REGULAR COVERS OF OPEN RELATIVELY COMPACT SUBANALYTIC SETS

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Abstract. Let $U$ be an open relatively compact subanalytic subset of a real analytic manifold. We show that there exists a finite linear covering (in the sense of Guillermou and Schapira) of $U$ by subanalytic open subsets of $U$ homeomorphic to a unit ball.

We also show that the algebra of open relatively compact subanalytic subsets of a real analytic manifold is generated by subsets subanalytically and bi-lipschitz homeomorphic to a unit ball.

Let $M$ be a real analytic manifold of dimension $n$. In this paper we study the algebra $S(M)$ of relatively compact open subanalytic subsets of $M$. As we show this algebra is generated by sets with Lipschitz regular boundaries. More precisely, we call a relatively compact open subanalytic subset $U \subset M$ an open subanalytic Lipschitz ball if its closure is subanalytically bi-Lipschitz homeomorphic to the unit ball of $\mathbb{R}^n$. Here we assume that $M$ is equipped with a Riemannian metric. Any two such metrics are equivalent on relatively compact sets and hence the above definition is independent of the choice of a metric.

Theorem 0.1. The algebra $S(M)$ is generated by open subanalytic Lipschitz balls.

That is to say if $U$ is a relatively compact open subanalytic subset of $M$ then the characteristic function $1_U$ is a linear combination of functions of the form $1_{W_1}, \ldots, 1_{W_m}$, where the $W_j$ are open subanalytic Lipschitz balls. Note that, in general, $U$ cannot be covered by subanalytic Lipschitz balls, as it is easy to see for $\{(x,y) \in \mathbb{R}^2; y^2 < x^3, x < 1\}$, $M = \mathbb{R}^2$, due to the presence of cusps. Nevertheless we show the existence of a ”regular” cover in the sense that we control the distance to the boundary.

Theorem 0.2. Let $U \in S(M)$. Then there exist a finite cover $U = \bigcup_i U_i$ by open subanalytic sets such that:

1. every $U_i$ is subanalytically homeomorphic to an open $n$-dimensional ball;
2. there is $C > 0$ such that for every $x \in U$, $\text{dist}(x, M \setminus U) \leq C \max_i \text{dist}(x, M \setminus U_i)$

The proofs of Theorems 0.1 and 0.2 are based on the regular projection theorem, cf. [6], [7], [8], the classical cylindrical decomposition, and the L-regular decomposition of subanalytic sets, cf. [4], [8], [9]. L-regular sets are natural multidimensional generalization of classical cusps. We recall them briefly in Subsection 1.6. We show also the following strengthening of Theorem 0.2.

Theorem 0.3. In Theorem 0.2 we may require additionally that all $U_i$ are open L-regular cells (i.e. interiors of L-regular sets).

For an open $U \subset M$ we denote $\partial U = \overline{U} \setminus U$. 

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1. Proofs

1.1. Reduction to the case $M = \mathbb{R}^n$. Let $U \in \mathcal{S}(M)$. Choose a finite covering $\overline{U} \subset \bigcup_i V_i$ by open relatively compact sets such that for each $V_i$ there is an open neighborhood of $\overline{V_i}$ analytically diffeomorphic to $\mathbb{R}^n$. Then there are finitely many open subanalytic $U_{ij}$ such that $U_{ij} \subset V_i$ and $1_U$ is a combination of $1_{U_{ij}}$. Thus it suffices to show Theorem 0.1 for relatively compact open subanalytic subsets of $\mathbb{R}^n$.

Similarly, it suffices to show Theorems 0.2 and 0.3 for $M = \mathbb{R}^n$. Indeed, it follows from the observation that the function

$$x \to \max_i \text{dist}(x, M \setminus V_i)$$

is continuous and nowhere zero on $\bigcup_i V_i$ and hence bounded from below by a nonzero constant $c > 0$ on $\overline{U}$. Then

$$\text{dist}(x, M \setminus U) \leq C_1 \leq c^{-1}C_1 \max_i \text{dist}(x, M \setminus V_i)$$

where $C_1$ is the diameter of $\overline{U}$ and hence, if $c^{-1}C_1 \geq 1$,

$$\text{dist}(x, M \setminus U) \leq c^{-1}C_1 \max_i \text{dist}(x, M \setminus V_i) \leq C c^{-1}C_1 \max_{ij} \text{dist}(x, M \setminus U_{ij}).$$

Thus the cover $U_{ij}$ satisfies the claim of Theorem 0.2, resp. Theorem 0.3.

1.2. Regular projections. We recall after [7], [8] the subanalytic version of the regular projection theorem of T. Mostowski introduced originally in [6] for complex analytic sets germs.

Let $X \subset \mathbb{R}^n$ be subanalytic. For $\xi \in \mathbb{R}^{n-1}$ we denote by $\pi_{\xi} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the linear projection parallel to $(\xi, 1) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Fix constants $C, \varepsilon > 0$. We say that $\pi = \pi_{\xi}$ is $(C, \varepsilon)$-regular at $x_0 \in \mathbb{R}^n$ (with respect to $X$) if

(a) $\pi|_X$ is finite;

(b) the intersection of $X$ with the open cone

\[
C_\varepsilon(x_0, \xi) = \{x_0 + \lambda(\eta, 1); |\eta - \xi| < \varepsilon, \lambda \in \mathbb{R} \setminus 0\}
\]

is empty or a finite disjoint union of sets of the form

\[
\{x_0 + \lambda_i(\eta)(\eta, 1); |\eta - \xi| < \varepsilon\},
\]

where $\lambda_i$ are real analytic nowhere vanishing functions defined on $|\eta - \xi| < \varepsilon$.

(c) the functions $\lambda_i$ from (b) satisfy for all $|\eta - \xi| < \varepsilon$

$$\|\text{grad } \lambda_i(\eta)\| \leq C|\lambda_i(\eta)|,$$

We say that $\mathcal{P} \subset \mathbb{R}^{n-1}$ defines a set of regular projections for $X$ if there exists $C, \varepsilon > 0$ such that for every $x_0 \in \mathbb{R}^n$ there is $\xi \in \mathcal{P}$ such that $\pi_{\xi}$ is $(C, \varepsilon)$-regular at $x_0$. 


Theorem 1.1 ([7], [8]). Let \( X \) be a compact subanalytic subset of \( \mathbb{R}^n \) such that \( \dim X < n \). Then the generic set of \( n + 1 \) vectors \( \xi_1, \ldots, \xi_{n+1} \), \( \xi_i \in \mathbb{R}^{n-1} \), defines a set of regular projections for \( X \).
(Here by generic we mean in the complement of a subanalytic nowhere dense subset of \( (\mathbb{R}^{n-1})^{n+1} \).)

1.3. Cylindrical decomposition. We recall the first step of a basic construction, the cylindrical algebraic decomposition, for details see for instance [2], [3].

Set \( X = \overline{U} \setminus U \). Then \( X \) is a compact subanalytic subset of \( \mathbb{R}^n \) of dimension \( n-1 \). We denote by \( Z \subset X \) the set of singular points of \( X \) that is the complement in \( X \) of the set

\( \text{Reg}(X) := \{ x \in X; (X, x) \) is the germ of a real analytic submanifold of dimension \( n-1 \}. \)

Then \( Z \) is closed in \( X \), subanalytic and \( \dim Z \leq n-2 \).

Assume that the standard projection \( \pi: \mathbb{R}^n \to \mathbb{R}^{n-1} \) restricted to \( X \) is finite. Denote by \( \Delta_\pi \subset \mathbb{R}^{n-1} \) the union of \( \pi(Z) \) and the set of critical values of \( \pi|_{\text{Reg}(X)} \). Then \( \Delta_\pi \), called the discriminant set of \( \pi \), is compact and subanalytic. It is clear that \( \pi(U) = \pi(U) \cup \Delta_\pi \).

Proposition 1.2. Let \( U' \subset \pi(U) \setminus \Delta_\pi \) be open and connected. Then there are finitely many bounded real analytic functions \( \varphi_1 < \varphi_2 < \cdots < \varphi_k \) defined on \( U' \), such that \( X \cap \pi^{-1}(U') \) is the union of graphs of \( \varphi_i \)'s. In particular, \( U \cap \pi^{-1}(U') \) is the union of the sets

\[ \{(x', x_n) \in \mathbb{R}^n; x' \in U', \varphi_1(x') < x_n < \varphi_{i+1}(x') \}, \]

and moreover, if \( U' \) is subanalytically homeomorphic to an open \((n-1)\)-dimensional ball, then each of these sets is subanalytically homeomorphic to an open \( n \)-dimensional ball.

1.4. The case of a regular projection. Fix \( x_0 \in U \) and suppose that \( \pi: \mathbb{R}^n \to \mathbb{R}^{n-1} \) is \((C, \varepsilon)\)-regular at \( x_0 \in \mathbb{R}^n \) with respect to \( X \). Then the cone (1.1) contains no point of \( Z \). By [8] Lemma 5.2, this cone contains no critical point of \( \pi|_{\text{Reg}(X)} \), provided \( \varepsilon \) is chosen sufficiently small (for fixed \( C \)). In particular, \( x'_0 = \pi(x_0) \notin \Delta_\pi \).

In what follows we fix \( C, \varepsilon > 0 \) and suppose \( \varepsilon \) small. We denote the cone (1.1) by \( \mathcal{C} \) for short. Then for \( \tilde{C} \) sufficiently large, that depends only on \( C \) and \( \varepsilon \), we have

\[ \text{dist}(x_0, X \setminus \mathcal{C}) \leq \tilde{C} \text{dist}(x'_0, \pi(X \setminus \mathcal{C})) \leq \tilde{C} \text{dist}(x'_0, \Delta_\pi). \]

The first inequality is obvious, the second follow from the fact that the singular part of \( X \) and the critical points of \( \pi|_{\text{Reg}(X)} \) are both outside the cone.

1.5. Proof of Theorem 0.2. Induction on \( n \). Set \( X = \overline{U} \setminus U \) and let \( \pi_{\xi_1}, \ldots, \pi_{\xi_{n+1}} \) be a set of \((C, \varepsilon)\)-regular projections with respect to \( X \). To each of these projections we apply the cylindrical decomposition. More precisely, let us fix one of these projections that for simplicity we suppose standard and denote it by \( \pi \). Then we apply the inductive assumption to \( \pi(U) \setminus \Delta_\pi \). Thus let \( \pi(U) \setminus \Delta_\pi = \bigcup U'_i \) be a finite cover satisfying the statement of Theorem 0.2. Applying to each \( U'_i \) Proposition 1.2 we obtain a family of cylinders that cover \( U \setminus \pi^{-1}(\Delta_\pi) \). In particular they cover the set of those points of \( U \) at which \( \pi \) is \((C, \varepsilon)\)-regular.

Lemma 1.3. Suppose \( \pi \) is \((C, \varepsilon)\)-regular at \( x_0 \in U \). Let \( U' \) be an open subanalytic subset of \( \pi(U) \setminus \Delta_\pi \) such that \( x'_0 = \pi(x_0) \in U' \) and

\[ \text{dist}(x'_0, \Delta_\pi) \leq \tilde{C} \text{dist}(x'_0, \partial U'). \]
with $\bar{C} \geq 1$ for which (1.2) holds. Then

\begin{equation}
\text{dist}(x_0, X) \leq (\bar{C})^2 \text{dist}(x_0, \partial U_1),
\end{equation}

where $U_1$ is the member of cylindrical decomposition of $U \cap \pi^{-1}(U')$ containing $x_0$.

\textbf{Proof.} We decompose $\partial U_1$ into two parts. The first one is vertical, i.e. contained in $\pi^{-1}(\partial U')$ and the second part is contained in $X$. The distance to the first one from $x_0$ equals to the horizontal distance, that is $\text{dist}(x_0', \partial U')$. Thus we have

\begin{equation}
\text{dist}(x_0, \partial U_1) = \min \{ \text{dist}(x_0, X), \text{dist}(x_0', \partial U') \}.
\end{equation}

If $\text{dist}(x_0, \partial U_1) = \text{dist}(x_0, X)$ then (1.4) holds with $\bar{C} = 1$, otherwise $\text{dist}(x_0, \partial U_1) = \text{dist}(x_0', \partial U') \leq \text{dist}(x_0, X)$ and then by (1.3) and (1.2)

\begin{equation}
\text{dist}(x_0, X \setminus \mathcal{C}) \leq \bar{C} \text{dist}(x_0', \Delta_{\pi}) \leq (\bar{C})^2 \text{dist}(x_0', \partial U') \leq (\bar{C})^2 \text{dist}(x_0, X).
\end{equation}

Thus to complete the proof of Theorem 0.2 it suffices to show that the assumptions of Lemma 1.3 are satisfied. This follows from the inclusion $\partial \pi(U) \subset \Delta_{\pi}$ that gives $\text{dist}(x_0', \Delta_{\pi}) \leq \text{dist}(x_0', \partial \pi(U))$, and from $\text{dist}(x_0', \partial \pi(U)) \leq \bar{C} \text{dist}(x_0', \partial U')$ that holds by the inductive assumption. This ends the proof of Theorem 0.2.

1.6. \textbf{L-regular sets.} Let $Y \subset \mathbb{R}^n$ be subanalytic, $\dim Y = n$. Then $Y$ is called \textit{L-regular (with respect to given system of coordinates)} if

(1) if $n = 1$ then $Y$ is a non-empty closed bounded interval;

(2) if $n > 1$ then $Y$ is of the form

\begin{equation}
Y = \{ (x', x_n) \in \mathbb{R}^n; f(x') \leq x_n \leq g(x'), x' \in Y' \},
\end{equation}

where $Y' \subset \mathbb{R}^{n-1}$ is L-regular, $f$ and $g$ are continuous subanalytic functions defined in $Y'$. It is also assumed that on the interior of $Y$, $f$ and $g$ are analytic, satisfy $f < g$, and have bounded first order partial derivatives.

If $\dim Y = k < n$ then we say that $Y$ is \textit{L-regular (with respect to given system of coordinates)} if

\begin{equation}
Y = \{ (y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}; z = h(y), y \in Y' \},
\end{equation}

where $Y' \subset \mathbb{R}^k$ is L-regular, $\dim Y' = k$, $h$ is a continuous subanalytic map defined on $Y'$, such that $h$ is real analytic on the interior of $Y$, and has the first order partial derivatives bounded.

We say that $Y$ is \textit{L-regular} if it is L-regular with respect to a linear (or equivalently orthogonal) system of coordinates on $\mathbb{R}^n$.

We say that $A \subset \mathbb{R}^n$ is an \textit{L-regular cell} if $A$ is the relative interior of an L-regular set. That is, it is the interior of an L-regular set if $\dim A = n$, and it is the graph of $h$ restricted to $\text{Int}(Y')$ for an L-regular set of the form (1.8). By convention, every point is a zero-dimensional L-regular cell.

By [4], see also Lemma 2.2 of [8] and Lemma 1.1 of [5], L-regular sets and L-regular cells satisfy the following property, called in [4] quasi-convexity. We say that $Z \subset \mathbb{R}^n$ is \textit{quasi-convex} if there is a constant $C > 0$ such that every two points $x, y$ of $Z$ can be connected in
Z by a continuous subanalytic arc of length bounded by $C\|x-y\|$. It can be shown that for an L-regular set or cell $Y$ the constant $C$ depends only on $n$ and the bounds on first order partial derivatives of functions describing $Y$ in the above definition. By Lemma 2.2 of [8], an L-regular cell is homeomorphic to the (open) unit ball.

Let $Y$ be a subanalytic subset of a real analytic manifold $M$. We say that $Y$ is $L$-regular if there exists its neighborhood $V$ in $M$ and an analytic diffeomorphism $\varphi : V \to \mathbb{R}^n$ such that $\varphi(Y)$ is $L$-regular. Similarly we define an $L$-regular cell in $M$.

1.7. Proof of Theorem 0.3. Fix a constant $C_1$ sufficiently large and a projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ that is assumed, for simplicity, to be the standard one. We suppose that $\pi$ restricted to $\dot{X} = \partial U$ is finite. We say that $x' \in \pi(U) \setminus \Delta_\pi$ is $C_1$-regularly covered if there is a neighborhood $\dot{U}'$ of $x'$ in $\pi(U) \setminus \Delta_\pi$ such that $X \cap \pi^{-1}(\dot{U}')$ is the union of graphs of analytic functions with all first order partial derivatives bounded (in the absolute value) by $C_1$. Denote by $U'(C_1)$ the set of all $x' \in \pi(U) \setminus \Delta_\pi$ that are $C_1$ regularly covered. Then $U'(C_1)$ is open (if we use strict inequalities while defining it) and subanalytic. By Lemma 5.2 of [8], if $\pi$ is a $(C,\varepsilon)$-regular projection at $x_0$ then $x'_0$ is $C_1$-regularly covered, for $C_1$ sufficiently big $C_1 \geq C_1(C,\varepsilon)$.

Moreover we have the following result.

Lemma 1.4. Given positive constants $C,\varepsilon$. Suppose that the constants $C$ and $C_1$ are chosen sufficiently big, $C_1 \geq C_1(C,\varepsilon), C \geq C(C,\varepsilon)$. Let $\pi$ be $(C,\varepsilon)$-regular at $x_0 \notin X$ and let $V' = \{x' \in \mathbb{R}^{n-1}; \text{dist}(x',x_0') < (\tilde{C})^{-1}\text{dist}(x_0, X \cap \mathcal{C})\}$. Then $\pi^{-1}(V') \cap X \cap \mathcal{C}$ is the union of graphs of $\varphi_i$ with all first order partial derivatives bounded (in the absolute value) by $C_1$. Moreover, then either $\pi^{-1}(V') \cap (X \setminus \mathcal{C}) = \emptyset$ or $\text{dist}(x_0',\Delta_\pi) = \text{dist}(x_0', \pi(X \setminus \mathcal{C})) \leq \text{dist}(x_0', \partial U'(C_1))$.

Proof. We only prove the second part of the statement since the first part follows from Lemma 5.2 of [8]. If $\pi^{-1}(V') \cap X \setminus \mathcal{C} \neq \emptyset$ then any point of $\pi(X \setminus \mathcal{C})$ realizing $\text{dist}(x_0', \pi(X \setminus \mathcal{C}))$ must be in the discriminant set $\Delta_\pi$.

We now apply to $U'(C_1)$ the inductive hypothesis and thus assume that $U'(C_1) = \bigcup U'_i$ is a finite regular cover by open L-regular cells. Fix one of them $U'$ and let $U_1$ be a member of the cylindrical decomposition of $U \cap \pi^{-1}(U')$. Then $U_1$ is an L-regular cell. Let $x_0 \in U_1$. We apply to $x_0$ Lemma 1.4.

If $\pi^{-1}(V') \cap (X \setminus \mathcal{C}) = \emptyset$ then

$$\text{dist}(x_0, X) \leq \text{dist}(x_0, X \cap \mathcal{C}) \leq \tilde{C} \text{dist}(x_0', \partial U'(C_1)) \leq \tilde{C}^2 \text{dist}(x_0', \partial U'),$$

where the second inequality follows from the first part of Lemma 1.4 and the last inequality by the induction hypothesis. Then dist$(x_0, X) \leq \tilde{C}^2 \text{dist}(x_0, \partial U_1)$ follows from (1.5).

Otherwise, $\text{dist}(x_0',\Delta_\pi) \leq \text{dist}(x_0', \partial U'(C_1)) \leq \tilde{C} \text{dist}(x_0', \partial U')$ and the claim follows from Lemma 1.3. This ends the proof.

1.8. Proof of Theorem 0.1. The proof is based on the following result.

Theorem 1.5. [Theorem A of [4]] Let $Z_i \subset \mathbb{R}^n$ be a finite family of subanalytic sets. Then there is be a finite disjoint collection $\{A_j\}$ of L-regular cells such that each $Z_i$ is the disjoint union of some of $A_j$. 

Similar results in the (more general) o-minimal set-up are proven in [5] and [9].

Let \( U \) be a relatively compact open subanalytic subset of \( \mathbb{R}^n \). By Theorem 1.5, \( U \) is a disjoint union of \( L \)-regular cells and hence it suffices to show the statement of Theorem 0.1 for a relatively compact, not necessarily open, \( L \)-regular cell. We consider first the case of an open \( L \)-regular cell. Thus suppose that

\[
U = \{ (x', x_n) \in \mathbb{R}^n; f(x') < x_n < g(x'), x' \in U' \},
\]

where \( U' \) is a relatively compact \( L \)-regular cell, \( f \) and \( g \) are subanalytic and analytic functions on \( U' \) with the first order partial derivatives bounded. Then, by the quasi-convexity of \( U' \), \( f \) and \( g \) are Lipschitz. Thanks to the classical result of Banach, cf. [1] (7.5) p. 122, we may suppose that \( f \) and \( g \) are restrictions to \( U' \) of Lipschitz subanalytic functions, denoted also by \( f \) and \( g \), defined everywhere on \( \mathbb{R}^{n-1} \) and satisfying \( f \leq g \). Indeed, Banach gives the following formula for such an extension of a Lipschitz function \( f \) defined on a subset of a metric space

\[
\tilde{f}(p) = \sup_{q \in B} f(q) - L \|p - q\|,
\]

where \( L \) is the Lipschitz constant of \( f \). Then \( \tilde{f} \) is Lipschitz with the same constant as \( f \) and subanalytic if so was \( f \). By the inductive assumption on dimension we may assume that \( U \) is given by (1.9) with \( U' \) an \( L \)-regular cell. Denote \( U \) by \( U_{f,g} \) to stress its dependence on \( f \) and \( g \) (with \( U' \) fixed). Then

\[
1_{U_{f,g}} = 1_{U_{f-1,g}} + 1_{U_{f,g+1}} - 1_{U_{f-1,g+1}}
\]

and \( U_{f-1,g}, U_{f,g+1}, \) and \( U_{f-1,g+1} \) are open subanalytic Lipschitz balls.

Suppose now that

\[
U = \{ (y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}; z = h(y), y \in U' \},
\]

where \( U' \) is a relatively compact open \( L \)-regular cell of \( \mathbb{R}^k \), \( h \) is a subanalytic and analytic map defined on \( U' \) with the first order partial derivatives bounded. Hence \( h \) is Lipschitz. We may again assume that \( h \) is the restriction of a Lipschitz subanalytic map \( h : \mathbb{R}^k \to \mathbb{R}^{n-k} \) and then, by the inductive hypothesis, that \( U' \) is a subanalytic Lipschitz ball. Let

\[
U_{\emptyset} = \{ (y, z) \in U' \times \mathbb{R}^{n-k}; h_i(y) - 1 < z_i < h_i(y) + 1, i = 1, ..., n - k \}
\]

For \( I \subset \{1, ..., n - k\} \) we denote

\[
U_I = \{ (y, z) \in U_{\emptyset}; z_i \neq h_i(y) \text{ for } i \in I \}.
\]

Note that each \( U_I \) is the disjoint union of \( 2^{|I|} \) of open subanalytic Lipschitz balls and that

\[
1_U = \sum_{I \subset \{1, ..., n - k\}} (-1)^{|I|} 1_{U_I}.
\]

This ends the proof.
REFERENCES


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