$ appears once in $X_1$ and $k \twoheadrightarrow i$. If label 3 appears under $Q_i$, then $m_k$ appears once in $X_2$. Hence, $m_k$ does not show up in the expression $X$. If there is no label 3 under $Q_i$, then $m_k$ does not appear in $X_2$ such that this $m_k$ appears with coefficient $-1$ in $X$.

- There exists no point $Q_j$ such that $j \to i$ and $k \twoheadrightarrow j$. Then $k \succ i$ and $m_k$ do not appear in $X_1$. The number of times that $m_k$ appears in $X_2$ depends on the labels below $Q_i$. It can be once, twice or thrice.

Notice that the multiplicities $m_k$ of the points $Q_k$ with $k \to i$ but not $k \twoheadrightarrow i$ do not appear in $X$. To analyse further the formula $X$, we now take the labels into account that appear below $Q_i$.

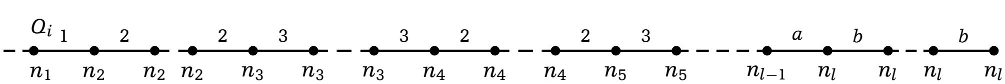
If $Q_i$ is the origin, then the points $Q_k$ for which $k \succ i$ appear with coefficient $-2$ in $X$. The other points $Q_j$ for which $j \twoheadrightarrow i$ have coefficient $-1$. Hence,

$$X = 3m_i - \sum_{k \twoheadrightarrow i} m_k - \sum_{k \succ i} m_k = r_{12} + r_{13} + r_{23} = R.$$

Also in the other cases, one can check that $X = R$: when 1 is the only label below $Q_i$, then $X = r_{12} + r_{13} = R$. If the labels showing up below $Q_i$ are 1 and 3, then $X = r_{13} = R$. If three labels show up under $Q_i$, then $X = 0 = R$. ■

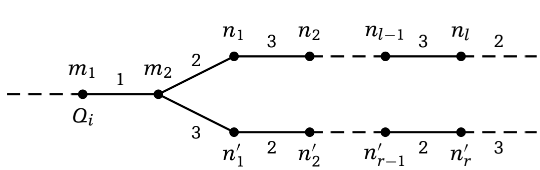
Notice that it follows from the linear proximity relations that $R \geq 0$. Formula (1) in Section 2.3 shows that $D \geq 0$ and from Table 1 it follows that $0 \leq T \leq 3$. In order to find the cases where $\chi(E_i^\circ) \leq 0$, we will investigate when $R \geq D$. We want to give an estimation for $D$. In particular, we will determine a lower bound $L$ for $D$ and then we will check when $R \geq L$. We introduce some terminology.

**Definition 10.** Let $l \in \mathbb{Z}_{>1}$ and let $n_1, \ldots, n_l, h_1, \ldots, h_{l-1} \in \mathbb{Z}_{>0}$ such that $n_j = h_j n_{j+1} + n_{j+2}$ where $0 < n_{j+2} < n_{j+1}$, for $1 \leq j \leq l-2$, and such that $n_{l-1} = h_{l-1} n_l$. If $l$ is even, then set $(a, b) = (3, 2)$. If $l$ is odd, we set $(a, b) = (2, 3)$. Let $\mathcal{A}$ be an idealistic cluster

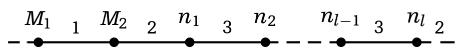



where $n_j$ appears $h_{j-1}$ consecutive times, $2 \leq j \leq l$. We call $\mathcal{A}$ a *Euclidean cluster starting in* $Q_i$. □

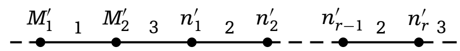
**Definition 11.** Let $\mathcal{A}$ be a cluster of the form



such that

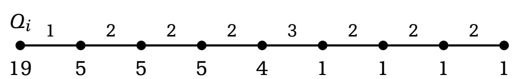



and

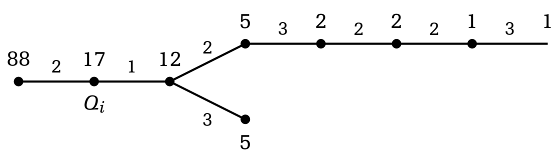



are Euclidean clusters, where $M_1 := m_1 - \sum_{j=1}^{r} n'_j$, $M_2 := m_2 - \sum_{j=1}^{r} n'_j$ and $M'_1 := m_1 - \sum_{j=1}^{l} n_j$, $M'_2 := m_2 - \sum_{j=1}^{l} n_j$. We call the cluster $\mathcal{A}$ a bi-Euclidean cluster starting in $Q_i$. □

**Example 5.** The cluster




is a Euclidean cluster starting in $Q_i$. The cluster



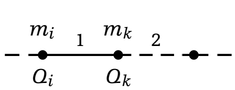

is a bi-Euclidean cluster starting in $Q_i$. □

**Definition 12.** Suppose that $Q$ is a point different from the origin in a three-dimensional toric constellation $\mathcal{C}$. Let $a \in \{1,2,3\}$ such that $Q = P(a)$ for a point $P \in \mathcal{C}$ and suppose that there exists $b \in \{1,2,3\}, a \neq b$, such that $Q(b) \in \mathcal{C}$. Then we call $Q$ a *switch point*. □

**Proposition 13.** Let $\mathcal{A} = (\mathcal{C}, \underline{m})$ be a three-dimensional toric idealistic cluster. Let $Q_i \in \mathcal{C}$ and suppose that there exists exactly one point $Q_k \in \mathcal{C}$ for which $k \succ i$. Then the following properties hold:

1. $m_i m_k \geq \sum_{j \to i} m_j^2$;
2. $m_i m_k = \sum_{j \to i} m_j^2$ if and only if $\mathcal{A}$ is a Euclidean cluster or $\mathcal{A}$ is a bi-Euclidean cluster starting in $Q_i$. □

**Proof.** Case 1: There exists at most one point $Q_l$ in $\mathcal{C}$ that is proximate to $Q_i$ and such that $Q_l \succ Q_k$. Then we can suppose that the cluster is of the form



We have $m_i \geq M_{Q_i}(1,2)$ and thus

$$m_k m_i \geq \sum_{t \geq 0} m_k m_{Q_i(1,2^t)}. \tag{4}$$

We give lower bounds for the terms $m_k m_{Q_i(1,2^t)}$ in (4) depending on whether $Q_i(1,2^t)$ is a switch point or not. If $Q_i(1,2^t) = Q_k$, then $m_k m_{Q_i(1,2^t)} = m_k^2$. If $P^t := Q_i(1,2^t)$, $t \neq 0$, is a switch point, then

$$m_k m_{P^t} \geq \left( \sum_{s \geq 0} m_{P^{t-1}(2,3^s)} + \sum_{s \geq 0} m_{P^{t-1}(3,2^s)} \right) m_{P^t}.$$

If $P^t$ is not a switch point, then we estimate $m_k m_{P^t} \geq m_{P^t} m_{P^t}$. We fill in these lower bounds in (4) and we get

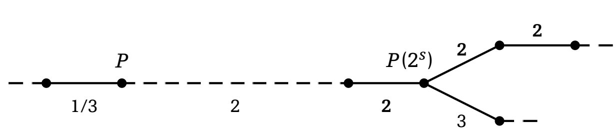
$$m_k m_i \geq m_k^2 + \sum_{\substack{P^t \text{ not} \\ \text{switch point}}} m_{P^t}^2 + \sum_{\substack{P^t \text{ switch} \\ \text{point, } t \neq 0}} \left( \sum_{s \geq 0} m_{P^{t-1}(2,3^s)} + \sum_{s \geq 0} m_{P^{t-1}(3,2^s)} \right) m_{P^t}.$$

We iterate this process: whenever we have a product $m_{Q_j} m_{Q_l}$ with $j < l$, we use the estimations described above for $m_{Q_j} m_{Q_l}$ according to whether $Q_l$ is a switch point or not, i.e., if $Q_l$ is a switch point and if $P \in \mathcal{C}$ is such that $Q_l \succ P$, then set $m_{Q_j} m_{Q_l} \geq (\sum_{t \geq 0} m_{P(2,3^t)} + \sum_{t \geq 0} m_{P(3,2^t)}) m_{Q_l}$. If $Q_l$ is not a switch point, then we set $m_{Q_j} m_{Q_l} \geq m_{Q_l} m_{Q_l}$. This is obviously a finite process and it shows that

$$m_i m_k \geq \sum_{j \to i} m_j^2.$$

We now study when $\sum_{j \to i} m_j^2 = m_i m_k$.

- If $\mathcal{C}$ is a chain, then it is not difficult to see that $\sum_{j \to i} m_j^2 = m_i m_k$ if and only if $\mathcal{A}$ is a Euclidean cluster.
- If $\mathcal{A}$ contains a subcluster of the form

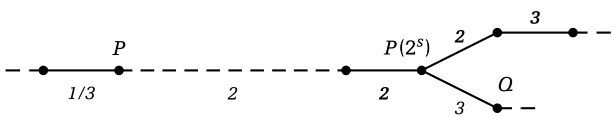


where $P(2^{s+1})$ is not a switch point, then at some moment in the process we get

$$
\begin{aligned}
m_i m_k &\geq \cdots + m_P \left( \sum_{t \geq 0} m_{P(2^t)} \right) \\
&> \cdots + \sum_{t=0}^{s-1} m^2_{P(2^t)} + m_{P(2^s)} \left( \sum_{t \geq 0} m_{P(2^s,3^t)} \right) + \sum_{t=s+1} m^2_{P(2^t)}.
\end{aligned}
$$

Indeed, $m_P > m_{P(2^{s+1})}$.

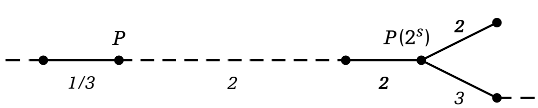
- If $\mathcal{A}$ contains a subcluster of the form




then at some moment in the process we get

$$
\begin{aligned}
m_i m_k &\geq \cdots + m_P \left( \sum_{t \geq 0} m_{P(2^t)} \right) \\
&\geq \cdots + \sum_{t=0}^{s-1} m^2_{P(2^t)} + m_{P(2^s)} \left( \sum_{t \geq 0} m_{P(2^s,3^t)} \right) \\
&\quad + m_{P(2^{s+1})} \left( \sum_{t \geq 0} m_{P(2^{s+1},3^t)} + \sum_{t \geq 0} m_{Q(2^t)} \right) \\
&\geq \sum_{j \to i} m_j^2 + m_{P(2^{s+1})} \sum_{t \geq 0} m_{Q(2^t)} \\
&> \sum_{j \to i} m_j^2.
\end{aligned}
$$

- If $\mathcal{A}$ contains a subcluster of the form

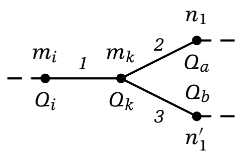



then at some moment in the process we get

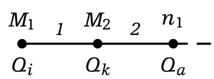
$$
\begin{aligned}
m_i m_k &\geq \cdots + m_P \left( \sum_{t \geq 0} m_{P(2^t)} \right) \\
&> \cdots + \sum_{t=0}^{s-1} m^2_{P(2^t)} + m_{P(2^s)} \sum_{t \geq 0} m_{P(2^s,3^t)} + m^2_{P(2^{s+1})}.
\end{aligned}
$$

Indeed, $m_P > m_{P(2^{s+1})}$.

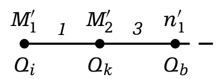
Case 2: There exist two points $Q_a$ and $Q_b$ in $\mathcal{C}$ that are proximate to $Q_i$ and such that $Q_a \succ Q_k$ and $Q_b \succ Q_k$. Then we may suppose that the cluster is of the form




Define $t := m_k - M_{Q_k}(2,3) - M_{Q_k}(3,2)$. As the cluster is idealistic, $t \geq 0$. Then also the clusters




and




with $M_1 := m_i - M_{Q_k}(2,3) - t$ and $M_2 := M_{Q_k}(2,3)$, $M'_1 := m_i - M_{Q_k}(3,2) - t$ and $M'_2 := M_{Q_k}(3,2)$ are idealistic. They are clusters of the form as in Case 1, therefore we can use the bound that we obtained there

$$\begin{aligned}\sum_{j\to i} m_j^2 &\leq M_1M_2 + M'_1M'_2 - M_2^2 - {M'_2}^2 + m_k^2\\ &= (M_2 + M'_2)(m_i - m_k) + m_k^2\\ &= m_im_k - t(m_i - m_k)\\ &\leq m_im_k.\end{aligned}$$

From the previous computations, it follows that $\sum_{j\to i} m_j^2 = m_im_k$ if and only if the cluster is a bi-Euclidean cluster starting in $Q_i$. ■

This combinatoric result is the key to determine the sign of $\chi(E_i^\circ)$.

## 6 Determination of the Sign of $\chi(E_i^\circ)$

In this section, we classify the irreducible exceptional components $E_i, 1 \leq i \leq r$, that arise in the blowing up of some three-dimensional toric idealistic cluster according to the sign of $\chi(E_i^\circ)$. As in Lemma 9, we write $\chi(E_i^\circ)$ as $D - R + T$. For the points $Q_i$ in the clusters of List 1, we give a lower bound $L$ for $D$. We will use very frequently Proposition 13. As upper bound for $R$, we use that $R \leq r_{12} + r_{13} + r_{23}$. We will study for which clusters in List 1 it holds that $r_{12} + r_{13} + r_{23} \geq L$. We mark the name of the constellation by a star if there exists a cluster with that underlying constellation that yields $\chi(E_i^\circ) \leq 0$. We refer to Table 1 for the values of $T$.

Let us first make the following observation.

**Remark 1.** Suppose $\mathcal{A} = (\mathcal{C}, \underline{m})$ is a three-dimensional toric idealistic cluster. Let $Q_i$ be a point of the constellation $\mathcal{C}$. We define a subconstellation $S^i\mathcal{C}$ of $\mathcal{C}$ as follows: the origin of $S^i\mathcal{C}$ is $Q_i$ and $Q_j \in S^i\mathcal{C}$ if and only if $j \to i$ in $\mathcal{C}$ or $j = i$. Suppose now that $Q_k \in S^i\mathcal{C}$, $Q_k \neq Q_i$. We define a cluster $S^i_k\mathcal{C} = (S^i\mathcal{C}, \underline{n})$ with underlying constellation $S^i\mathcal{C}$: for each point $Q_j \in S^i\mathcal{C}$, $j \neq k$, set its multiplicity $n_j := m_j$ and set $n_k := m_k + 1$. If $S^i_k\mathcal{C}$ is idealistic, then there always exists an idealistic cluster $\tilde{\mathcal{A}} = (\mathcal{C}, \underline{\tilde{m}})$ that contains $S^i_k\mathcal{C}$ as a subcluster. Blowing up the constellation $\mathcal{C}$ of cluster $\tilde{\mathcal{A}}$ then yields

$$\begin{aligned}\chi(\tilde{E}_i^\circ) &= \tilde{D} - \tilde{R} + \tilde{T} \\ &= D - 2m_k - 1 - (R - x) + T \\ &= \chi(E_i^\circ) - 2m_k - 1 + x,\end{aligned}$$

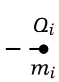
where $x$ is equal to $0, 1$ or $2$ depending on the constellation $\mathcal{C}$.

It follows that $\chi(\tilde{E}_i^\circ) < \chi(E_i^\circ)$. □

This will make it possible to simplify computations. Indeed, as described above, when we let increase the values of the multiplicities such that the cluster stays idealistic and if $\chi(\tilde{E}_i^\circ) \geq 0$, then $\chi(E_i^\circ) > 0$.

We now proceed to the classification. Firstly, we investigate the constellations of List 1 where at most one edge is going out of $Q_i$. Then we consider the ones where exactly two edges leave out of $Q_i$. We treat constellation II11 and we draw conclusions about the subconstellations of II11 if possible. We will have to investigate constellation II7 separately and then we also get the classification for the constellations II1 and II3. Studying constellation III9 will be enough to classify the constellations where three edges are going out of $Q_i$.

**01***

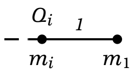



$D = m_i^2$. Does there exist a positive integer $m_i$ such that $3m_i \geq m_i^2$?

We study the exact value of $\chi(E_i^\circ)$ if $m_i \in \{1, 2, 3\}$. If $Q_i$ is the origin, then $T = 3$ and $\chi(E_i^\circ) = m_i^2 - 3m_i + 3 > 0$. If there exists exactly one point $Q_j$ such that $i \to j$, then

$T = 1$ and $\chi(E_i^\circ) = m_i^2 - 2m_i + 1$. We find that $\underline{\chi(E_i^\circ) = 0}$ if $m_i = 1$. If there exist exactly two points in the constellation to which $Q_i$ is proximate, then $T = 0$ and $\chi(E_i^\circ) = m_i^2 - m_i$. We again find that $\underline{\chi(E_i^\circ) = 0}$ if $m_i = 1$. If there are three points to which $Q_i$ is proximate, then $T = 0$ and $\chi(E_i^\circ) = m_i^2 > 0$.

**I1***




$D = m_i^2 - m_1^2$. Do there exist positive integers $m_i$ and $m_1$ such that $(m_i - m_1) + (m_i - m_1) + m_i \geq m_i^2 - m_1^2$?

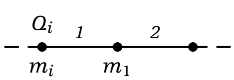
If $m_1 = m_i$, this inequality holds. Then $R = \hat{r}_{23}$ and thus, if there is a label 1 under $Q_i$, one has that $R = 0$ and $\underline{\chi(E_i^\circ) = T = 0}$. If there is no label 1 under $Q_i$, then $R = r_{23} = m_i$, so $\underline{\chi(E_i^\circ) = T - m_i}$. If $Q_i$ is the origin, then we have $\underline{\chi(E_i^\circ) = 2 - m_i}$. If only label 2 or only label 3 appears under $Q_i$, then $\underline{\chi(E_i^\circ) = 1 - m_i}$. If label 2 as well as label 3 are present under $Q_i$, then $\underline{\chi(E_i^\circ) = -m_i}$.

Suppose now that $m_1 < m_i$ and that the inequality holds. This implies that

$$(m_i - 1)(m_i - 3) \geq m_1(m_1 - 2) \geq m_i(m_i - 3).$$

Then $(m_i, m_1) = (3, 2)$ or $(m_i, m_1) = (2, 1)$. If $(m_i, m_1) = (3, 2)$, then $\chi(E_i^\circ) = 5 - R + T$ and $R \leq 5$. If $R = 5$, then $Q_i$ is the origin. Then $T = 2$ and thus $\chi(E_i^\circ) > 0$. If $(m_i, m_1) = (2, 1)$, then $\chi(E_i^\circ) = 3 - R + T$ and $R \leq 4$. If $R \geq 3$, then we should have $\hat{r}_{23} = r_{23}$ and also say $\hat{r}_{12} = r_{12}$. Thus, label 1 and label 3 do not appear under $Q_i$. Then only label 2 appears below $Q_i$ or $Q_i$ is the origin. However, also under these conditions we have $\chi(E_i^\circ) > 0$.

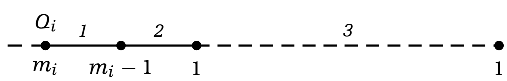
**I2***




$L = m_i^2 - m_i m_1$. Do there exist multiplicities for which $(m_i - \sum_{t \geq 0} m_{Q_i(1,2^t)}) + (m_i - m_1) + m_i \geq m_i^2 - m_i m_1$?

We rewrite this inequality as $m_1(m_i - 2) - \sum_{t \geq 1} m_{Q_i(1,2^t)} \geq m_i(m_i - 3)$.

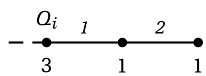
- If $m_1 = m_i - 1$, the cluster becomes



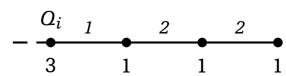

where label 3 appears, say $k$ times, with $0 \leq k \leq m_i - 2$. Then $\chi(E_i^\circ) = m_i^2 - (m_i - 1)^2 - (k + 1) - R + T$ and $R \leq m_i + 1$, so $\chi(E_i^\circ) \geq 2m_i - 2 - k - m_i - 1 + T = m_i - 3 - k + T$. If $k = m_i - 3$, then $\chi(E_i^\circ)$ could only be 0 if $R = m_i + 1$ and

$T = 0$ but this is impossible. If $k = m_i - 2$ and $\chi(E_i^\circ) \leq 0$, then $R$ should be $m_i$ or $m_i + 1$. If $R = m_i$, then $\chi(E_i^\circ) = T$. We find that $\underline{\chi(E_i^\circ) = 0}$ if labels 2 and 3 appear below $Q_i$. When $R = m_i + 1$, then $\chi(E_i^\circ) = -1 + T$. Then $\underline{\chi(E_i^\circ) = 0,}$ if we only have label 3 under $Q_i$.

- If $m_i \geq 3$ and if the inequality holds, then certainly $m_1 > m_i - 3$. So suppose now that $m_1 = m_i - 2$. Then the inequality becomes $(m_i - 2)(m_i - 2) - \sum_{t\geq 1} m_{Q_i(1,2^t)} \geq m_i(m_i - 3)$ or $4 - \sum_{t\geq 1} m_{Q_i(1,2^t)} \geq m_i$ and so $m_i = 3$. The cluster is then of the form

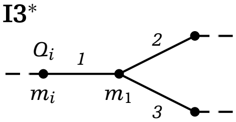



or




In the first picture $\chi(E_i^\circ) = 7 - R + T$ and $R \leq 6$, and thus $\chi(E_i^\circ) > 0$. In the picture at the right, $\chi(E_i^\circ) = 6 - R + T$ and $R \leq 5$, and thus again $\chi(E_i^\circ) > 0$.

**I3***



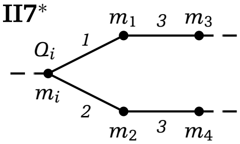

$L = m_i^2 - m_i m_1$. Do there exist multiplicities such that $(m_i - \sum_{t\geq 0} m_{Q_i(1,2^t)}) + (m_i - \sum_{t\geq 0} m_{Q_i(1,3^t)}) + m_i \geq m_i^2 - m_i m_1$?

We rewrite the inequality as follows:

$$m_1(m_i - 2) - \sum_{t\geq 1} m_{Q_i(1,2^t)} - \sum_{t\geq 1} m_{Q_i(1,3^t)} \geq m_i(m_i - 3).$$

If this inequality holds, then certainly $m_1 = m_i - 1$. Let $k \in \{2, \ldots, m_i - 1\}$ be the number of points that are proximate to $Q_i$ and that are different from $Q_i(1)$. Then we find that $\chi(E_i^\circ) = m_i^2 - (m_i - 1)^2 - k - R + T$. As $R \leq m_i$, we have $\chi(E_i^\circ) \geq m_i - 1 - k + T$. It follows that $\underline{\chi(E_i^\circ) = 0}$ if $k = m_i - 1$, $R = m_i$ and when labels 2 and 3 appear below $Q_i$.

**II7***




$L = m_i^2 - m_1 \sum_{t\geq 0} m_{Q_i(1,3^t)} - m_2 \sum_{t\geq 0} m_{Q_i(2,3^t)}$. We allow that $m_3$ and $m_4$ are 0; thus we include the constellations II1 and II3.

- Suppose $r_{12} = 0$. Can the following inequality hold?

$$\left(m_i - \sum_{t\geq 0} m_{Q_i(1,3^t)}\right) + \left(m_i - \sum_{t\geq 0} m_{Q_i(2,3^t)}\right) \geq m_i^2 - m_1 \sum_{t\geq 0} m_{Q_i(1,3^t)} - m_2 \sum_{t\geq 0} m_{Q_i(2,3^t)}$$

$$\Updownarrow$$

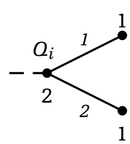
$$(m_1 - 1)\sum_{t\geq 0} m_{Q_i(1,3^t)} + (m_2 - 1)\sum_{t\geq 0} m_{Q_i(2,3^t)} \geq m_i(m_i - 2).$$

On the other hand, we have

$$\begin{aligned} m_i(m_i-2) &= m_i(m_1 - 1 + m_2 - 1) \\ &\geq (m_1 - 1)\sum_{t\geq 0} m_{Q_i(1,3^t)} + (m_2 - 1)\sum_{t\geq 0} m_{Q_i(2,3^t)}, \end{aligned}$$

and thus $m_i(m_i-2) = (m_1-1)\sum_{t\geq 0} m_{Q_i(1,3^t)} + (m_2-1)\sum_{t\geq 0} m_{Q_i(2,3^t)}$. The cluster has then one of the following forms:

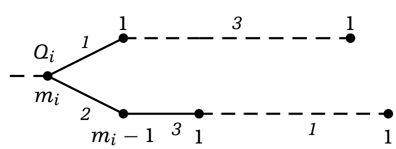
– $m_1 = m_2 = 1$: if the cluster is




then $\chi(E_i^\circ) = 2 - R + T$ with $R \leq 2$. However, if $R = 2$, then $T > 0$; hence $\chi(E_i^\circ) > 0$.

The other clusters for which $m_1 = m_2 = 1$ will be treated in the next cases.

– $m_1 = 1$ and $\sum_{t\geq 0} m_{Q_i(2,3^t)} = m_i$:




Suppose that the multiplicity 1 appears $k \in \{1, \ldots, m_i\}$ times in the upper chain and that the label 1 appears $l-1$ times in the lower chain, $1 \leq l \leq m_i - 1$. We have $\chi(E_i^\circ) = m_i^2 - (m_i-1)^2 - l - k - R + T = 2m_i - 1 - l - k - R + T$. As $R \leq m_i - k$, we get $\chi(E_i^\circ) \geq m_i - 1 - l + T$. If $\chi(E_i^\circ) \leq 0$, then we must have that $R = m_i - k$, $l = m_i - 1$, and $T = 0$.

If $k < m_i$, then label 2 may not appear under $Q_i$ (indeed, $R = \check{r}_{13}$) and label 1 should certainly appear under $Q_i$ (see Table 1). We then have that $\underline{\chi(E_i^\circ) = 0}$. If $k = m_i$, then also $\sum_{t\geq 0} m_{Q_i(1,3^t)} = m_i$. This cluster will be treated later.

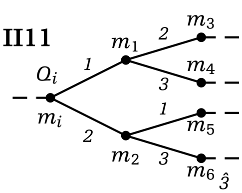
- $m_2 = 1$ and $\sum_{t\geq 0} m_{Q_i(1,3^t)} = m_i$: up to permutation this case is the same as the previous case.
- $\sum_{t\geq 0} m_{Q_i(1,3^t)} = m_i$ and $\sum_{t\geq 0} m_{Q_i(2,3^t)} = m_i$: in this case $R = 0$ and therefore $\underline{\chi(E_i^\circ) = 0}$ if and only if $D = T = 0$. From Proposition 13, it follows that both chains that leave out of $Q_i$ should be Euclidean clusters. To have $T = 0$, one needs at least two labels under $Q_i$ or exactly one label under $Q_i$ that then should be 1 or 2.

- If $r_{12} \neq 0$, we may suppose that $r_{13} = r_{23} = 0$. We study if the following inequality can hold:

$$m_i - m_1 - m_2 \geq m_i^2 - m_i m_1 - m_i m_2.$$

We rewrite the inequality as $(m_1 + m_2)(m_i - 1) \geq m_i(m_i - 1)$. This gives a contradiction to $r_{12} \neq 0$.

**II11**




We estimate roughly and get $L > m_i^2 - m_i m_1 - m_i m_2$.

As described in Remark 1, let the value of $m_2$ increase as long as the cluster stays idealistic.

- Suppose that $r_{12} = 0$. We study if the following inequality can occur:

$$\left(m_i - \sum_{t\geq 0} m_{Q_i(1,3^t)}\right) + (m_i - m_2 - m_6) > m_i^2 - m_i m_1 - m_i m_2.$$

We rewrite it as

$$-\sum_{t\geq 0} m_{Q_i(1,3^t)} - m_2 - m_6 > m_i(m_i - 2 - m_1 - m_2). \tag{5}$$

As $2 + m_1 + m_2 \leq m_i$, this inequality can never hold.

- Suppose that $r_{12} \neq 0$ and that $r_{23} = 0$.Moreover, we can suppose that $r_{13} = 0$ (we let increase the value of $m_1$). We investigate the inequality

$$\left(m_i - m_1 - \sum_{t\geq 1} m_{Q_i(1,2^t)} - m_2 - \sum_{t\geq 1} m_{Q_i(2,1^t)}\right) > m_i^2 - m_i m_1 - m_i m_2.$$

We rewrite the inequality as

$$-\sum_{t\geq 0} m_{Q_i(1,2^t)} - \sum_{t\geq 0} m_{Q_i(2,1^t)} > m_i(m_i - m_1 - m_2 - 1) \tag{6}$$

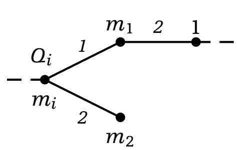
and we see again that this can never happen.

**Remark 2.** If we allow the multiplicities for the constellation II11 to be 0, except for $m_i, m_1$, and $m_2$, and if we also suppose that both $m_3$ and $m_5$ are not 0, then we also have $L > m_i^2 - m_i m_1 - m_i m_2$. For the clusters with underlying constellation II2, II4, II5, II6 or II8, we may suppose that $r_{12} = 0$. It follows then from the inequality (5) that $\chi(E_i^\circ) > 0$ for the clusters with underlying constellations II5 and II8. When $r_{12} \neq 0$, then it follows from inequality (6) that $\chi(E_i^\circ) > 0$ for the clusters with underlying constellation II9 or II10. □

From this remark, it follows that we should study the case $r_{12} = 0$, $m_i < m_1 + m_2 + 2$ for the clusters with underlying constellation II2, II4, II6, II9 or II10. If $m_i = m_1 + m_2$, then we have a cluster whose underlying constellation is a subconstellation of II7, and thus already treated. So suppose that $m_i = m_1 + m_2 + 1$. Then inequality (5) becomes

$$1 - \sum_{t\geq 1} m_{Q_i(1,3^t)} - m_6 > 0.$$

It follows that $\sum_{t\geq 1} m_{Q_i(1,3^t)} = m_6 = 0$ and that the cluster is like



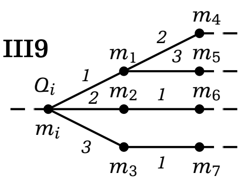

We have $L = m_i^2 - m_2^2 - (m_i - m_2)m_1$.

Can the following inequality hold:

$$m_i - m_1 + m_i - m_2 \geq m_i^2 - m_2^2 - (m_i - m_2)m_1?$$

Substituting $m_i$ by $m_1+m_2+1$, we get $1 \geq 2m_1m_2+m_2$. This contradiction allows us to conclude that $\chi(E_i^\circ) > 0$.

**III9**




A rough estimate gives $L > m_i^2 - m_im_1 - m_2(m_i - \sum_{t\geq 0} m_{Q_i(1,2^t)} - r_{12}) - m_3(m_i - \sum_{t\geq 0} m_{Q_i(1,3^t)} - r_{13})$.

- We let increase the value of $m_1$; suppose that we then get $r_{13}=0$. We also let $m_2$ increase; suppose that $r_{12}$ becomes 0. Can the following inequality then hold:

$$m_i - m_2 - m_3 > m_i^2 - m_im_1 - m_2\left(m_i - \sum_{t\geq 0} m_{Q_i(1,2^t)}\right) - m_3\left(m_i - \sum_{t\geq 0} m_{Q_i(1,3^t)}\right)?$$

  We rewrite the inequality as follows:

$$0 > (m_i - m_1 - 1)(m_i - m_2 - m_3) + m_2\sum_{t\geq 1} m_{Q_i(1,2^t)} + m_3\sum_{t\geq 1} m_{Q_i(1,3^t)}. \tag{7}$$

  This inequality can never be true.

- We let increase $m_1$; suppose that we get $r_{13}=0$. Then we let increase the value of $m_2$ and $r_{23}$ becomes 0: can

$$r_{12} > m_i^2 - m_im_1 - m_2\left(m_i - \sum_{t\geq 0} m_{Q_i(1,2^t)} - r_{12}\right) - m_3\left(m_i - \sum_{t\geq 0} m_{Q_i(1,3^t)}\right)?$$

  As $m_i = m_2 + m_3$, we get

$$r_{12} > r_{12}m_2 + m_2\sum_{t\geq 1} m_{Q_i(1,2^t)} + m_3\sum_{t\geq 1} m_{Q_i(1,3^t)}, \tag{8}$$

  which is never satisfied.

- We let increase $m_1$; suppose that $r_{12}$ becomes 0. Now we let increase $m_3$ and suppose $r_{23}$ becomes 0 (we already treated $r_{12}=r_{13}=0$). Can

$$r_{13} > m_i^2 - m_im_1 - m_2\left(m_i - \sum_{t\geq 0} m_{Q_i(1,2^t)}\right) - m_3\left(m_i - \sum_{t\geq 0} m_{Q_i(1,3^t)} - r_{13}\right)?$$

As $m_i = m_2 + m_3$, we get

$$r_{13} > r_{13}m_3 + m_2 \sum_{t\geq 1} m_{Q_i(1,2^t)} + m_3 \sum_{t\geq 1} m_{Q_i(1,3^t)}, \tag{9}$$

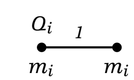
which cannot hold.

**Remark 3.** Notice that one can use the same lower bound for $L$ for the subconstellations IIIx with $1 \leq x \leq 8$ of constellation III9 and that the inequalities (7), (8), and (9) neither hold for them. □

This closes the computational part that yields the classification of the $\chi(E_i^\circ)$. In particular, we get the following results.

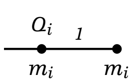
**Theorem 14.** Let $f$ be a polynomial map that is general with respect to a three-dimensional toric idealistic cluster $\mathcal{A} = (\mathcal{C}, \underline{m})$. If $Q_i \in \mathcal{C}$, then $\chi(E_i^\circ) < 0$ if and only if the configuration in $E_i \cong \mathbb{P}^2$ consists of (at least three) lines—possibly exceptional—that are all going through the same point, i.e., if and only if $Q_i$ appears in a subcluster of List 2 in $\mathcal{A}$. □

**C1**



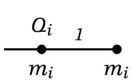

If $Q_i$ is the origin, then $\chi(E_i^\circ) = 2 - m_i$. Thus, if $m_i \geq 3$, then $\chi(E_i^\circ) < 0$.

**C2**




If only label 2 or only label 3 appears under $Q_i$, then $\chi(E_i^\circ) = 1 - m_i$. So, if $m_i \geq 2$, then $\chi(E_i^\circ) < 0$.

**C3**




If only label 2 and label 3 appear under $Q_i$, then $\chi(E_i^\circ) = -m_i$ and thus $\chi(E_i^\circ) < 0$.

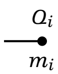
*List* 2

**Example 6.** The surface with equation $x^{2m_i} + y^{m_i} + z^{m_i} = 0$ is an example of a surface that is general with respect to the cluster **C1**. □

In the general case of surfaces, there exist much more configurations that yield a negative $\chi(E_i^\circ)$. Such examples are given in [30].

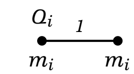
**Theorem 15.** Let $f$ be a polynomial map that is general with respect to a three-dimensional toric idealistic cluster $\mathcal{A} = (\mathcal{C}, \underline{m})$. If $Q_i \in \mathcal{C}$, then $\chi(E_i^\circ) = 0$ if and only if $Q_i$ appears in a subcluster of List 3 in $\mathcal{A}$. $\square$

**C4**



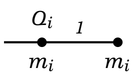

If there exists exactly one or exactly two points to which $Q_i$ is proximate and if $m_i = 1$, then $\chi(E_i^\circ) = 0$.

**C5**



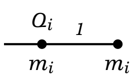

If $Q_i$ is the origin and if $m_i = 2$, then $\chi(E_i^\circ) = 0$.

**C6**



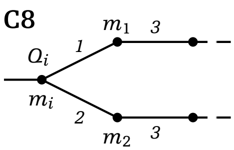

If only label 2 or only label 3 appears under $Q_i$ and if $m_i = 1$, then $\chi(E_i^\circ) = 0$.

**C7**




If at least label 1 appears under $Q_i$, then $\chi(E_i^\circ) = 0$.

**C8**

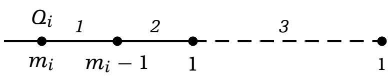



If $m_1 + m_2 = m_i$, if the upper chain and the lower chain are Euclidean clusters and

**A**. if only label 1 or only label 2 appears under $Q_i$, then $\chi(E_i^\circ) = 0$; or

**B**. if at least two different labels appear under $Q_i$, then $\chi(E_i^\circ) = 0$.

**C9**

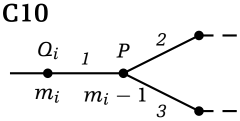



If label 3 appears $m_i - 2$ times and

**A**. if only label 3 appears under $Q_i$, then $\chi(E_i^\circ) = 0$; or

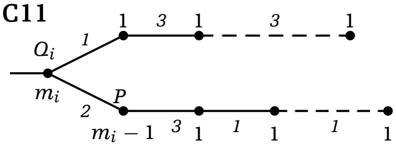
**B**. if only labels 2 and 3 appear under $Q_i$, then $\chi(E_i^\circ) = 0$.

**C10**




If $\#\{s \mid s \in \mathbb{Z}_{\geq 0}, P(2, 3^s) \in \mathcal{C}\} + \#\{s \mid s \in \mathbb{Z}_{\geq 0}, P(3, 2^s) \in \mathcal{C}\} = m_i - 1$ and if only label 2 and label 3 appear under $Q_i$, then $\chi(E_i^\circ) = 0$.

**C11**




If $\#\{s \mid s \in \mathbb{Z}_{\geq 0}, P(3, 1^s) \in \mathcal{C}\} = m_i - 1$, if $\#\{s \mid s \in \mathbb{Z}_{\geq 0}, Q_i(1, 3^s) \in \mathcal{C}\} = k$ for $1 \leq k \leq m_i$ and

**A**. if only label 1 appears under $Q_i$, then $\chi(E_i^\circ) = 0$; or

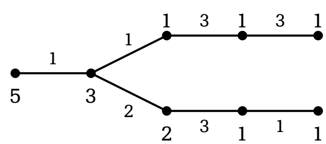
**B**. if only label 1 and label 3 appear under $Q_i$, then $\chi(E_i^\circ) = 0$.

*List* 3

**Example 7.** The ideal

$$I = (x^9, y^5, z^5, x^6y, x^5y^2, x^3y^3, x^2y^4, y^4z, y^3z^2, y^2z^3, yz^4, xz^4,$$
$$x^2z^3, x^5z^2, x^7z, xyz^3, xy^2z^2, xy^3z, x^3yz^2, x^3y^2z, x^5yz)$$

is the complete finitely supported ideal that corresponds to the cluster

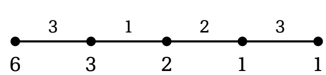



A general element of $I$ illustrates a surface with a singularity as in cluster C8.

Let $J$ be the ideal

$$(x^6, y^6, z^9, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^5z, y^4z^2, y^3z^3, y^2z^5, yz^7, x^5z, x^4z^3, x^3z^4,$$
$$x^2z^6, xz^7, xyz^6, xy^2z^4, xy^3z^2, xy^4z, x^2yz^4, x^2y^2z^2, x^2y^3z, x^3yz^2, x^3y^2z, x^4yz).$$

This is the complete finitely supported ideal corresponding with the cluster



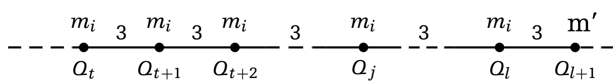

A general element of $J$ illustrates a surface with a singularity as in cluster C9. □

**Remark 4.** Let $Q_l$ be a point with multiplicity 1 in a three-dimensional toric idealistic constellation $\mathcal{C}$ and let $Q_k \in \mathcal{C}$ be such that $Q_l \succ Q_k$. Suppose that $Q_l$ is lying only on the irreducible exceptional component $E_k \cong \mathbb{P}^2$. Then obviously $C_k$ has normal crossings in $Q_l$. Suppose that $Q_l$ is lying on exactly two exceptional components $E_k \cong \mathbb{P}^2$ and $E_j$. If $C_k$ does not have normal crossings in $Q_l$ then $E_k \cap E_j$ should be the tangent line to $C_k$ in $Q_l$. After blowing up in the point $Q_l$, one needs at least one more blowing up to obtain an embedded resolution. By iterating this argument, we can conclude that studying the cluster C9 is enough to know the poles of the topological zeta function associated with the blowing up of the clusters C11. Neither we have to consider the cluster C4 and the cluster C6. □

## 7 The Monodromy Conjecture for Candidate Poles of Order 1

For the sake of completeness, we recall the short proof of the next lemma (see also [18]). Recall that, given a candidate pole $-\nu_j/N_j = a/b$ with $a$ and $b$ coprime, $J_b$ then denotes the subset of indices $\{1 \le i \le r \mid b \text{ divides } N_i\}$.

**Lemma 16.** Let $\chi(E_t^\circ) < 0$ such that we are in the situation




(11)

where $Q_t$ is the point in the chain with the lowest level for which an edge with label 3 is leaving and where $Q_l$ is the point in this chain with the highest level for which its multiplicity is equal to $m_i$.

1. If a set $J_b$ contains the index $t$, then it also contains the indices in $\{t+1, \ldots, l\}$.
2. If $\frac{\nu_l}{N_l} = \frac{c}{d}$ with $c$ and $d$ coprime, then $t \notin J_d$. □

**Proof.** If we denote the numerical data of $E_t$ by $(\nu, N)$, then, independent of the number of points $Q_s$ for which $t \to s$, one easily computes that the numerical data for $i \in \{t+1, \ldots, l\}$ are

$$E_i((i-t+1)\nu - (i-t), (i-t+1)N).$$

Now the first assertion follows immediately.

To see the second claim, suppose that $t \in J_d$. Then $d \mid N$ which implies that

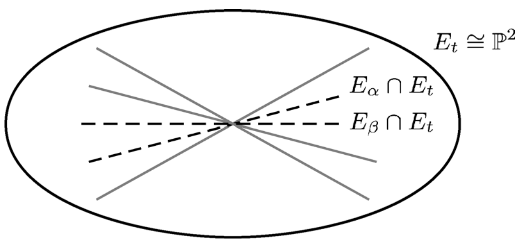
$$l - t + 1 | (l - t + 1)\nu - (l - t).$$

This contradiction closes the proof. ■

We can now prove one of the most important properties concerning the surfaces we study. (We proved this result for a more restricted class of surfaces in [18].)

**Theorem 17.** If $\chi(E_j^\circ) > 0$, then $e^{-2\pi i \frac{\nu_j}{N_j}}$ is an eigenvalue of monodromy of $f$. □

**Proof.** Suppose that $E_j$ is an exceptional component for which $\chi(E_j^\circ) > 0$. To prove that $e^{-2\pi i\nu_j/N_j}$ is an eigenvalue of monodromy of $f$, we show that $e^{-2\pi i\nu_j/N_j}$ is a pole of $\zeta_f$. We write $\nu_j/N_j$ as $a/b$ with $a$ and $b$ coprime. If $J_b$ does not contain an index $t$ for which $\chi(E_t^\circ) < 0$, then there is nothing to verify. So suppose now that $\chi(E_t^\circ) < 0$ and that $t \in J_b$. From Lemma 16 it follows that $E_j \neq E_l$ and that $l \in J_b$. We will show that $\chi(E_t^\circ) + \chi(E_l^\circ) \geq 0$. The configuration in $E_t \cong \mathbb{P}^2$ is as follows:




1. If $Q_t$ is the origin of the constellation, then $\chi(E_t^\circ) = 2 - m_i$. For $\chi(E_l{}^\circ)$ we find that $\chi(E_l^\circ) = D - R + T$ with $D \geq m_i^2 - m_i m'$ and $R \leq 2m_i - 2m'$. We get

$$\begin{aligned}\chi(E_t^\circ) + \chi(E_l^\circ) &\geq 2 - m_i + m_i^2 - m_i m' - 2m_i + 2m' \\ &= 2 + 2m' + m_i(m_i - m' - 2).\end{aligned}$$

   If $m_i - m' \geq 2$, then $\chi(E_t^\circ) + \chi(E_l^\circ) > 0$. If $m_i - m' = 1$, then $\chi(E_t^\circ) + \chi(E_l^\circ) \geq 2 + 2(m_i - 1) - 2m_i = 0$. One can even check that also here $\chi(E_t^\circ) + \chi(E_l^\circ) > 0$. Hence, we always have $\chi(E_t^\circ) + \chi(E_l^\circ) > 0$.
2. If there is exactly one point, say $Q_\alpha$, for which $t \to \alpha$, then $\chi(E_t^\circ) = 1 - m_i$. For $\chi(E_l^\circ)$ we find that $\chi(E_l^\circ) = D - R + T$ with $D \geq m_i^2 - m_i m'$ and $R \leq m_i - m'$ and we obtain

$$\begin{aligned}\chi(E_t^\circ) + \chi(E_l^\circ) &\geq 1 - m_i + m_i^2 - m_i m' - m_i + m' \\ &= 1 + m' + m_i(m_i - m' - 2).\end{aligned}$$

   If $m_i - m' \geq 2$, then $\chi(E_t^\circ) + \chi(E_l^\circ) > 0$. If $m_i - m' = 1$, we get $\chi(E_t^\circ) + \chi(E_l^\circ) \geq 0$.

3. Finally, if there exist two points, say $Q_\alpha$ and $Q_\beta$, for which $t \to \alpha$ and $t \to \beta$, then $\chi(E_t^\circ) = -m_i$. In this case $R = 0$ and we get

$$\begin{aligned} \chi(E_t^\circ) + \chi(E_l^\circ) &\geq -m_i + m_i^2 - m_i m' \\ &= m_i(m_i - m' - 1) \\ &\geq 0. \end{aligned}$$

This study permits us to conclude that $\sum_{i \in J_b} \chi(E_i^\circ) > 0$. Hence, $e^{-2\pi i \frac{\nu_j}{N_j}}$ is an eigenvalue of monodromy of $f$. ■

In the general case of surfaces, it can happen that positive $\chi(E_j^\circ)$ does not imply that $e^{-2\pi i \nu_j/N_j}$ is an eigenvalue of monodromy of $f$.

**Corollary 18.** If $-\nu_j/N_j$ is a candidate pole of $Z_{top,f}$ of order 1 that is a pole, then $e^{-2\pi i \nu_j/N_j}$ is an eigenvalue of monodromy of $f$. □

**Proof.** In [32], it is shown that then there exists an exceptional component $E_k$ for which $\nu_k/N_k = \nu_j/N_j$ and $\chi(E_k^\circ) > 0$. The result follows now from Theorem 17. ■

The second author shows in [32] in particular that if $E_j$ is created by blowing up a point and if $\chi(E_j^\circ) < 0$, then the contribution of $E_j$ to the residue of $-\nu_j/N_j$ for $Z_{top,f}$ is equal to 0. In this particular setting, this is a consequence of Proposition 2.

**Corollary 19.** If $\chi(E_j^\circ) < 0$, then the contribution of $E_j$ to the residue of $-\nu_j/N_j$ for $Z_{top,f}$ is equal to 0. □

**Proof.** Denef and Loeser show in [11] that the poles of $Z_{top,f}$ are of the form $-\nu(a)/N(a)$ where $a$ is orthogonal to a facet of $\mathcal{N}(f)$. The compact facets of $\mathcal{N}(f)$ correspond to the Rees valuations of the complete ideal of hypersurfaces that pass through the points of the constellation with at least the given multiplicity. The result now follows from Proposition 2 and equation (2) in Section 2.3. Indeed, if $\chi(E_i^\circ) < 0$, then $m_i^2 = \sum_{j \to i} m_j^2$. ■

Although the surfaces that we work with are all nondegenerate with respect to their Newton polyhedron, our proof covers many new cases. We recall the numerical conditions that the nondegenerate polynomials should satisfy in the proof of the monodromy conjecture that Loeser gave for them. Suppose that the blowing-ups of $Q_i$ and $Q_j$ give rise to Rees valuations and thus to facets $F_i$ and $F_j$ of the Newton polyhedron.

Suppose that their equations are

$$a_1(F_i)x_1 + a_2(F_i)x_2 + a_3(F_i)x_3 = N_i$$
$$a_1(F_j)x_1 + a_2(F_j)x_2 + a_3(F_j)x_3 = N_j$$

and that these faces have a nonempty intersection. Let $a_{ij}$ be the greatest common divisor of the determinants of the $2 \times 2$-matrices in the matrix

$$\begin{pmatrix} a_1(F_i) & a_2(F_i) & a_3(F_i) \\ a_1(F_j) & a_2(F_j) & a_3(F_j) \end{pmatrix}.$$

Then to be covered by the proof of Loeser, it should hold that

$$\frac{\nu_i - \frac{\nu_j}{N_j} N_i}{a_{ij}} \notin \mathbb{Z} \qquad \text{and} \qquad \nu_i / N_i \notin \mathbb{Z}.$$

Very simple toric clusters, such as the blowing up of two points $Q_1$ and $Q_2$ with multiplicity $m_1 = 6$ and $m_2 = 2$, do not satisfy these conditions. Further, candidate poles of order at least 2 are not included.

## 8 The Monodromy Conjecture for Candidate Poles of Order 2 or 3

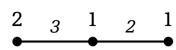
Let us now study when the topological zeta function can have a candidate pole of order at least 2. Suppose a three-dimensional toric idealistic cluster is given and suppose that the blowing up of the cluster provides an embedded resolution for the hypersurface $\{f = 0\}$. Let $s$ be a candidate pole of order at least 2 of the topological zeta function associated with $f$, say $s = -\nu_i/N_i = -\nu_j/N_j$, $1 \le i, j \le r$. We write $s$ as $a/b$ such that $a$ and $b$ are coprime. If $J_b$ is the set $\{j \in \{1, \ldots, r\} \mid b \text{ divides } N_j\}$, then we study when $\sum_{j \in J_b} \chi(E_j^\circ) = 0$. Recall that $e^{2i\pi s}$ is not an eigenvalue of monodromy if this sum is 0.

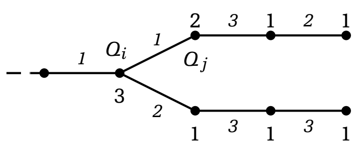
As we are looking for candidate poles of order at least 2 that are poles, it follows that $m_i^2$ should be different from $\sum_{j \to i} m_j^2$ for one of the exceptional components $E_i$ that yield that candidate pole. It follows now from Theorem 17 that we should study two cases. Firstly, there are the clusters with candidate poles of order at least 2 provided by intersecting exceptional components $E_i$ and $E_j$ for which $\chi(E_i^\circ) = \chi(E_j^\circ) = 0$. Secondly, we study the clusters with candidate poles of order at least 2 provided by intersecting exceptional components $E_i$ and $E_j$ for which $\chi(E_i^\circ) = 0$ and $\chi(E_j^\circ) < 0$. In the following subsections we proceed with the study of these cases.

### 8.1 $\chi(E_i^\circ) = \chi(E_j^\circ) = 0$

**Proposition 20.** If $s_0 = -\nu_i/N_i = -\nu_j/N_j$ is a candidate pole of $Z_{top,f}$ of order at least 2 that is a pole, and if $\chi(E_i^\circ) = \chi(E_j^\circ) = 0$, then $e^{2\pi i s_0}$ is an eigenvalue of monodromy of $f$. □

**Proof.** Suppose that $j \to i$. We study the possible combinations from List 3.

- C8A and C9A:
  We can only combine the cluster 
  
  of the form C9A with a cluster of the form C8A and then we get

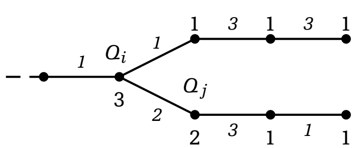
  
  

  Suppose only label 1 appears under $Q_i$.

  If not, the upper chain in C8A would not be a Euclidean cluster. We can write that the numerical data of $E_i$ are equal to $(2i+1, \sum_{l=1}^{i} m_l)$ and that the ones of $E_j$ are equal to $(2i+3, \sum_{l=1}^{i} m_l + 2)$. Hence, if $E_i$ and $E_j$ give rise to the same candidate pole, we should have

  $$\frac{2i+1}{\sum_{l=1}^{i} m_l} = \frac{2i+3}{\sum_{l=1}^{i} m_l + 2}.$$

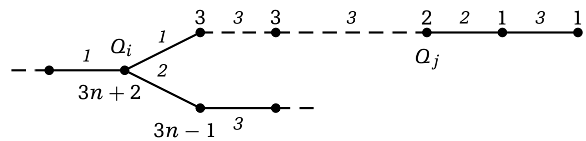
  If this equality holds, then $2i+1 = \sum_{l=1}^{i} m_l \geq 4(i-1)+3$ and then $i$ should be equal to 1. This is a contradiction because $Q_i$ is not the origin.
- C8A and C9B: There are two possibilities.

  1. 
  

  Suppose only label 1 appears under $Q_i$.

  We can write that the numerical data of $E_i$ are equal to $(2i+1, \sum_{l=1}^{i-1} m_l + 3)$. The numerical data of $E_j$ are then equal to $(4i+1, 2\sum_{l=1}^{i-1} m_l + 3 + 2)$. If these exceptional components give rise to the same candidate pole, then we find that

$2i - 2 = \sum_{l=1}^{i-1} m_l \geq 5(i-1)$. This can only hold when $i = 1$ but $Q_i$ is not the origin.

2. 


Suppose only label 1 appears under $Q_i$.

If the numerical data of $E_i$ are $(2i+1, \sum_{l=1}^{i} m_l)$ and if there are $n \geq 1$ points with multiplicity 3 between $Q_i$ and $Q_j$, then the numerical data of $E_j$ are $(2(n+1)i + (2n+3), (n+1)\sum_{l=1}^{i} m_l + (3n+2))$. If $E_i$ and $E_j$ give the same candidate pole, then one should have

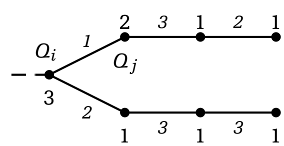
$$6in + 4i + 3n + 2 = (n+2)\sum_{l=1}^{i} m_l \geq (n+2)((i-1)(6n+1) + 3n + 2)$$

or

$$8n + 2i + 3n^2 \geq 7in + 6in^2.$$

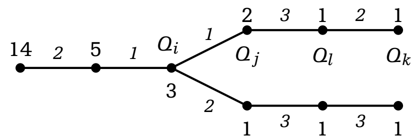
As $i \geq 2$, this inequality can never hold and thus $E_i$ and $E_j$ cannot give rise to the same candidate pole.

- C8B and C9B: Again there are two possibilities.

1. 


Suppose exactly label 1 and label 2 appear under $Q_i$.

In this situation, $E_i$ and $E_j$ can give rise to the same candidate pole, as shown in the following example:




We find $\nu_i/N_i = \nu_j/N_j = -1/4$ and

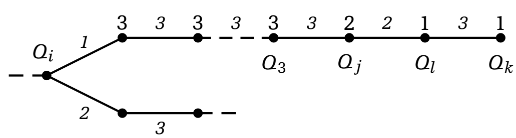
$$Z_{top}(s) = \frac{A(s)}{9(14s+3)(192s+47)(168s+43)(19s+5)(s+1)(103s+25)(4s+1)},$$

with $A$ a polynomial in $s$. However, we have $N_k = 192$ and thus also $k \in J_b$. As $\chi(E_k^\circ) = 1 > 0$, we can conclude that $e^{-2i\pi/4}$ is an eigenvalue of monodromy. This phenomenon is true in general as we will see now.

We call $Q_l := Q_j(3)$ and $Q_k := Q_l(2)$. We show that if $\nu_i/N_i = \nu_j/N_j = a/b$ with $a$ and $b$ coprime, then $b \mid N_k$. Let $Q_2$ be the point with the highest level under $Q_i$ for which $Q_2(2)$ is a point of the constellation. Let $(\nu_2, N_2)$ be the numerical data of the point $Q_2$. Then we have that

$$\begin{aligned}
N_k &= N_i + N_j + N_l + 1 \\
&= N_i + (N_i + N_2 + 2) + (N_i + (N_i + N_2 + 2) + N_2 + 1) + 1 \\
&= 4N_i + 3N_2 + 6 \\
&= N_i + 3N_j.
\end{aligned}$$

Since $b \mid N_i$ and $b \mid N_j$, also $b \mid N_k$. As $\chi(E_k^\circ) = 1 > 0$ and $E_k$ does not play the role of $E_l$ in cluster (11), it follows by the proof of Theorem 17 that $e^{2\pi i s_0}$ is always an eigenvalue of monodromy.

2. 

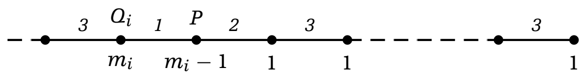

Suppose exactly label 1 and label 3 appear under $Q_i$.

We call $Q_l := Q_j(2)$ and $Q_k := Q_l(3)$. Let $Q_3$ be the point such that $Q_j = Q_3(3)$ and let its associated numerical data be $(\nu_3, N_3)$. Then we get

$$\begin{aligned}
N_k &= N_i + N_j + N_l + 1 \\
&= N_i + (N_i + N_3 + 2) + (N_i + N_3 + (N_i + N_3 + 2) + 1) + 1 \\
&= 4N_i + 3N_3 + 6 \\
&= N_i + 3N_j.
\end{aligned}$$

Again we can conclude that $e^{2\pi i s_0}$ is always an eigenvalue of monodromy.

- C9A and C7:




Suppose that only label 3 appears under $Q_i$.

Let $P := Q_i(1)$ and $Q_j$ be the point $P(2, 3^l)$ with $l \in \{0, 1, \dots, m_i - 3\}$. If the numerical data of $Q_i$ are equal to $(2i + 1, \sum_{s=1}^{i} m_s)$, then we have the following numerical data corresponding to the points

$$P : \left(4i + 1, 2\sum_{s=1}^{i-1} m_s + 2m_i - 1\right),$$

$$P(2) : \left(8i + 1, 4\sum_{s=1}^{i-1} m_s + 3m_i\right),$$

$$P(2, 3^l) : \left(i(8 + 6l) + 2l + 1, (4 + 3l)\sum_{s=1}^{i-1} m_s + (3 + 3l)m_i\right).$$

We check if there exists an $l \in \{0, 1, \dots, m_i - 2\}$ such that

$$\frac{2i + 1}{\sum_{s=1}^{i} m_s} = \frac{i(8 + 6l) + 2l + 1}{(4 + 3l)\sum_{s=1}^{i-1} m_s + (3 + 3l)m_i}.$$

If this equality holds, then

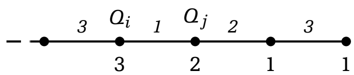
$$\begin{aligned} 2im_i - lm_i - 2m_i &= (l + 3)\sum_{s=1}^{i-1} m_s \\ &\geq (l + 3)(i - 1)(2m_i - 1) \\ &= 2ilm_i + 6im_i - 2lm_i - 6m_i - il - 3i + l + 3. \end{aligned}$$

We rewrite this and obtain

$$3(i - 1) + il \geq (m_i(2i - 1) + 1)l + 4m_i(i - 1).$$

As $3 < 4m_i$ and $i < m_i(2i - 1) + 1$, we get a contradiction. We conclude that $Q_i$ and $Q_j$ cannot give rise to the same candidate pole.

- C9A and C9B:



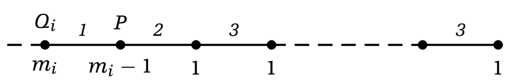

Suppose that only label 3 appears under $Q_i$.

If $E_i$ has numerical data $(2i+1, \sum_{s=1}^{i} m_s)$, then $E_j$ has numerical data $(4i+1, 2\sum_{s=1}^{i-1} m_s + 5)$. If they give rise to the same candidate pole, then one should have

$$2i - 2 = \sum_{s=1}^{i-1} m_s \geq 5(i-1).$$

As $Q_i$ is not the origin, this inequality can never be fulfilled.

- C9B and C7: There are two possibilities.

1. 


Suppose that exactly label 2 and label 3 appear under $Q_i$.

Let $P := Q_i(1)$ with numerical data $(\nu_1, N_1)$ and $Q_j$ be the point $P(2, 3^l)$ with $l \in \{0, 1, \ldots, m_i - 3\}$. Let $Q_3$, resp. $Q_2$, be the point with the highest level such that $i > 3$, resp. $i > 2$, and such that $Q_3(3)$, resp. $Q_2(2)$, is a point of the constellation. We denote its numerical data by $(\nu_3, N_3)$, resp. $(\nu_2, N_2)$. Suppose now that $\nu_i/N_i = \nu_j/N_j = a/b$ with $a$ and $b$ coprime. Let $Q_k := Q_1(2, 3^{k-1})$. We show that $b \mid N_k$ when $k > j$.
We have that

$$\begin{aligned} N_i &= N_3 + N_2 + m_i \quad \text{and} \\ N_1 &= 2N_2 + 2N_3 + 2m_i - 1 = 2N_i - 1. \end{aligned}$$

If $Q_j = Q_i(1, 2)$, then $N_j = N_3 + N_i + N_1 + 1$ and so

$$\begin{aligned} N_k &= (k-1)N_i + (k-1)N_1 + N_j + (k-1) \\ &= (k-1)N_i + (k-1)(2N_i - 1) + N_j + (k-1) \\ &= 3(k-1)N_i + N_j \end{aligned}$$

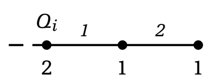
and we can conclude that $b \mid N_k$.

If $Q_j = Q_i(1, 2, 3^l)$ for $l \neq 0$, then $N_j = lN_i + lN_1 + l + N_3 + N_i + N_1 + 1 = (l+1)N_i + (l+1)N_1 + N_3 + (l+1) = (l+1)N_i + (l+1)(2N_i - 1) + N_3 + (l+1) = 3(l+1)N_i + N_3$ and so

$$\begin{aligned} N_k &= (k-1)N_i + (k-1)N_1 + (N_3 + N_i + N_1 + 1) + (k-1) \\ &= (k-1)N_i + (k-1)(2N_i - 1) + (N_3 + N_i + 2N_i) + (k-1) \\ &= 3kN_i + N_3. \end{aligned}$$

As $b \mid N_i$ and $b \mid N_j$, we have that also $b \mid N_3$ and so $b \mid N_k$. As $\chi(E_k^\circ) = 1 > 0$ for $k = m_i - 1$ and $Q_{m_i-1}$ cannot play the role of $Q_l$ in cluster (11), it follows that $e^{2\pi i s_0}$ is an eigenvalue of monodromy.

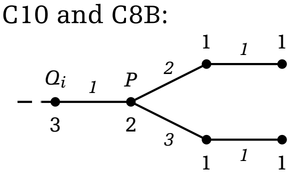
2. In the previous cluster, $Q_j$ can also be $Q_1$, but then $m_i$ should be equal to 2.




Suppose that exactly label 2 and label 3 appear under $Q_i$.

We then have that $\nu_i/N_i = \nu_1/N_1 = (2\nu_i - 1)/(2N_i - 1)$ if and only if $\nu_i/N_i = 1$. As 1 is always an eigenvalue of monodromy, this cluster does not pose any problem.

- C10 and C7:
  This case is completely analogous to the combination C9B and C7.
- C10 and C8B:




Only label 2 and label 3 appear under $Q_i$.

Let $Q_3$, resp. $Q_2$, be the point with the highest level such that $i > 3$, resp. $i > 2$, and such that $Q_3(3)$, resp. $Q_2(2)$, is a point of the constellation. We denote its numerical data by $(\nu_3, N_3)$, resp. $(\nu_2, N_2)$. Then we have that

$$\begin{aligned} N_i &= N_2 + N_2 + 3 \\ N_j &= N_i + N_2 + N_3 + 2 = 2N_i - 1 \\ \nu_i &= \nu_2 + \nu_3 + 1 \\ \nu_j &= \nu_i + \nu_2 + \nu_3 = 2\nu_i - 1. \end{aligned}$$

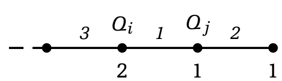
Hence, if $\nu_i/N_i = \nu_j/N_j$, then $-\nu_i/N_i = -1$ and 1 is always an eigenvalue of monodromy. ■

## 8.2 $\chi(E_i^\circ) = 0$ and $\chi(E_j^\circ) < 0$

**Proposition 21.** If $s_0 = -\nu_i/N_i = -\nu_j/N_j$ is a candidate pole of $Z_{top,f}$ of order at least 2 that is a pole, and if $\chi(E_i^\circ) = 0$ and $\chi(E_j^\circ) < 0$, then $e^{2\pi i s_0}$ is an eigenvalue of monodromy of $f$. □

**Proof.** We take List 2 and List 3 and look for the combinations that are possible to obtain $\sum_{k\in J_b} \chi(E_k^\circ) = 0$. Recall that we proved in Theorem 17 that $\sum_{k\in J_b} \chi(E_k^\circ) = 0$ implies that the value of $m'$ in cluster (11) should be equal to $m_i - 1$. The only possible combination where at least $\nu_i$ or at least $\nu_j$ is Rees, is the following one.

- C9A and C3:

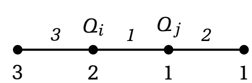



Suppose that only label 3 appears under $Q_i$.

If the numerical data of $E_i$ are equal to $(2i + 1, \sum_{s=1}^{i} m_s)$, then the ones of $Q_j$ are equal to $(4i + 1, 2\sum_{s=1}^{i-1} m_s + 2 + 1)$. If $E_i$ and $E_j$ give rise to the same candidate pole, then one should have

$$2i - 1 = \sum_{s=1}^{i-1} m_s \geq 3(i - 1)$$

which can only be true if $i = 2$, and if the multiplicity of the origin is 3. Then we have the cluster




The candidate pole provided by $E_i$ and $E_j$ is then equal to $-1$. Remember that 1 is an eigenvalue of monodromy. ■

Hence, we can conclude with the following result.

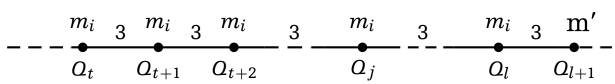
**Theorem 22.** If $s_0$ is a candidate pole of $Z_{top,f}$ of order at least 2 that is a pole, then $e^{2\pi i s_0}$ is an eigenvalue of monodromy of $f$. □

## 9 The Holomorphy Conjecture

To prove the holomorphy conjecture, we first prove the following lemma. It gives us a set of orders of eigenvalues of monodromy.

**Lemma 23.** If $\chi(E_j^\circ) > 0$, then $e^{2\pi i/N_j}$ is an eigenvalue of monodromy of $f$ at some point of the hypersurface $f = 0$. □

**Proof.** To prove that $e^{2\pi i/N_j}$ is an eigenvalue of monodromy, we will show that $\sum_{N_j|N_i} \chi(E_i^\circ) \neq 0$. So suppose that $N_j \mid N_t$ and $\chi(E_t^\circ) < 0$. Then we are in the situation



(11)

where $Q_t$ is the point in the chain with the lowest level for which an edge with label 3 is leaving and where $Q_l$ is the point in this chain with the highest level for which its multiplicity is equal to $m_i$.

In Lemma 16 we proved that then also $N_j \mid N_i$, for $i \in \{j+1,\dots,l\}$. As $N_l > N_t$, it follows that $N_l \nmid N_t$, and hence $E_j \neq E_l$. In Theorem 17, we proved that $\chi(E_t^\circ) + \chi(E_l^\circ) \geq 0$, and thus we obtain $\sum_{N_j|N_i} \chi(E_i^\circ) > 0$. ■

**Theorem 24.** If $r \in \mathbb{Z}_{>0}$ does not divide the order of any eigenvalue of monodromy of $f$ at some point of the hypersurface $f = 0$, then $Z_{top,f}^{(r)}$ is holomorphic on $\mathbb{C}$. □

**Proof.** Suppose that $Z_{top,f}^{(r)}$ is not holomorphic, hence has a pole, say $s_0$. Let $E_i$ be an exceptional component that gives rise to this pole of $Z_{top,f}^{(r)}$ and let $(\nu_i, N_i)$ be its numerical data. If $\chi(E_i^\circ) > 0$, then it follows from Lemma 23 that there is an eigenvalue of monodromy of order $N_i$. This contradicts the given condition on $r$.

If $\chi(E_i^\circ) < 0$, then we can set $E_i = E_t$ as in the cluster above. Thus, we also have $r \mid N_l$. However, as $\chi(E_l^\circ) > 0$, it follows that $N_l$ is the order of an eigenvalue of monodromy.

This implies that if $r \mid N_i$, then $\chi(E_i^\circ) = 0$. If all these components are disjoint, then we get $Z_{top,f}^{(r)} = 0$. We may now suppose that at least two such components intersect each other, and that at least one of them is Rees (it is shown in [11] that only facets in the Newton polyhedron can give rise to poles of $Z^{(r)}_{top,f}$). Then our cluster must contain one of the following combinations of subclusters (see also Section 8.1.).

- C8A and C9A: We computed $N_j = N_i + 2$, hence if $r \mid N_i$ and $r \mid N_j$, then $r \mid 2$. Set $Q_k := Q_j(3, 2)$, then $N_k = 4N_i + 6$ and $\chi(E^\circ_k) > 0$. Lemma 23 tells us that $N_k$ is the order of an eigenvalue of monodromy, which contradicts the choice of $r$.
- C8A and C9B:
    1. We obtained $N_j = 2N_i - 1$. If $r \mid N_i$ and $r \mid N_j$, then $r = 1$, which divides the order of any eigenvalue of monodromy.
    2. We had $N_j = (n + 1)N_i + (3n + 2)$. Set $Q_k := Q_j(2, 3)$, then $N_k = (3n + 4)N_i + 9n + 6$. If $r$ divides $N_i$ and $N_j$, then it follows that $r$ also divides $N_k$. As $\chi(E^\circ_k) > 0$, we can conclude by Lemma 23 that there is an eigenvalue of order $N_k$. Again we get a contradiction.
- C8B and C9B: Let $Q_k := Q_j(3, 2)$ as in that cluster in Section 8.1. We found $N_k = N_i + 3N_j$. Analogously, we find that $E_i$ and $E_j$ do not give rise to poles of $Z^{(r)}_{top,f}$, if $r \mid N_i$ and $r \mid N_j$.

  Also the other combination of C8B and C9B in Section 8.1 gives this contradiction.
- C9A and C7: For $Q_j = P(2, 3^l)$, we computed $N_j = (4 + 3l)N_i - m_i$. So if $r \mid N_i$ and $r \mid N_j$, then $r \mid m_i$. Let $Q_k := P(2, 3^{m_i-2})$ be the maximal point. Then $N_k = (4 + 3k)N_i - m_i$, hence $r \mid N_k$, but as $\chi(E^\circ_k) > 0$, we get a contradiction.
- C9A and C9B: In this cluster, we had $N_j = 2N_i - 1$, but then $r$ should be equal to 1.
- C9B and C7: Again we can use the maximal point $Q_k := P(2, 3^{m_i-2})$. In Section 8.1, we found that $\chi(E^\circ_k) > 0$ and if $r$ divides $N_i$ and $N_j$, that $r$ then also divides $N_k$.
- C10 and C7: This case is exactly the same as the previous one.
- C10 and C8B: We found that $N_j = 2N_i - 1$; thus it follows that when $r$ divides $N_i$ and $N_j$, then $r = 1$.

Hence, we find that $Z^{(r)}_{top,f}$ can neither have a pole coming from an exceptional component for which $\chi(E^\circ_i) = 0$. This ends the proof. ■

Notice that if $r \mid N_i$ and $r \mid N_j$ with $\chi(E^\circ_i) = \chi(E^\circ_j) = 0$ and $E_i \cap E_j \neq \emptyset$, then we found that $r = 1$ or that there exists another component $E_k$ with $r \mid N_k$ and $\chi(E^\circ_k) > 0$. For general surfaces, such a component $E_k$ does not necessarily exist.

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