Hawkes processes and application in neuroscience and genomic

Patricia Reynaud-Bouret

CNRS, Université de Nice Sophia Antipolis

Journées Aussois 2015, Modélisation Mathématique et Biodiversité

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Point process

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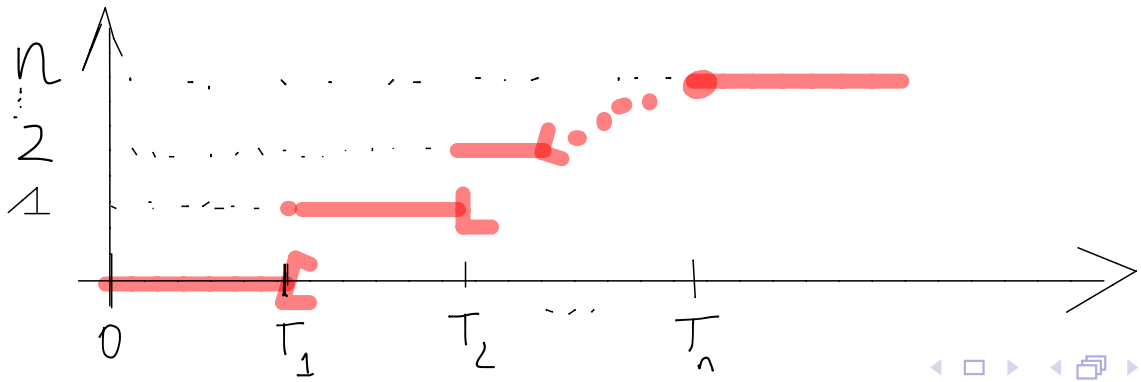
N_A number of points of N in A, $dN_t = \sum_{T \text{ point de } N} \delta_T$.
$\int f(t) dN_t = \sum_{T \in N} f(T)$

If $\mathbb{X} = \mathbb{R}^+$ and no accumulation, the counting process is $N_t = N_{[0,t]}$.



Examples

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- date / duration / position of phone calls
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Everything has been noted, probably link via a model between those quantities. No reason at all that they may be identically distributed and/or independent (not i.i.d.).

Life times

Historically, first evident records of statistics start with Graunt in 1662 and its life table.
NB : at the same time, Pascal and Euler started Probability ... Halley, Huyghens and many others also recorded life tables

"Why are people dying ?" $\rightarrow$ record of the time of deaths in London at that time.

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- Even if considered i.i.d., the interesting quantity is the hazard rate $q(t)$, with

$$q(t)dt \simeq \mathbb{P}(\text{ die just after } t \text{ given that alive in } t)$$

If f density, $q(t) = f(t)/S(t)$, with

$$S(t) = \int_t^{+\infty} f(s)ds.$$

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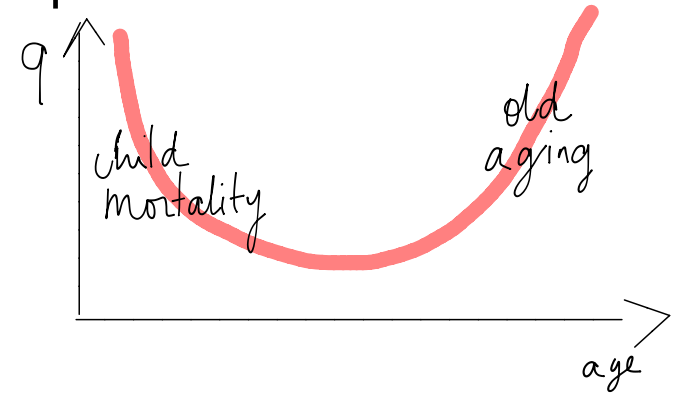
"Why are people dying ?" $\rightarrow$ record of the time of deaths in London at that time.

- Even if considered i.i.d., the interesting quantity is the hazard rate $q(t)$,
- $q = cte$: do not "age", no memory $\rightarrow$ exponential variable
- q increases : better young than old
- q decreases : better old than young.

classical shape for human



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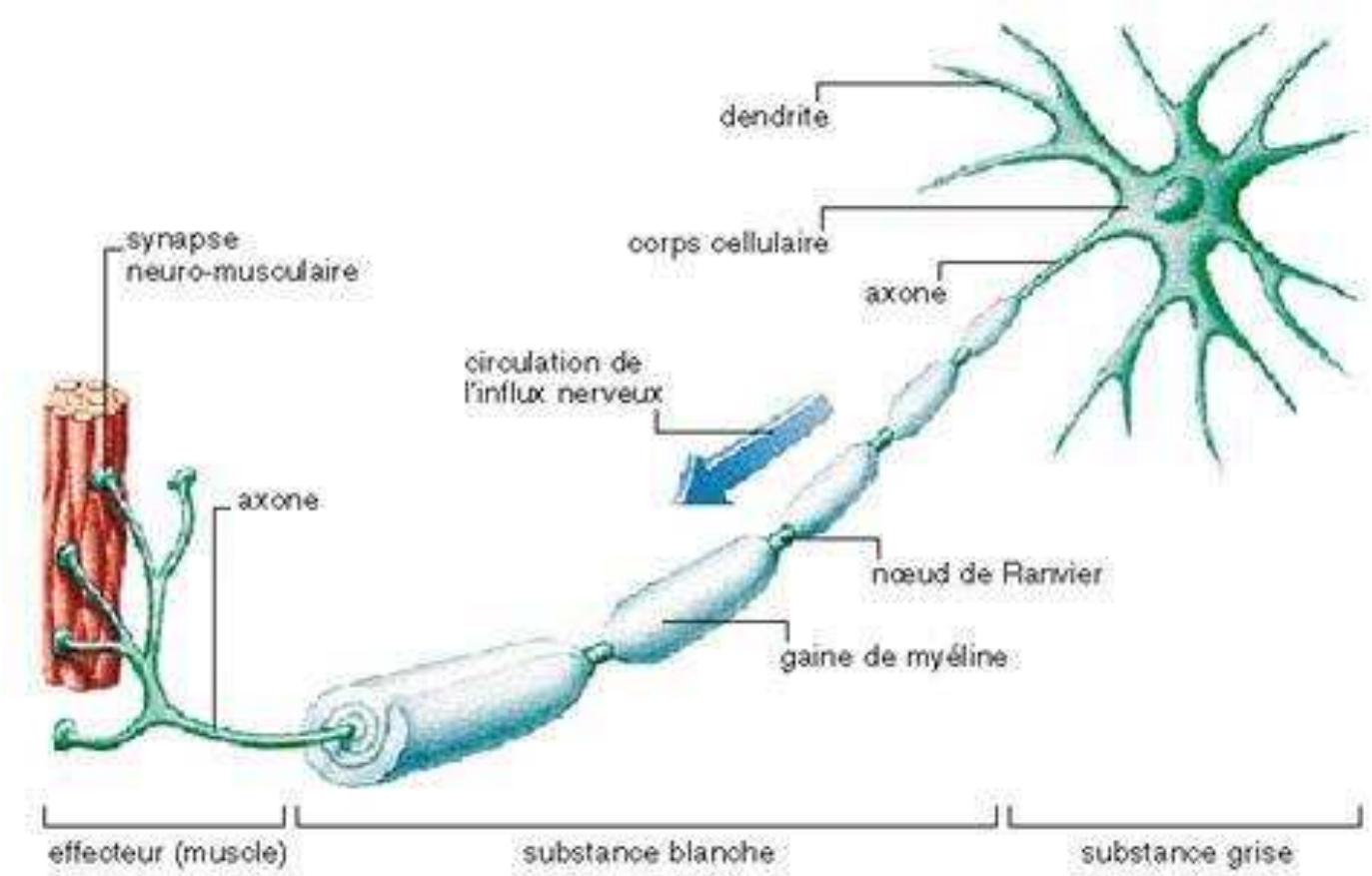
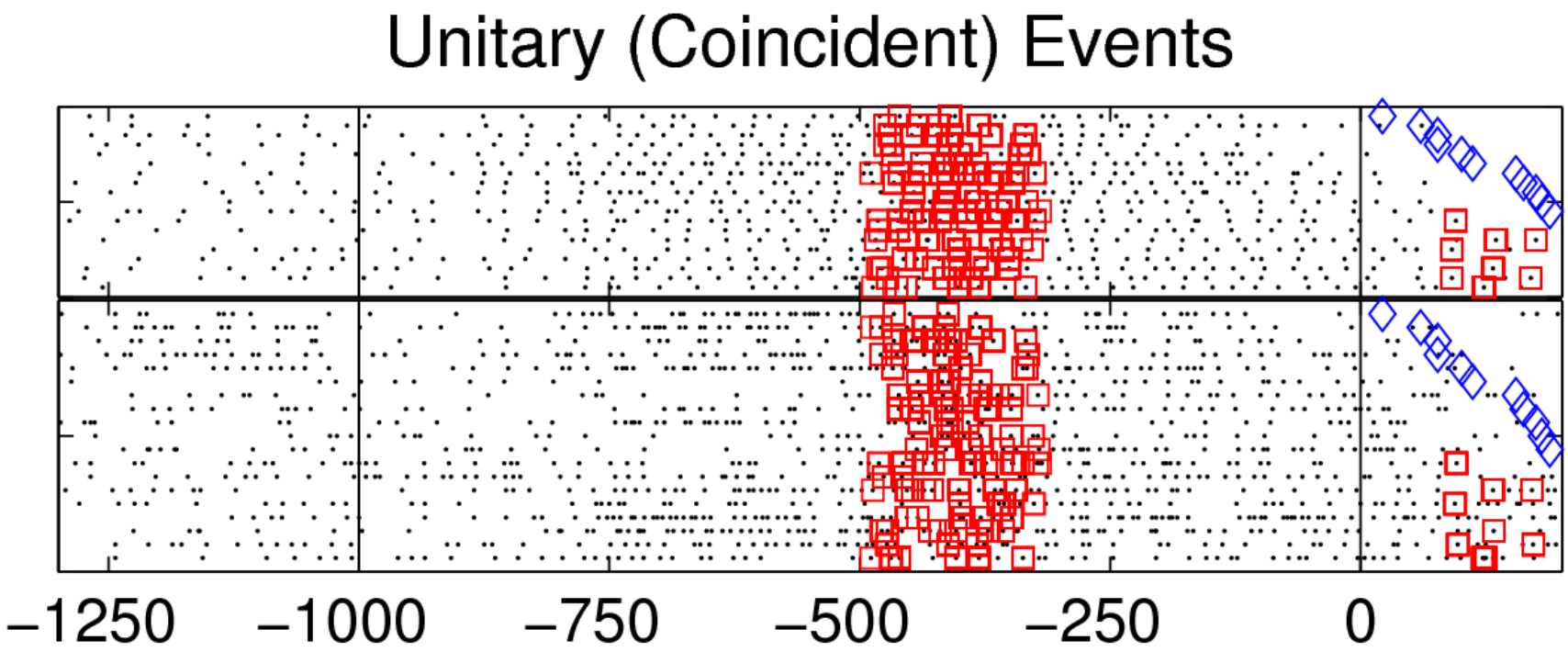
"Why are people dying ?" $\rightarrow$ record of the time of deaths in London at that time.

- Even if considered i.i.d., the interesting quantity is the hazard rate $q(t)$,
- not even clearly i.i.d. : people may move out before disease, may contaminate each other etc... $\rightarrow$ what is the interesting quantity ?

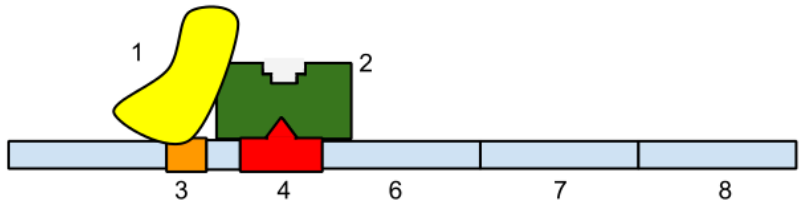
Neuroscience and neuronal unitary activity

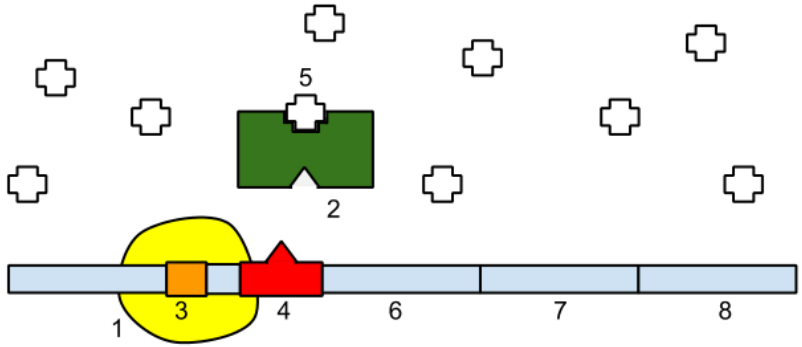


Neuronal data and Unitary Events



Genomics and Transcription Regulatory Elements





Definition

Poisson processes

- for all integer n, for all $A_1, \ldots, A_n$ disjoint measurable subsets of $\mathbb{X}$, $N_{A_1}, \ldots, N_{A_n}$ are independent random variables.

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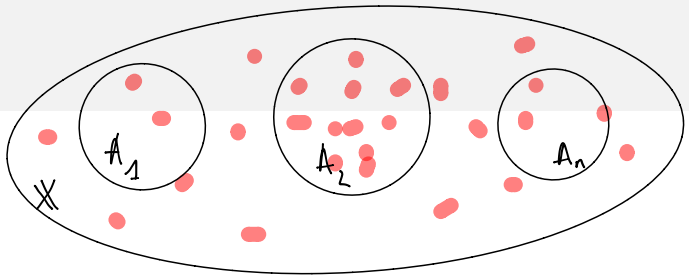

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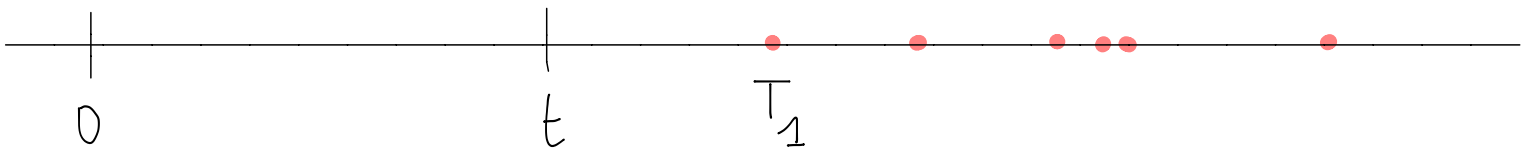

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↑ Poisson distribution with parameter $\lambda t = \theta$

$$P(N_k = k) = \frac{\theta^k}{k!} e^{-\theta}.$$

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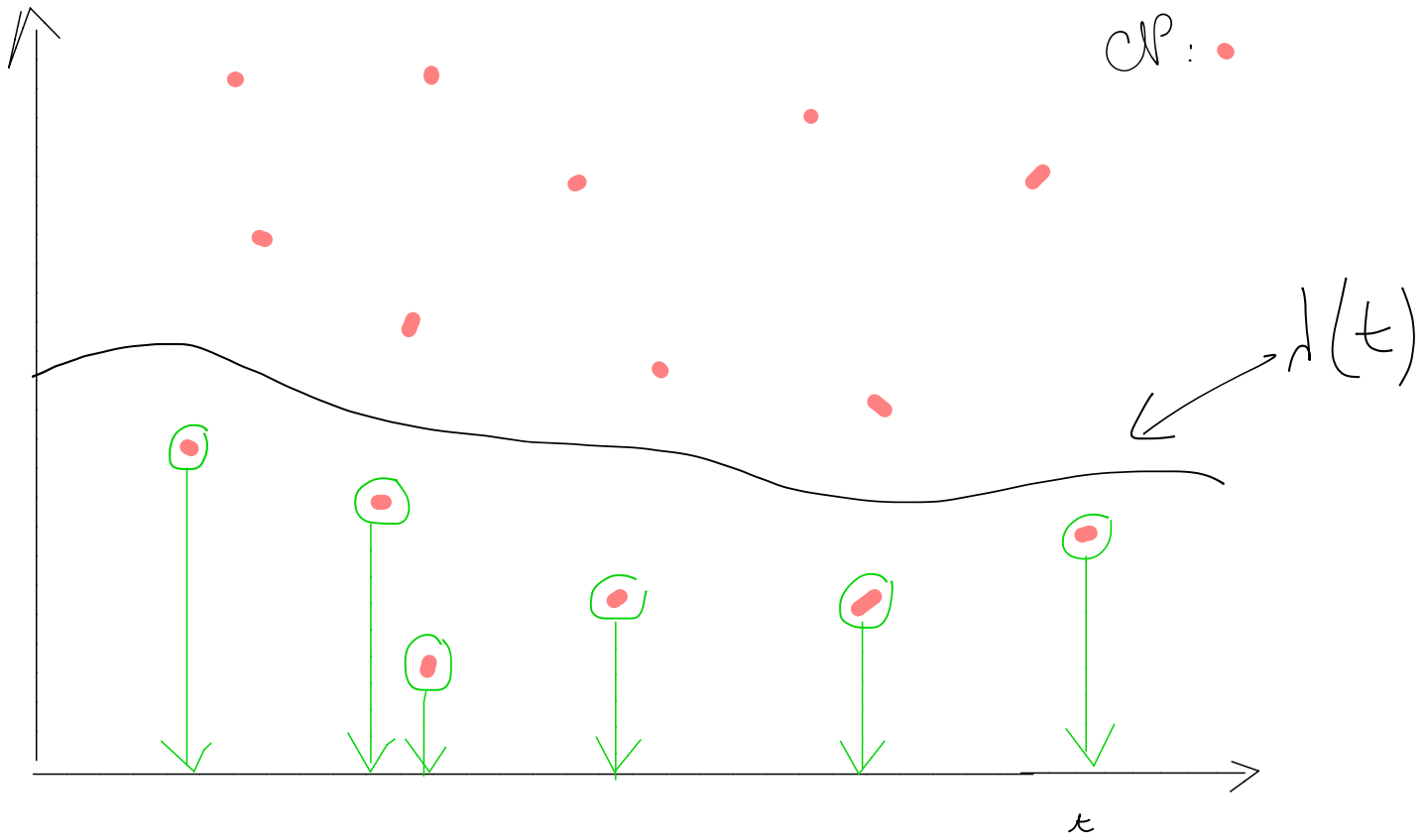
$\rightarrow$ first time is an exponential variable
All "intervals" are i.i.d and exponentially distributed.
No memory $\Longleftrightarrow$ independence to the past

If λ not constant

If $\ell([0,t]) = \int_0^t \lambda(s)ds$, let $\mathcal{N}$ be a Poisson process in $\mathbb{R}_+^2$ and



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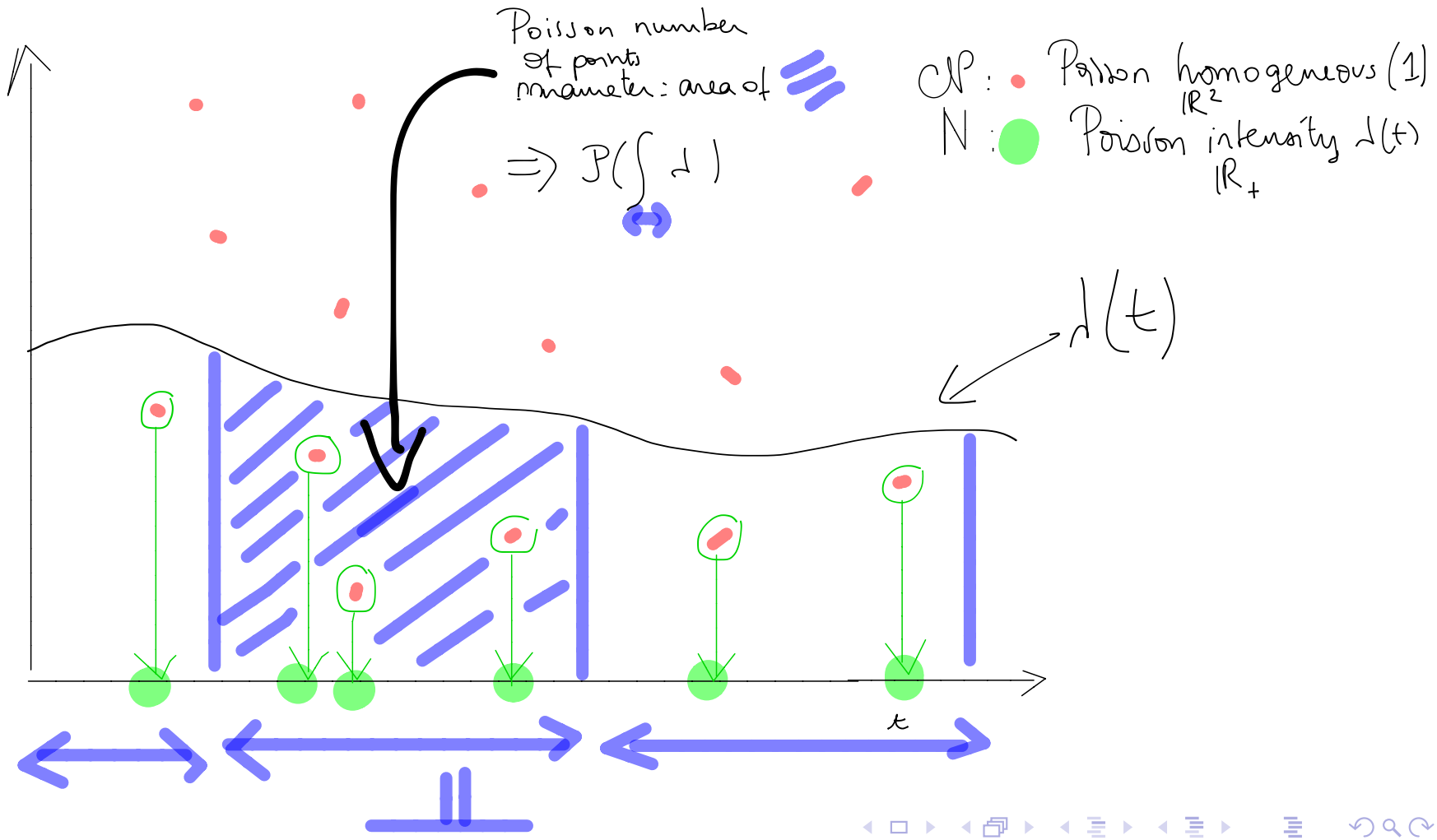

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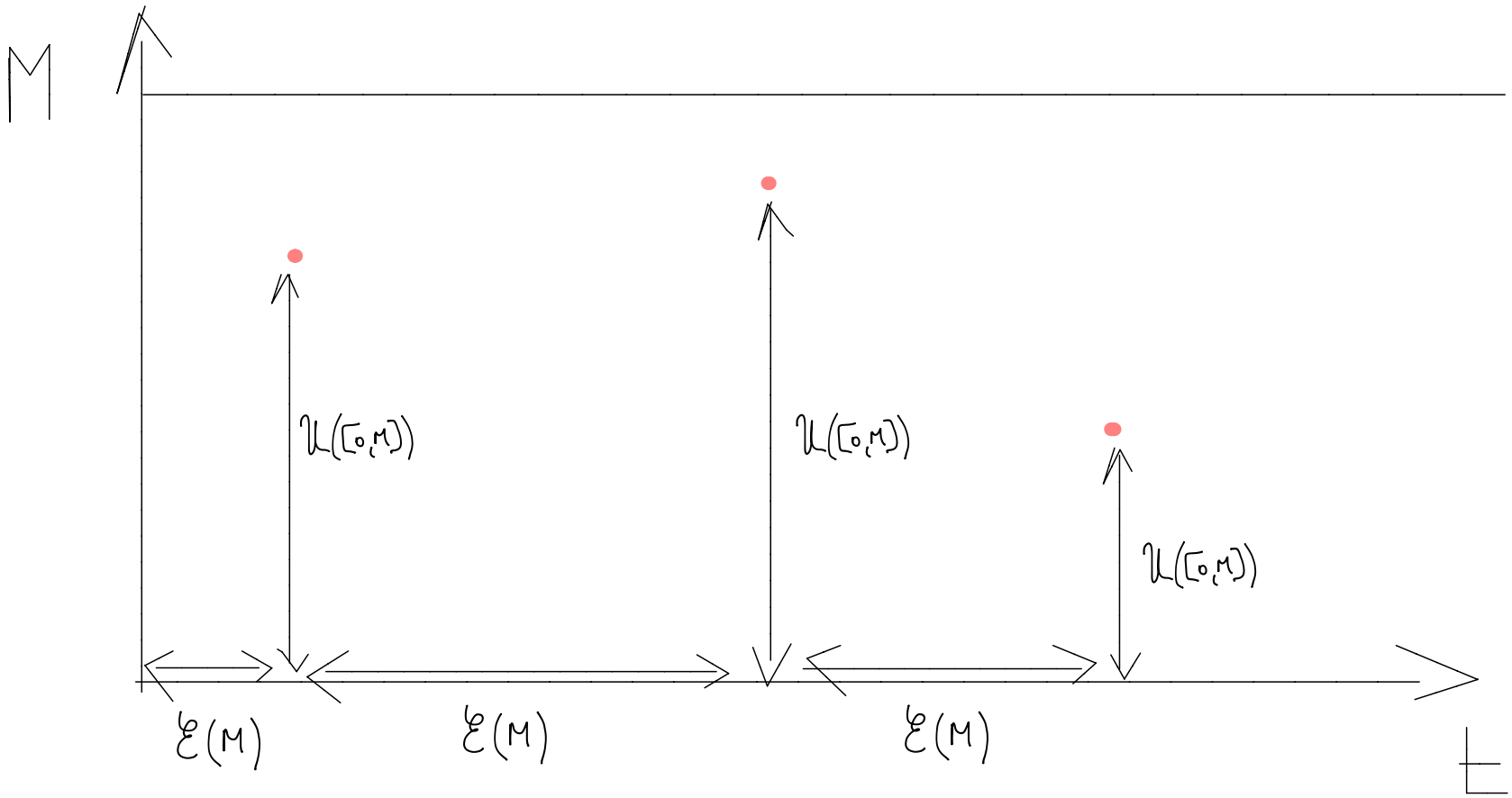

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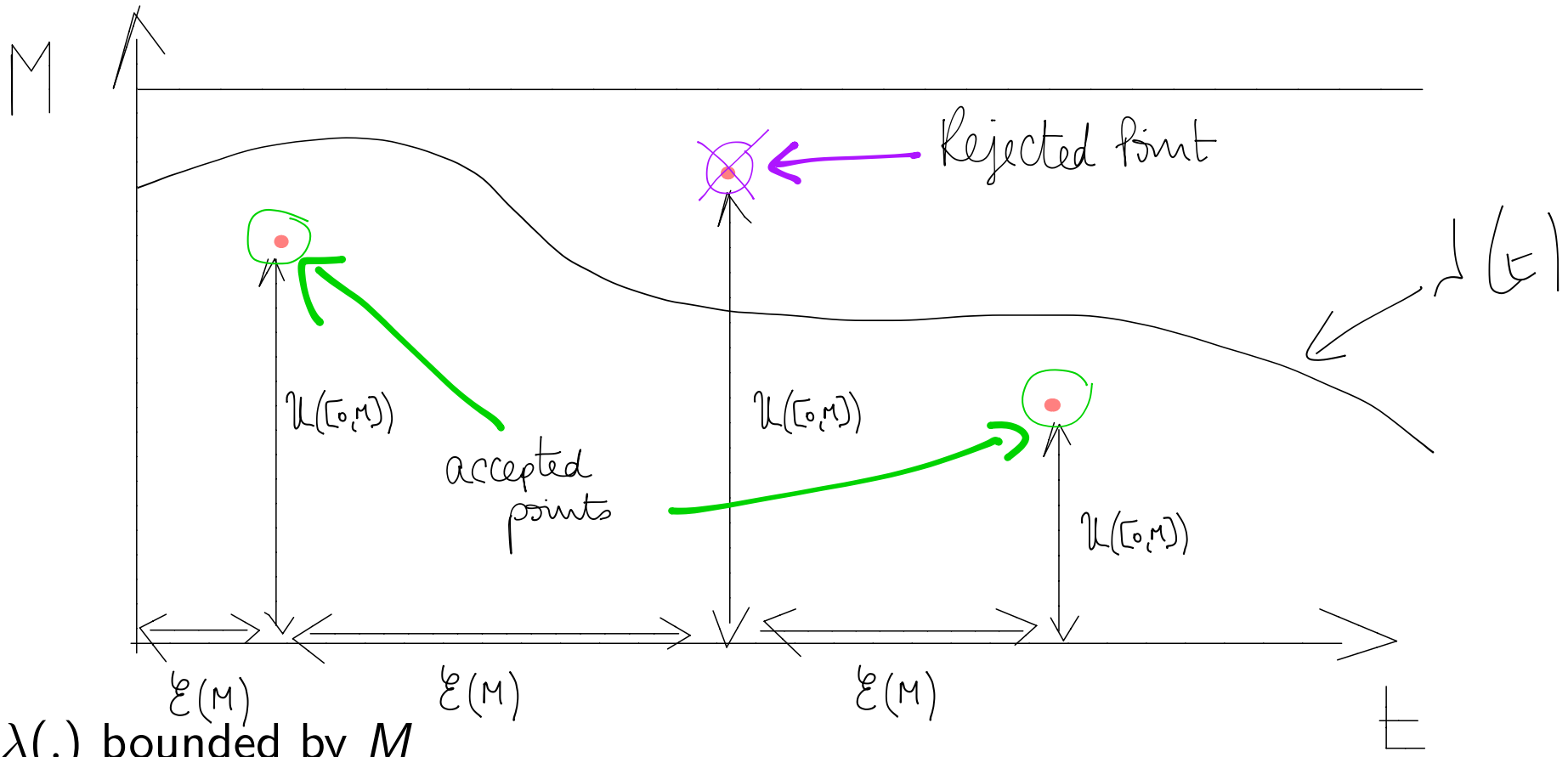

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Conditional intensity

Need of the same sort of interpretation as hazard rate but given "the Past" !

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$$\underbrace{dN_t}_{\text{Nbr observed points in } [t, t+dt]} = \overbrace{\underbrace{\lambda(t)\, dt}_{\text{Expected Number given the past before } t}}^{\textbf{intensity}} + \underbrace{\text{noise}}_{\textit{Martingales differences}}$$

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$\lambda(t)$ = instantaneous frequency

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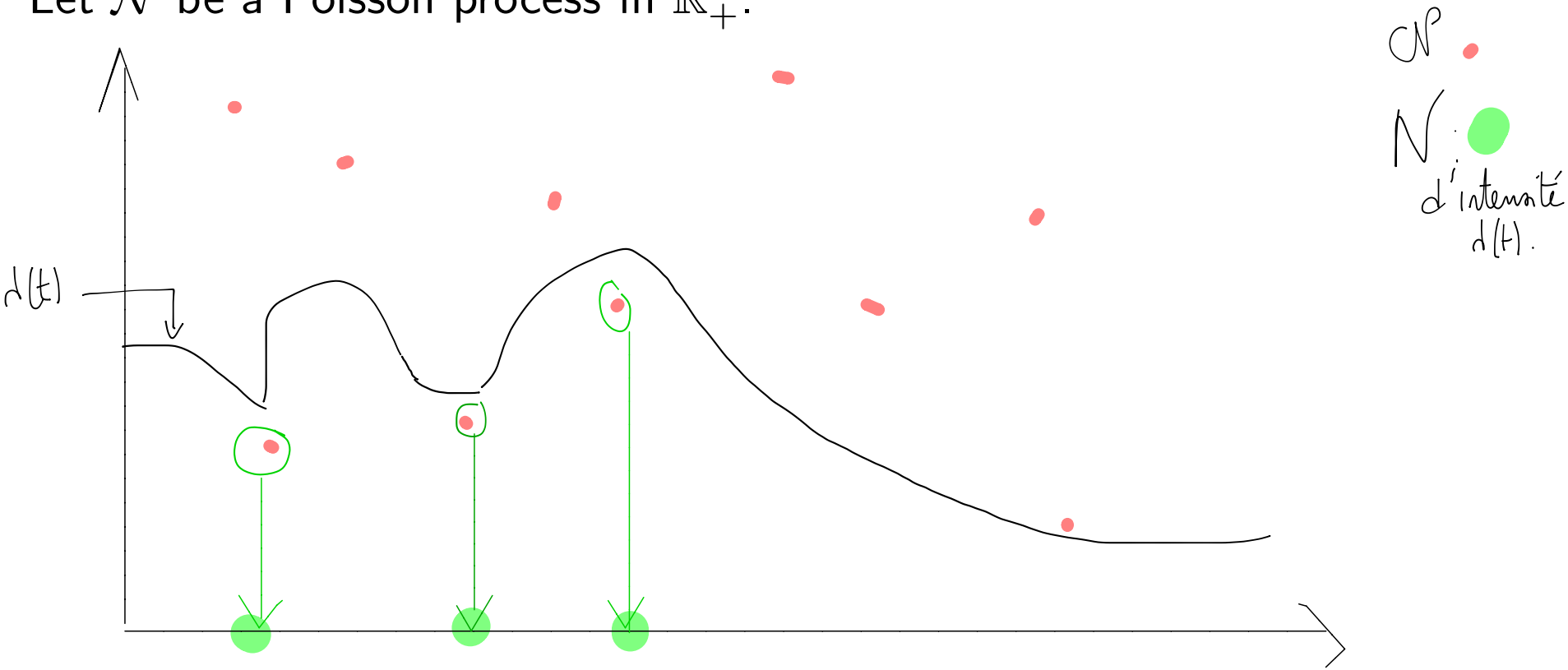
$$\begin{aligned} \lambda(t) &= \text{instantaneous frequency} \\ &= \textbf{random}, \text{ depends on the Past (previous points ...)} \\ &= \text{characterizes the distribution when exists} \end{aligned}$$

NB : if $\lambda(t)$ deterministic $\rightarrow$ Poisson process

Thinning

Let $\mathcal{N}$ be a Poisson process in $\mathbb{R}_+^2$.



$\lambda(t)$ is predictable, i.e. depends on the past only, can be drawn from left to right if one discovers one point at a time. Usually "c-à-g" for "continu à gauche" (left continuous)

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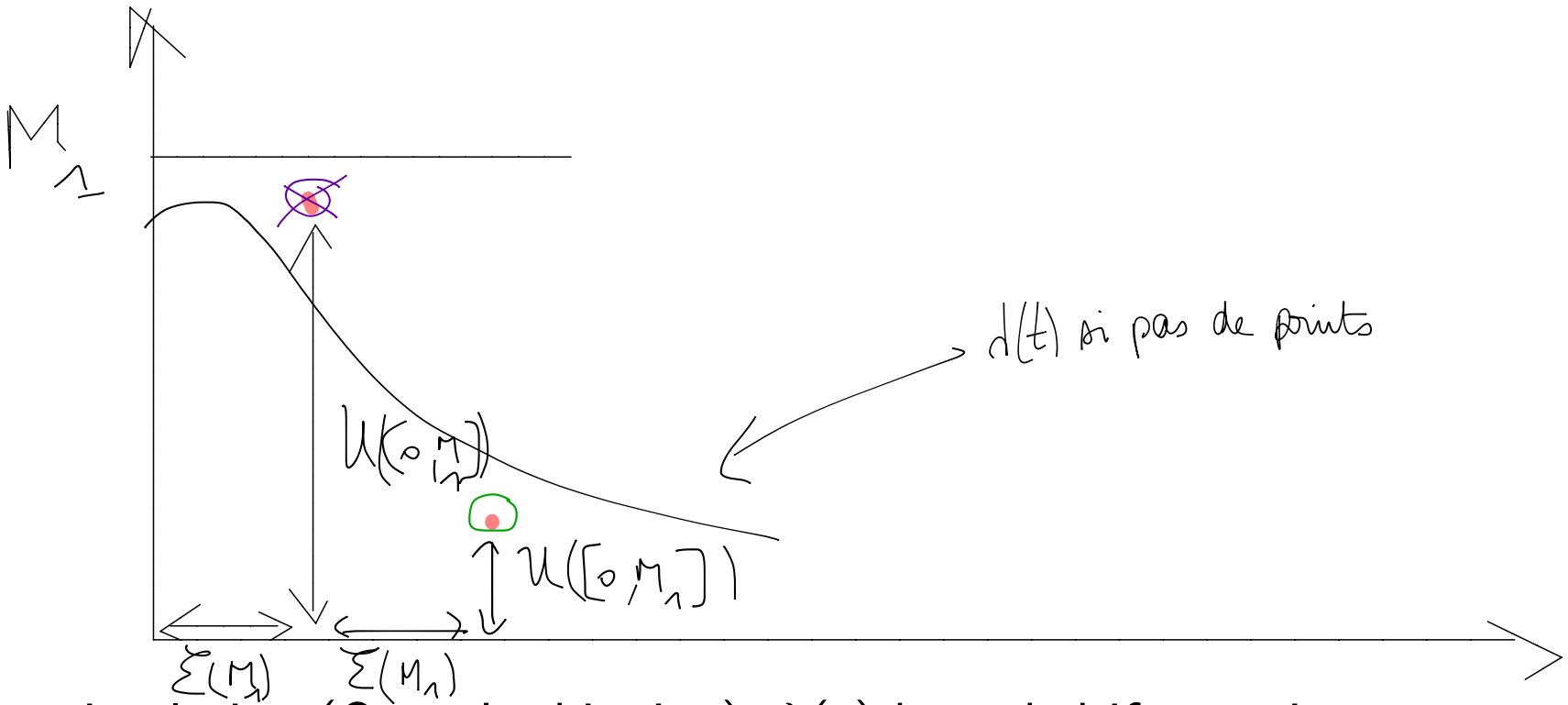

For simulation (Ogata's thinning), $\lambda(t)$ bounded if no points appear

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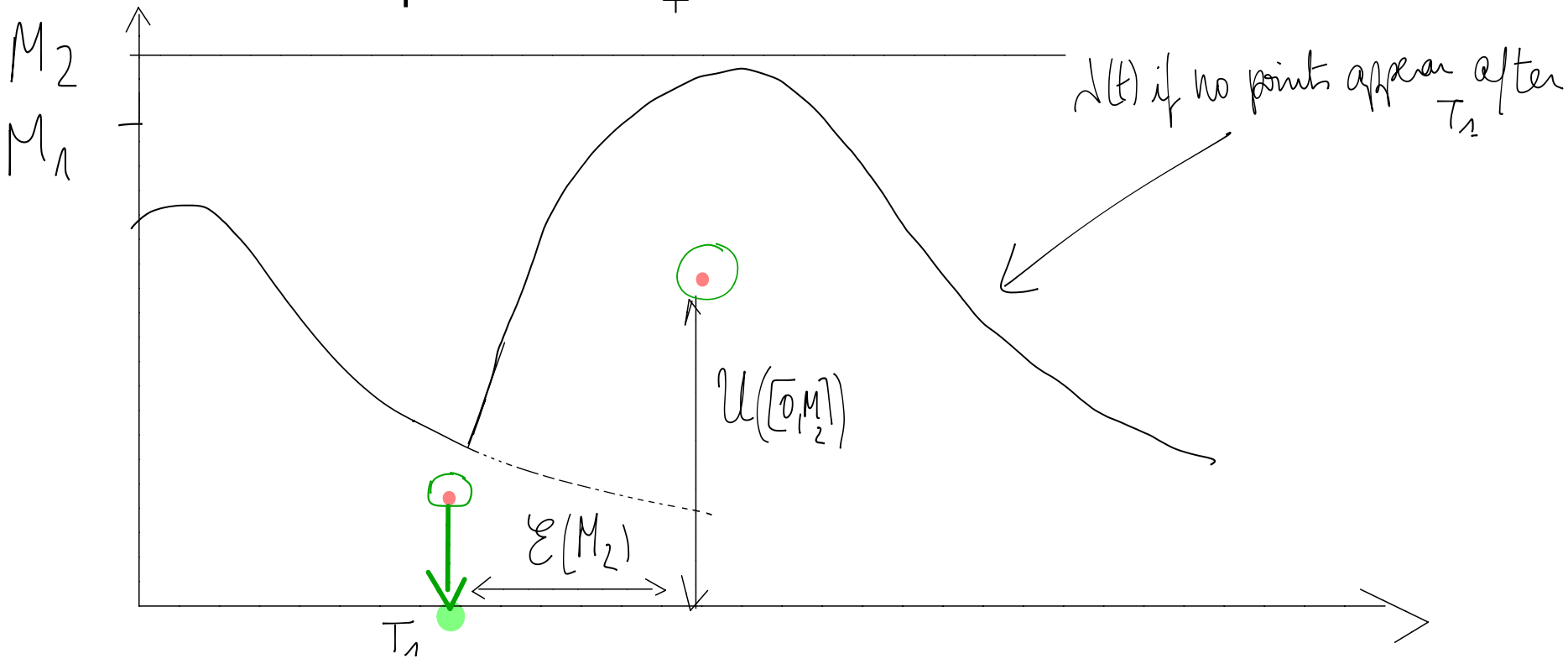

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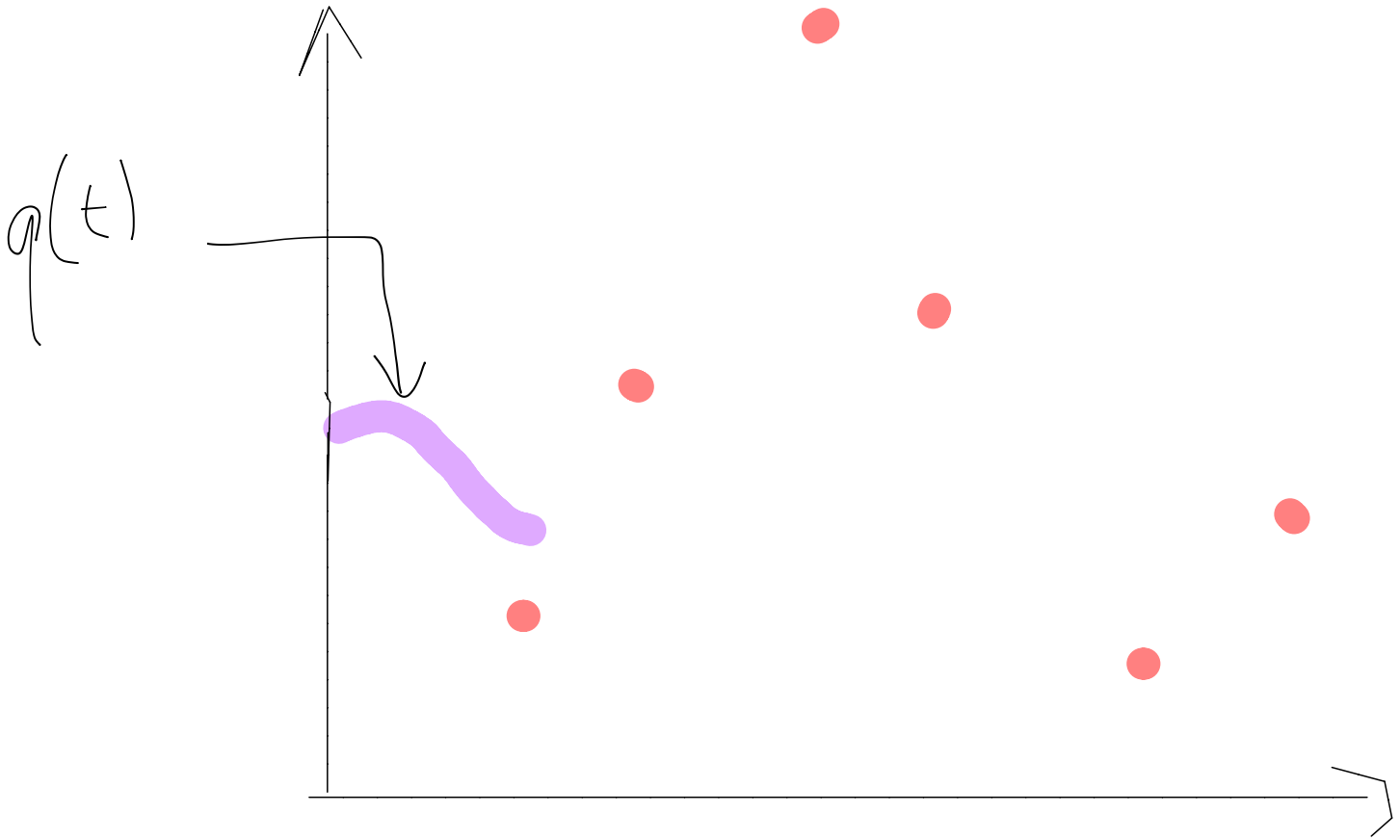
One life time

$N = \{T\}$, T of hazard rate q.



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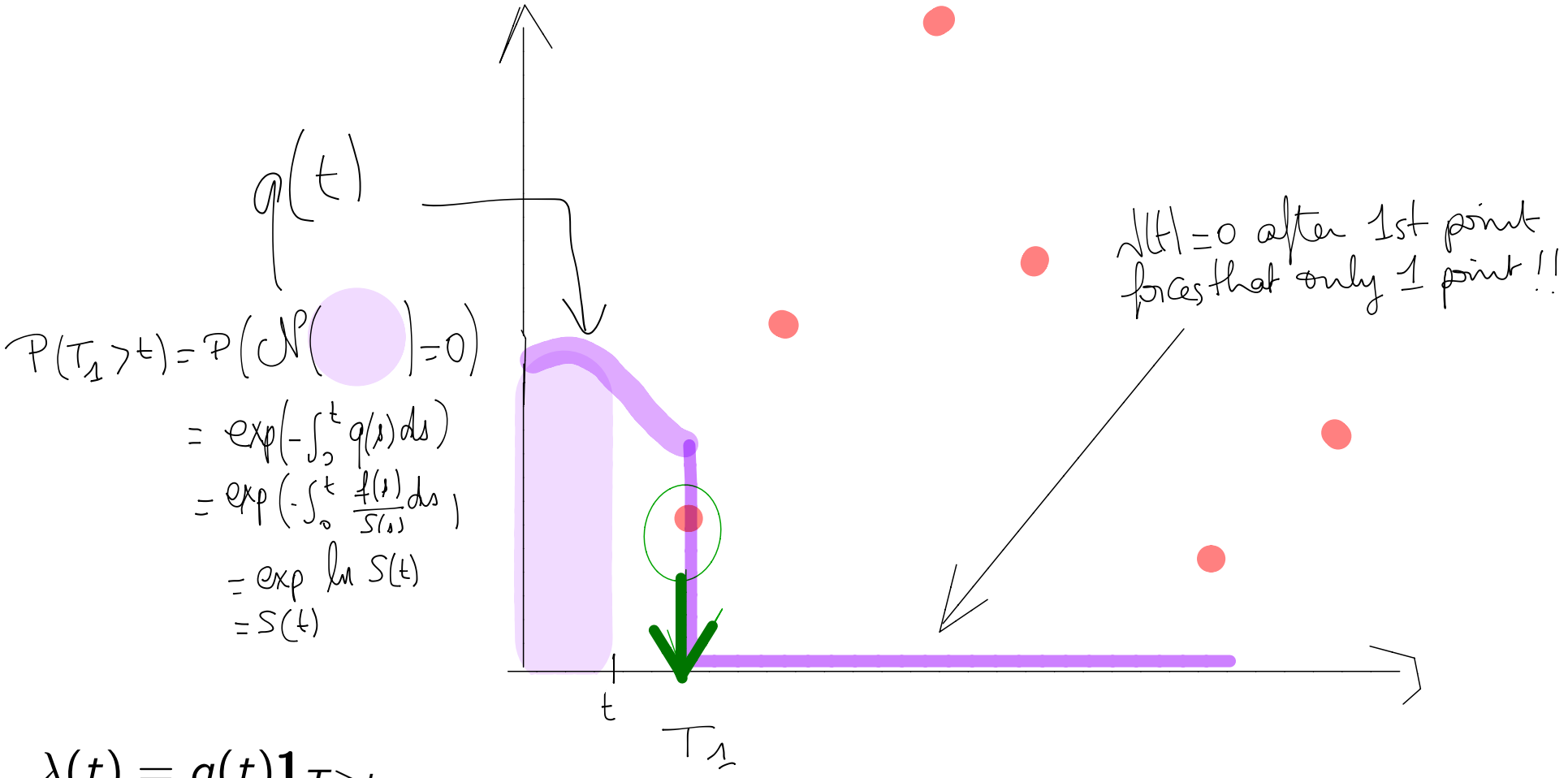
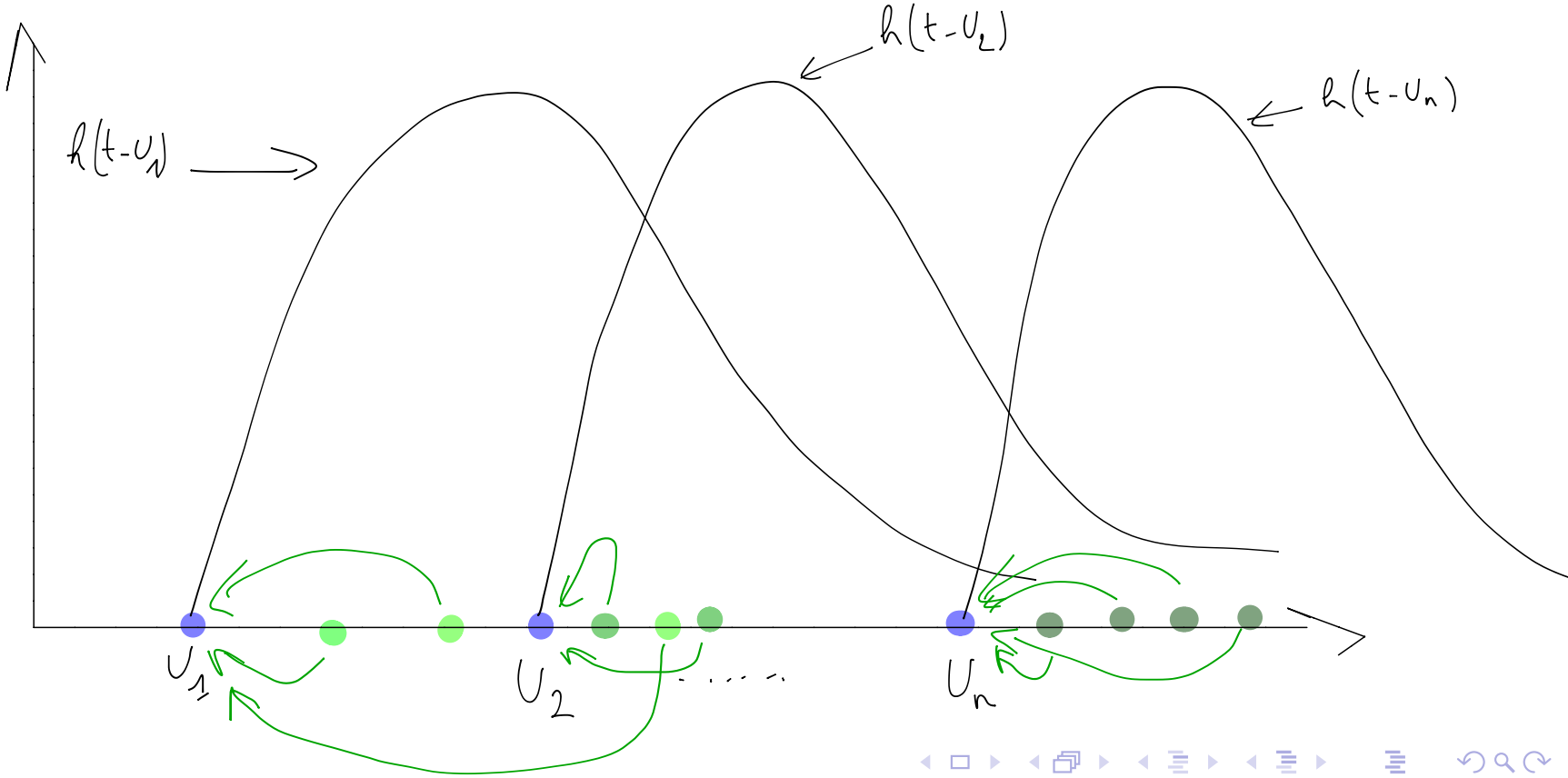


$$\lambda(t) = q(t)\mathbf{1}_{T \geq t}$$

Poissonian interaction

Parents : U_i either (uniform) i.i.d. or (homogeneous) Poisson process
Each parent gives birth according to a Poisson process of intensity h originated in U_i.
Process of the children ?



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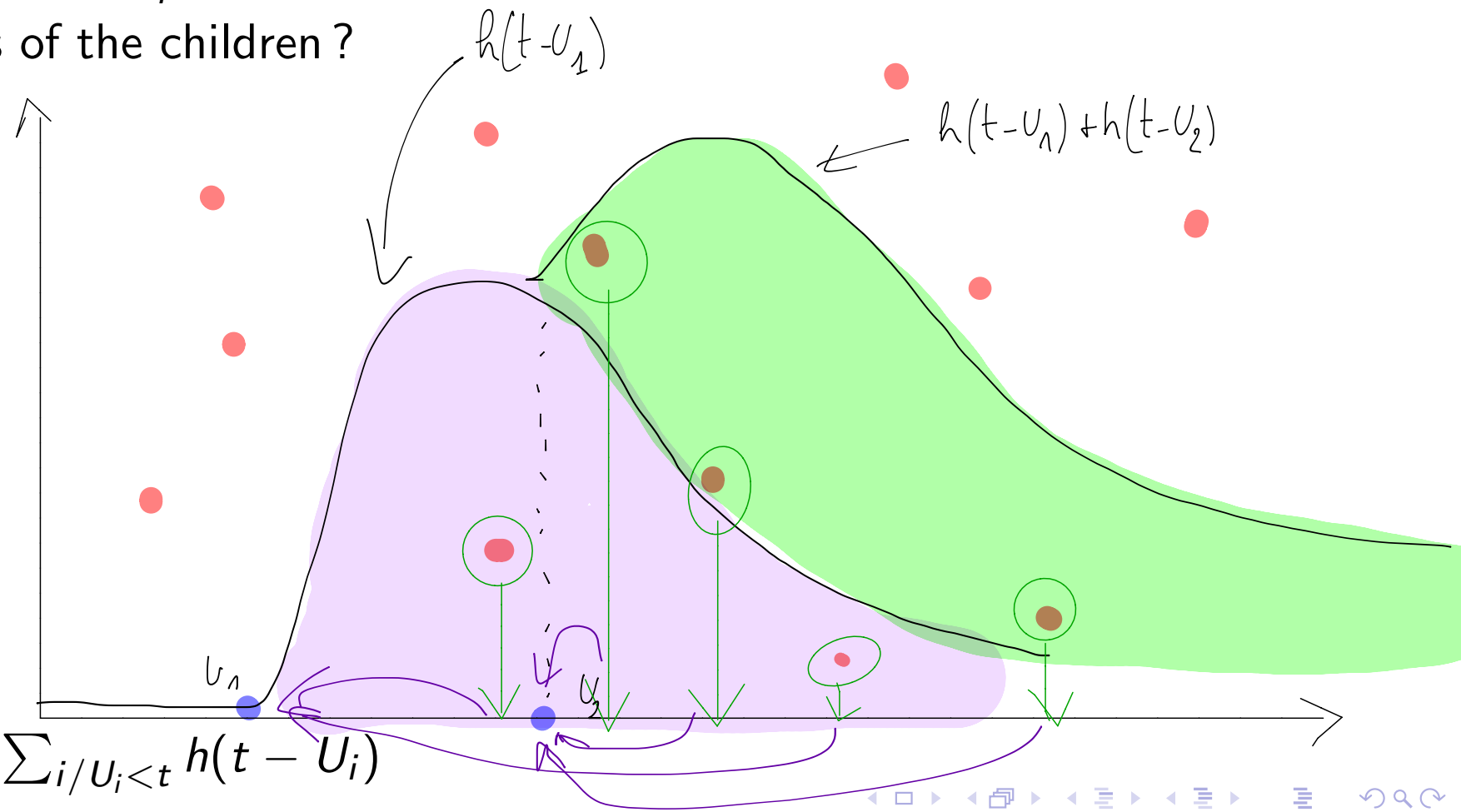

$$\lambda(t) = \sum_{i/U_i<t} h(t - U_i)$$

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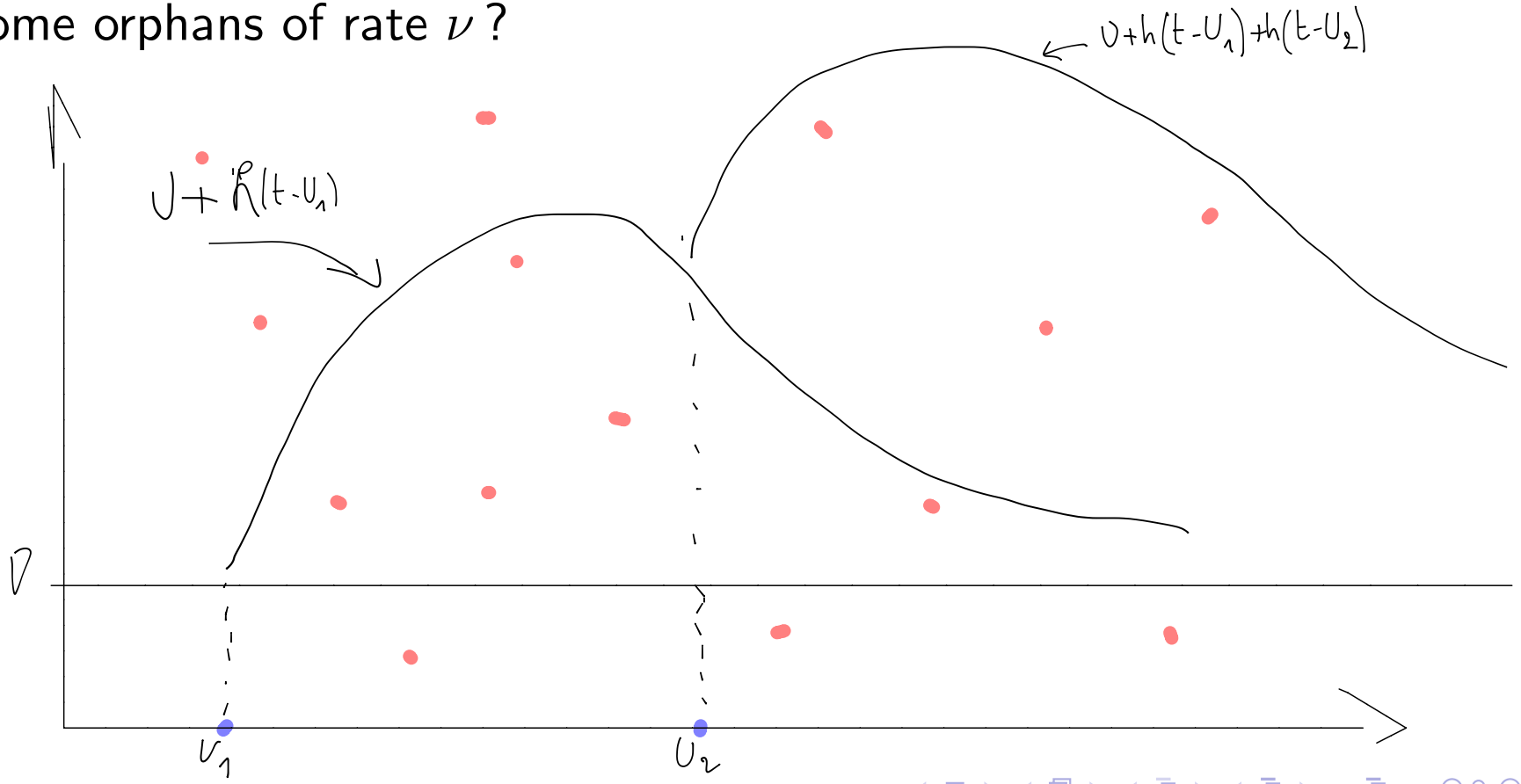

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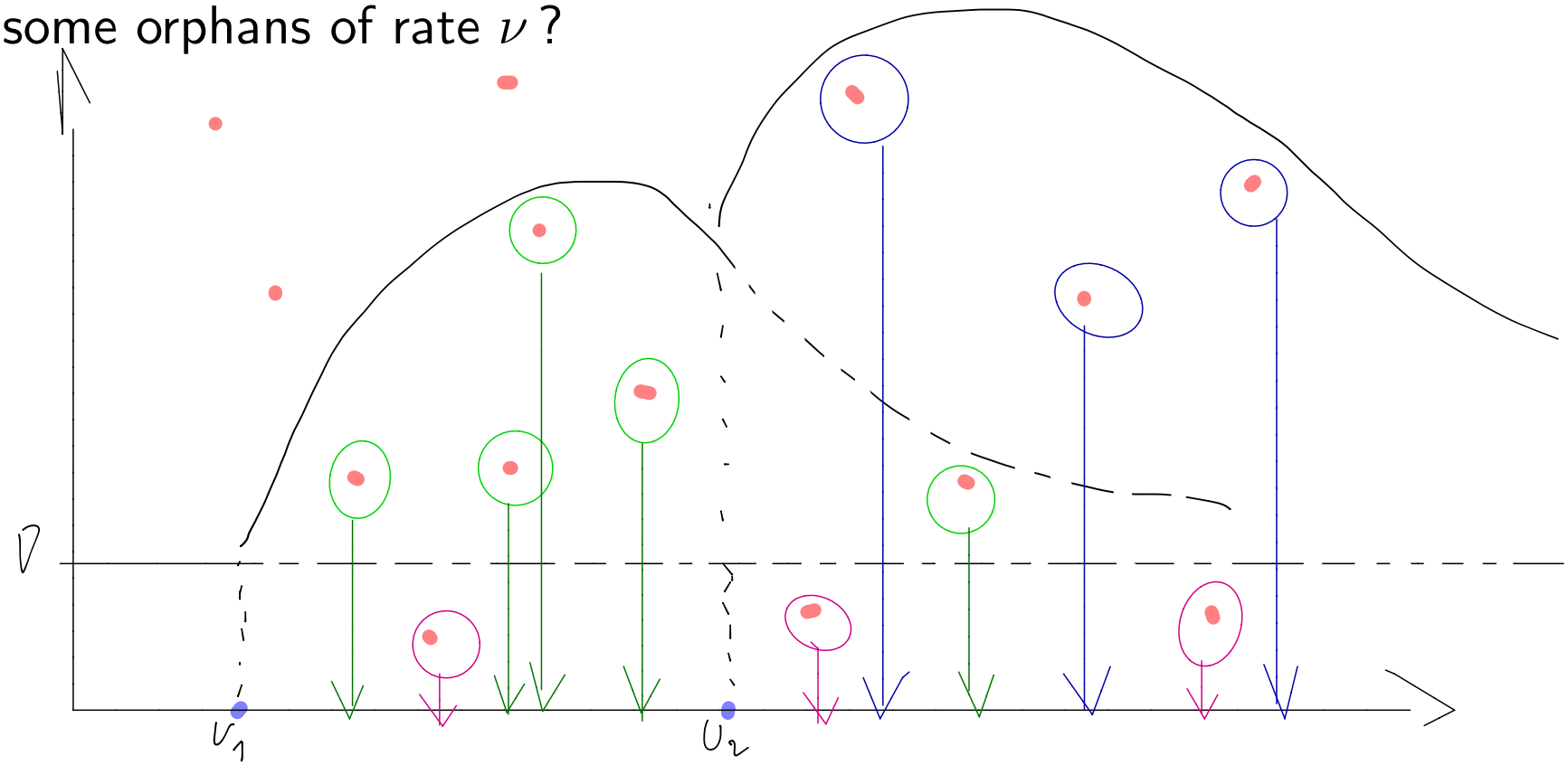
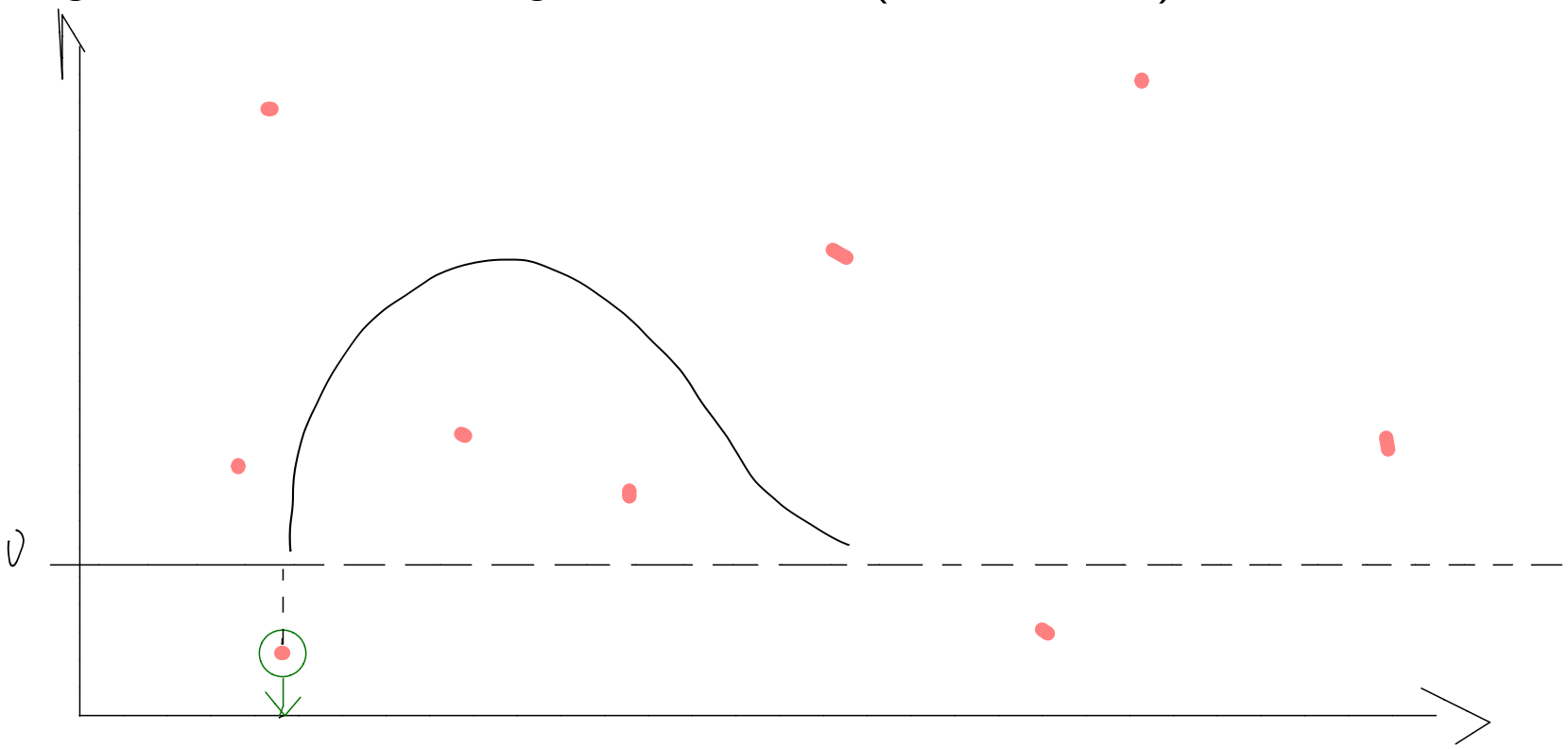


$\lambda(t) = \nu + \sum_{i/U_i<t} h(t - U_i)$

Hawkes process : linear case

A self-exciting process introduced by Hawkes in the 70's to model earthquakes.
Ancestors appear at rate ν.
Each point gives birth according to h etc etc (aftershocks)



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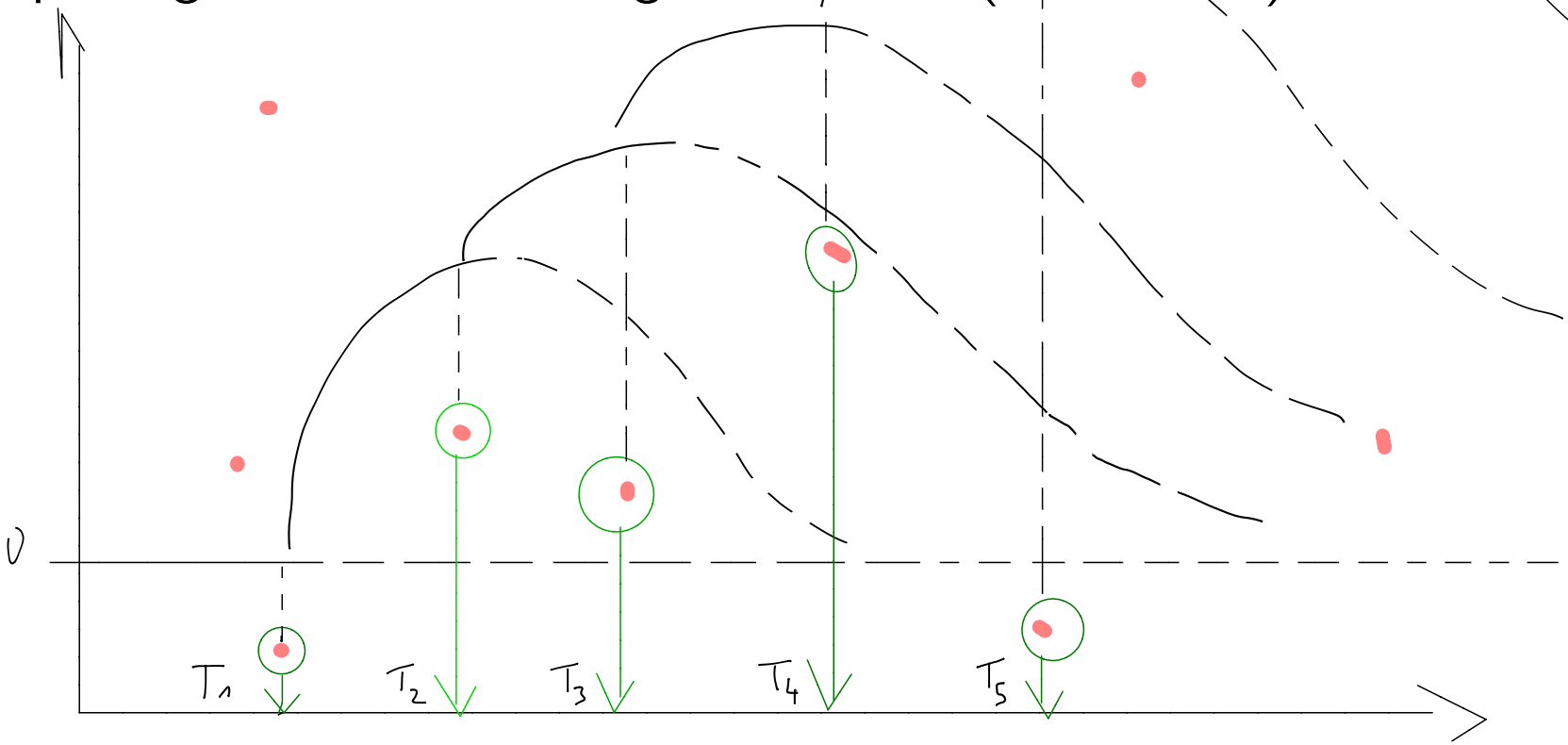

$$\lambda(t) = \nu + \sum_{T<t} h(t-T) = \nu + \int_{-\infty}^{t} h(t-u)dN_u$$

Hawkes and Modelisation

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- can therefore model interaction also (see later)
- generally, modelling people "want" stationnarity : if $\int h < 1$, the branching procedure ends $\Longleftrightarrow$ existence of stationnary version.
- for genomics and neuroscience, need of inhibition : possible to use

$$\lambda(t) = \Phi\Big(\int_{-\infty}^{t} h(t-u)dN_u\Big),$$

with Φ 1-Lipschitz positive and h of any sign. Stationnarity condition $\int |h| < 1$.

- In particular, $\Phi = (.)_+$ but loss of the branching structure (?)

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- Informally the likelihood of the observation N should give the probability to see (something near) N.

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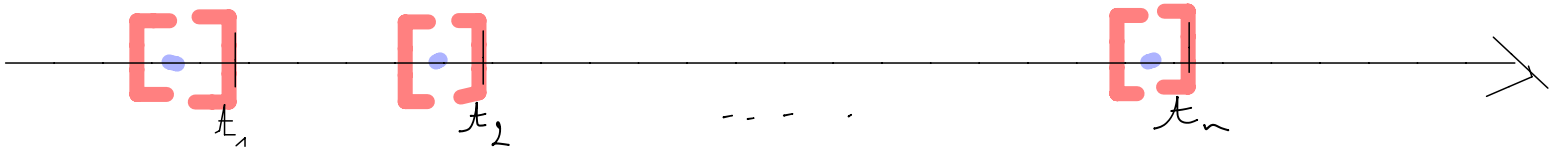

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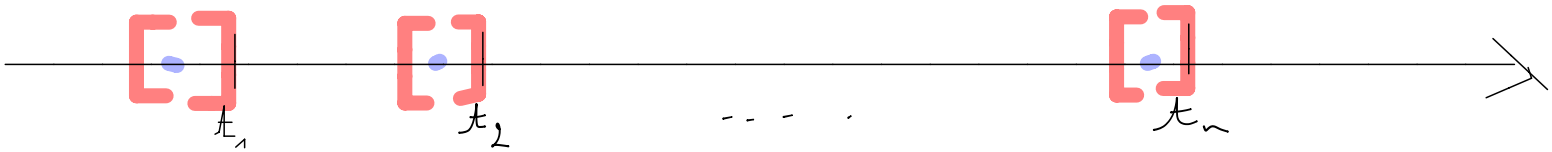
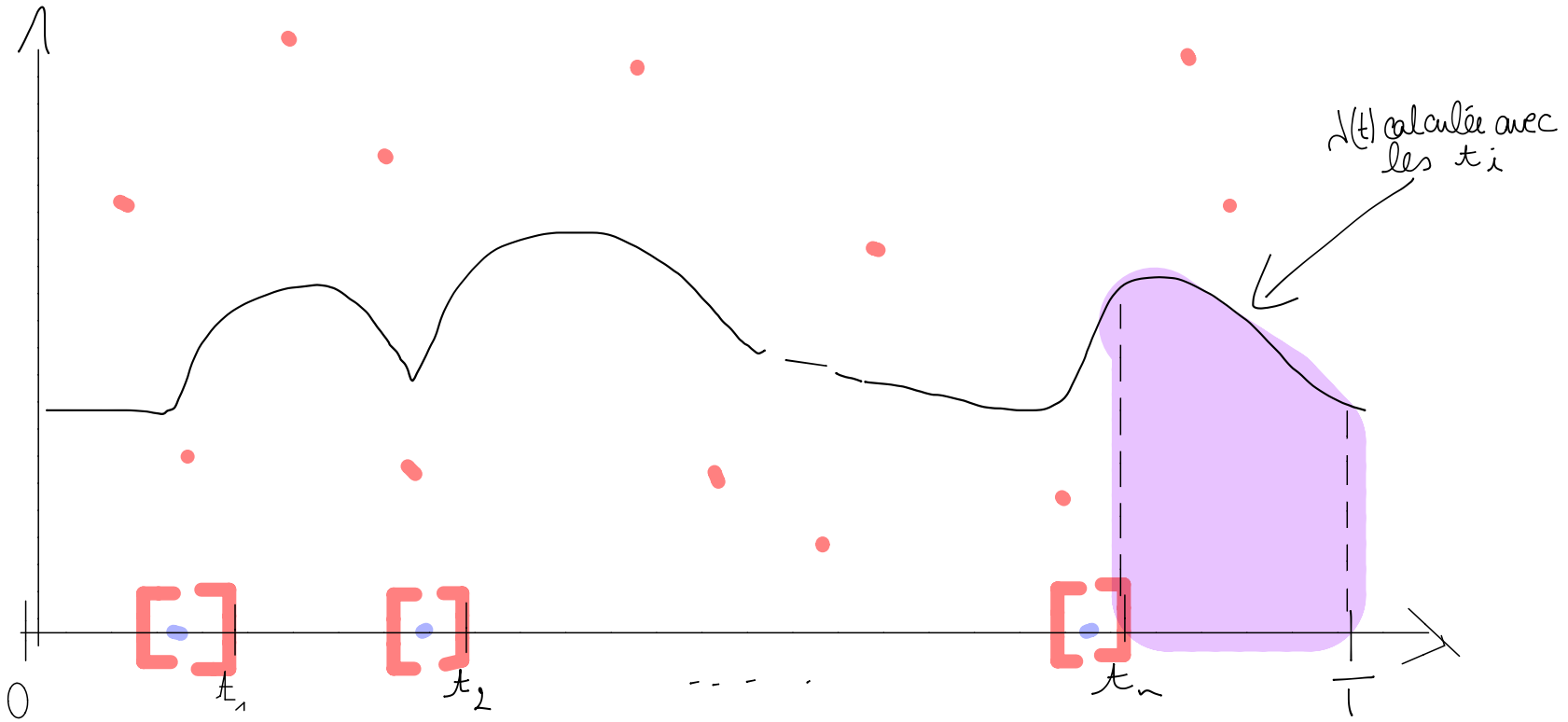


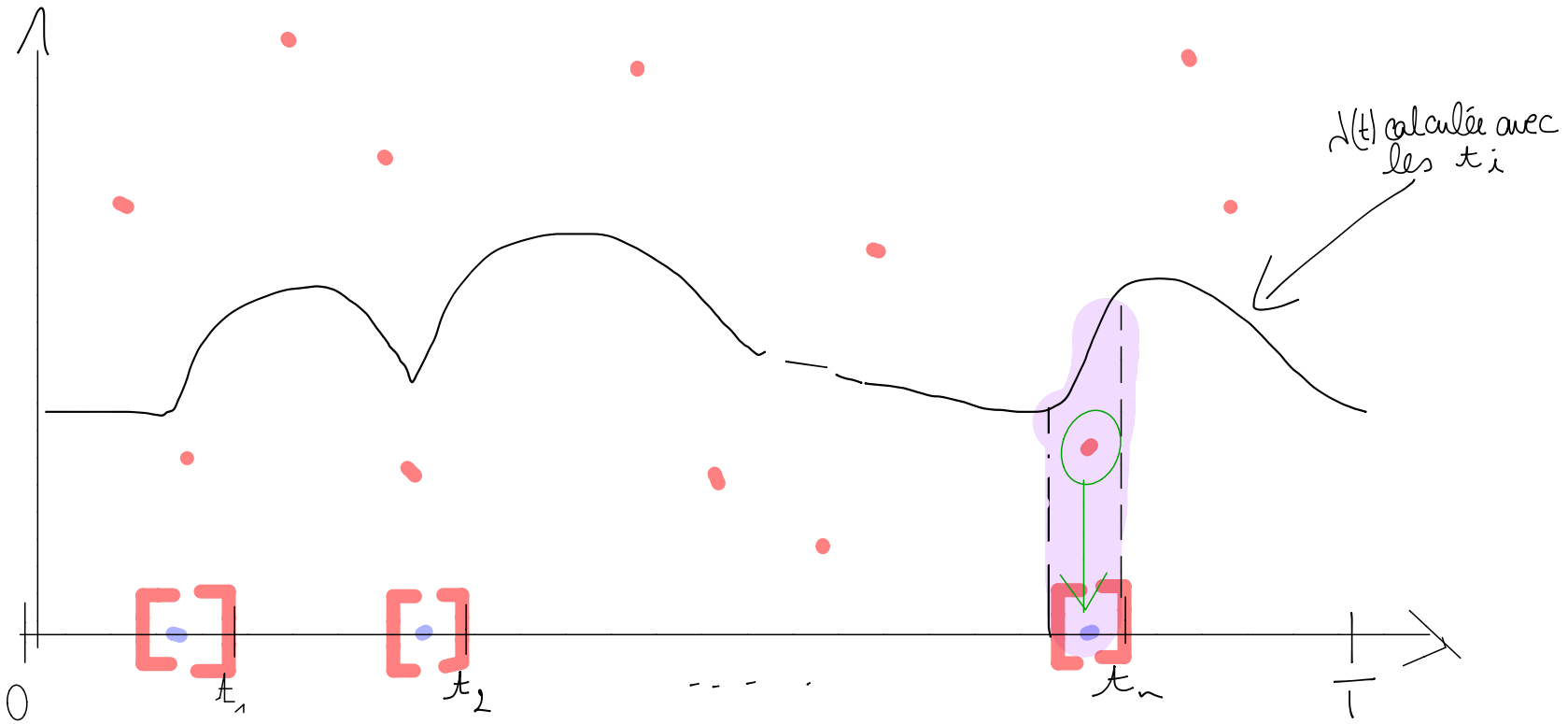
$$\mathbb{P}(\text{no point in } [0, t_1 - dt_1], 1 \text{ point in } [t_1 - dt_1, t_1], ..., \text{no point in} [t_n, T])$$

Likelihood(2)



$$\mathbb{P}(\text{no point in}[t_n, T] \Big| \text{past at } t_n) = e^{-\int_{t_n}^{T} \lambda(s)ds}$$

Likelihood(3)



$$\mathbb{P}(1 \text{ point in} [t_n - dt_n, t_n] \Big| \text{past at } t_n - dt_n) \simeq \lambda(t_n) dt_n e^{-\int_{t_n}^{t_n + dt_n} \lambda(s) ds}$$

Likelihood(4)

$$\mathbb{P}(\text{no point in } [0, t_1 - dt_1], 1 \text{ point in } [t_1 - dt_1, t_1], ..., \text{no point in} [t_n, T])$$

$$= \mathbb{E}\left(\mathbf{1}_{\text{no point in } [0, t_1 - dt_1]} \cdots \mathbf{1}_{1 \text{ point in } [t_n - dt_n, t_n]} \times \mathbb{P}(\text{no point in} [t_n, T] \Big| \text{past at } t_n)\right)$$

$$= \mathbb{E}\left(\mathbf{1}_{\text{no point in } [0, t_1 - dt_1]} \cdots \mathbf{1}_{1 \text{ point in } [t_n - dt_n, t_n]}\right) e^{-\int_{t_n}^{T} \lambda(s) ds}$$

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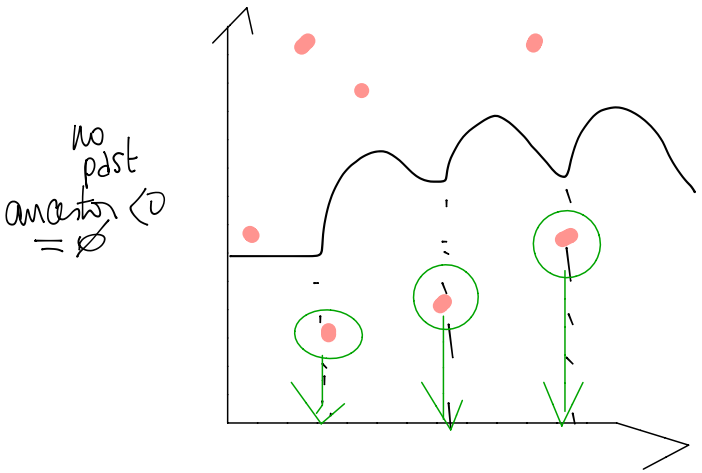
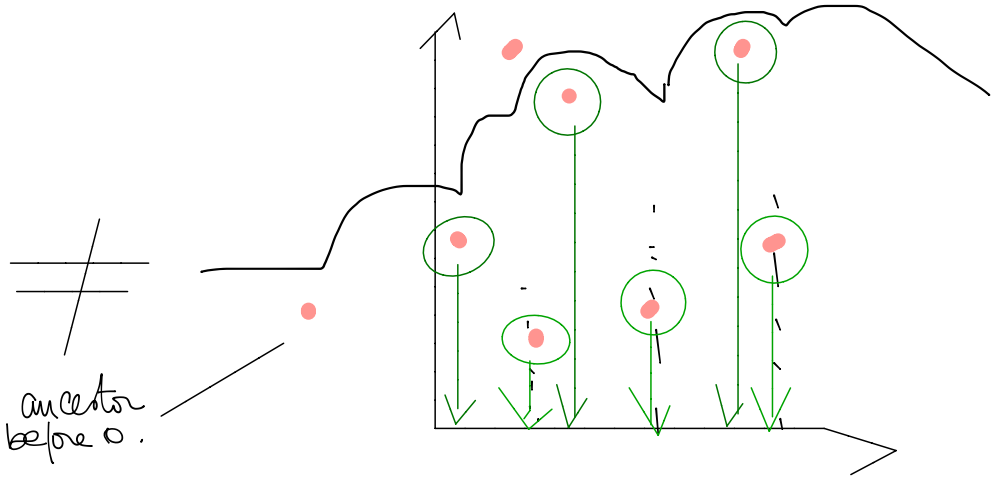
Likelihood and log-likelihood

(Pseudo)Likelihood

$$\prod_i \lambda(t_i) e^{-\int_0^T \lambda(s)ds}$$

dependance on the past not explicit (ex Past before 0 for Hawkes)





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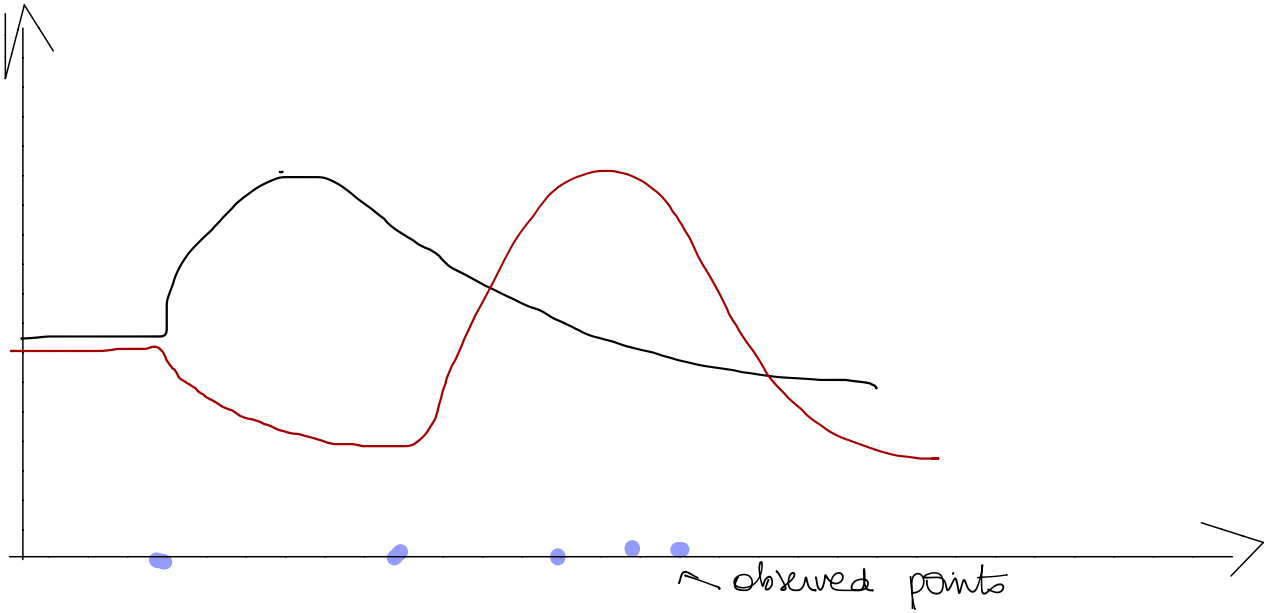
LogLikelihood

$$\ell_\lambda(N) = \int_0^T \log[\lambda(s)] dN_s - \int_0^T \lambda(s)ds$$

Maximum likelihood estimator

If model for $\lambda \to \lambda_\theta$, $\theta \in \mathbb{R}^d$, how to choose the best θ according to the observed data N on $[0, T]$?



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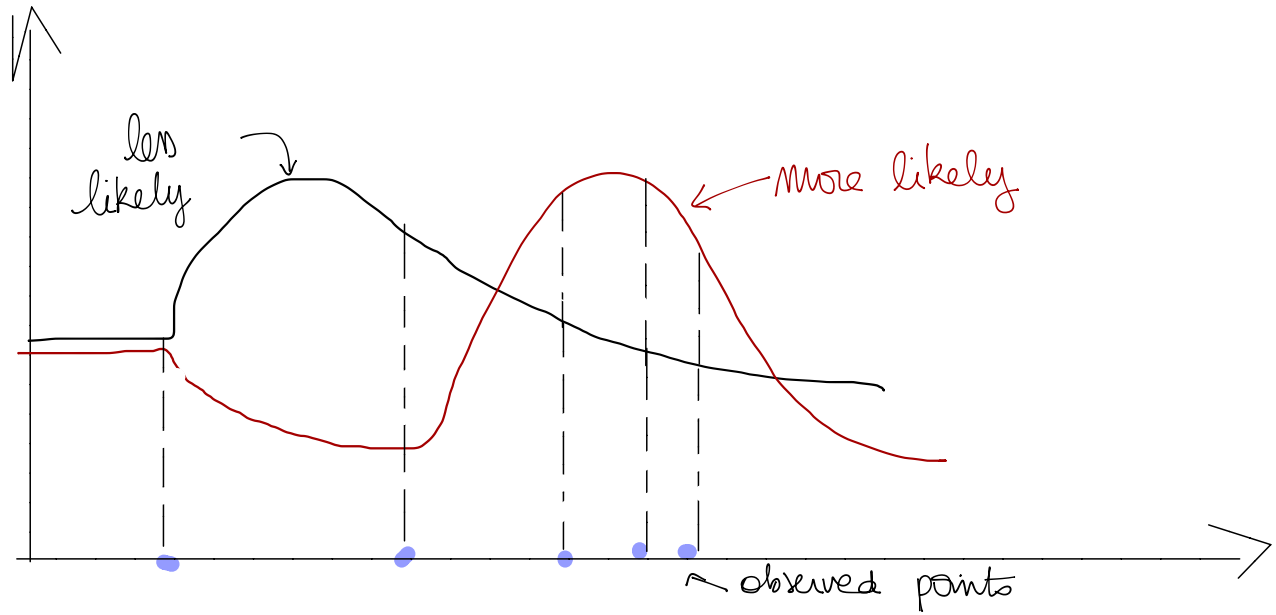

then $MLE = \hat{\theta} = \arg\max_\theta \ell_{\lambda_\theta}(N)$.

Maximum likelihood estimator(2)

If d, number of unknown parameter, fixed one can show usually nice asymptotic properties (in T ...) :

- consistency ($\hat{\theta} \to \theta$)
- asymptotic normality
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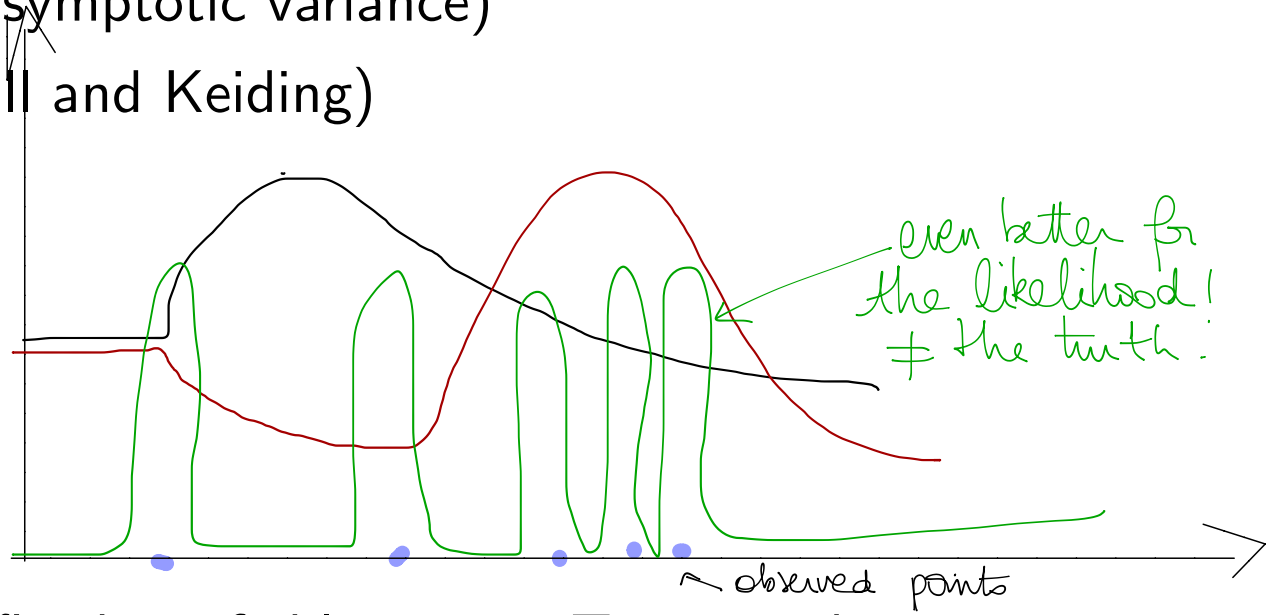
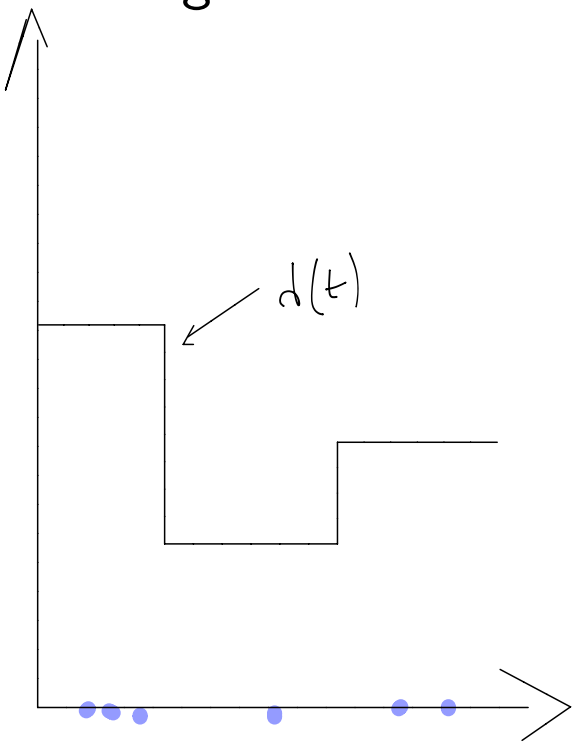


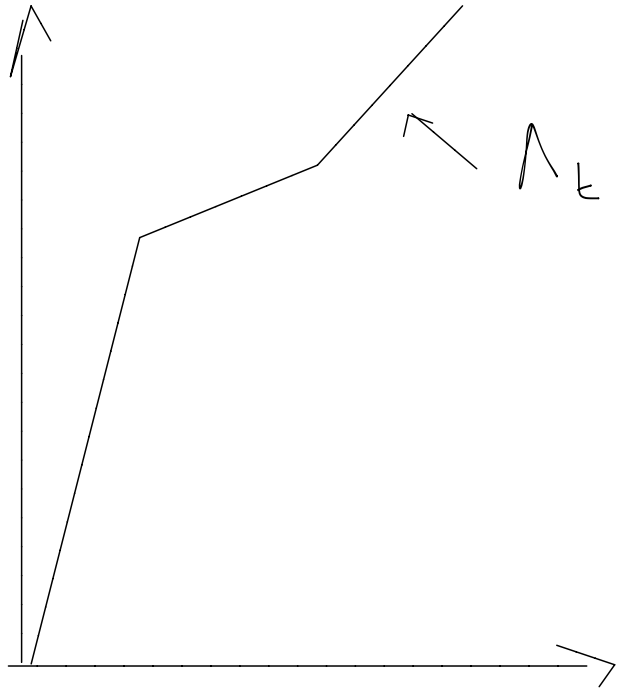
In practice, T and d are fixed $\to$ if d large wrt T, i.e. in the non asymptotic regime, over fitting.

Goodness-of-fit test

If $\Lambda_t = \int_0^t \lambda(s)ds$, then

- Λ_t is a nondecreasing process, predictable and its (pseudo)inverse exists.
- It is the compensator of the counting process N_t, i.e. $(N_t - \Lambda_t)_t$ martingale.





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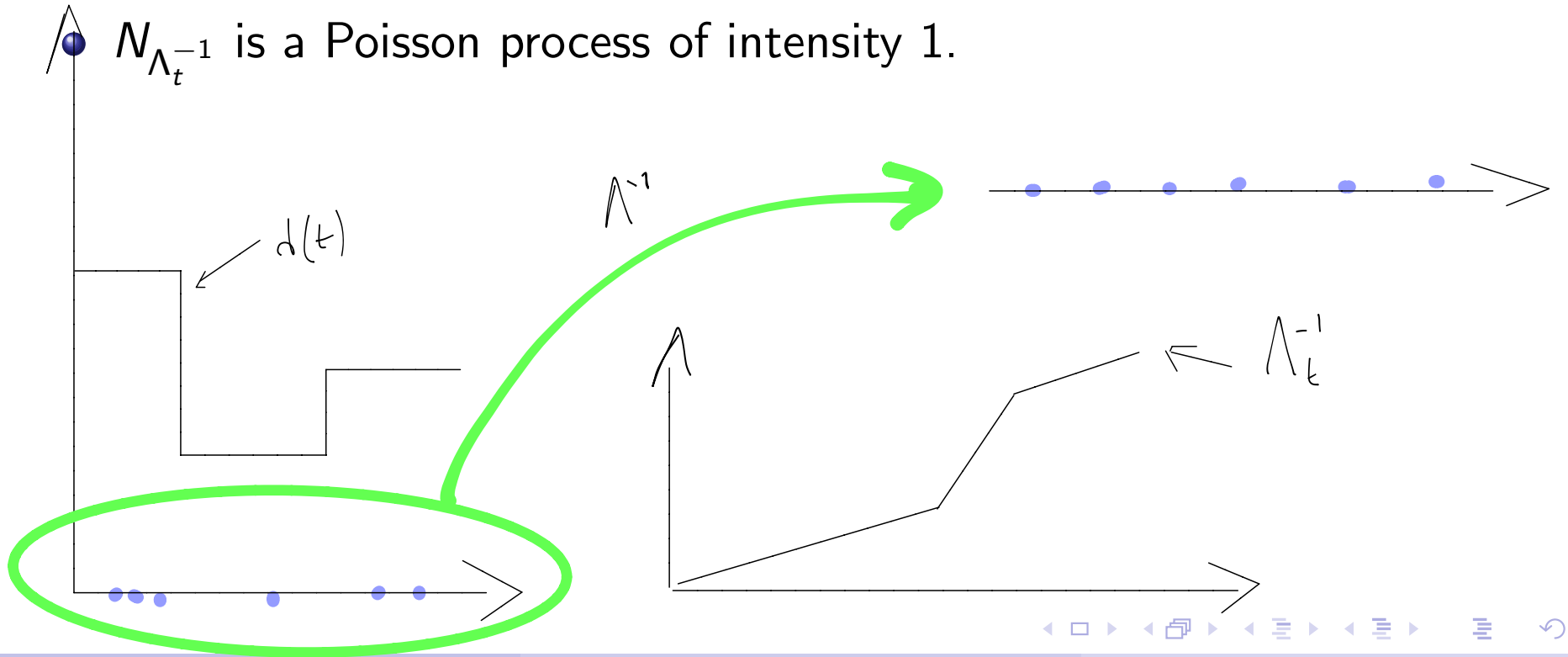

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- $\rightarrow$ test that N (observed) has the desired intensity λ.

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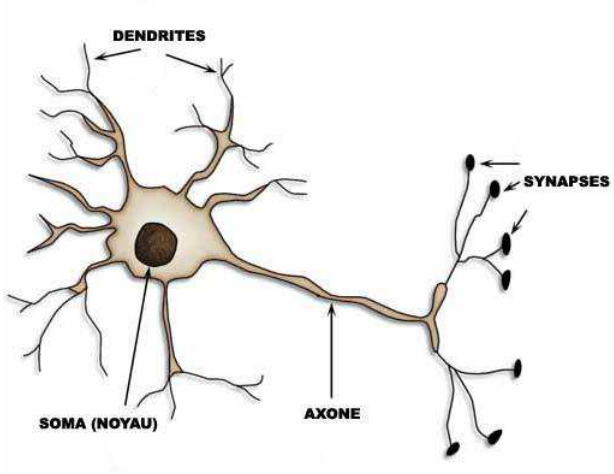
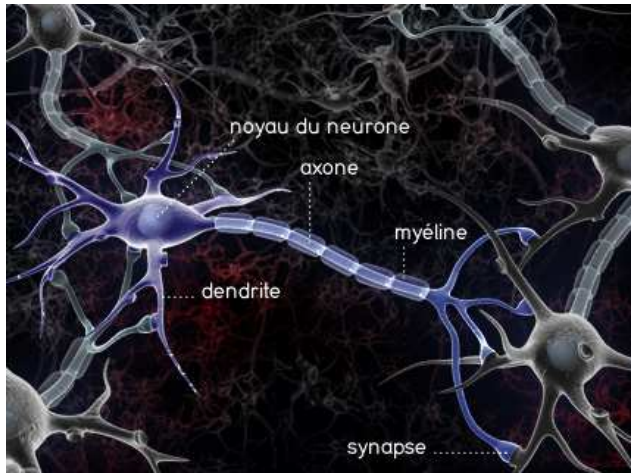
Biological framework



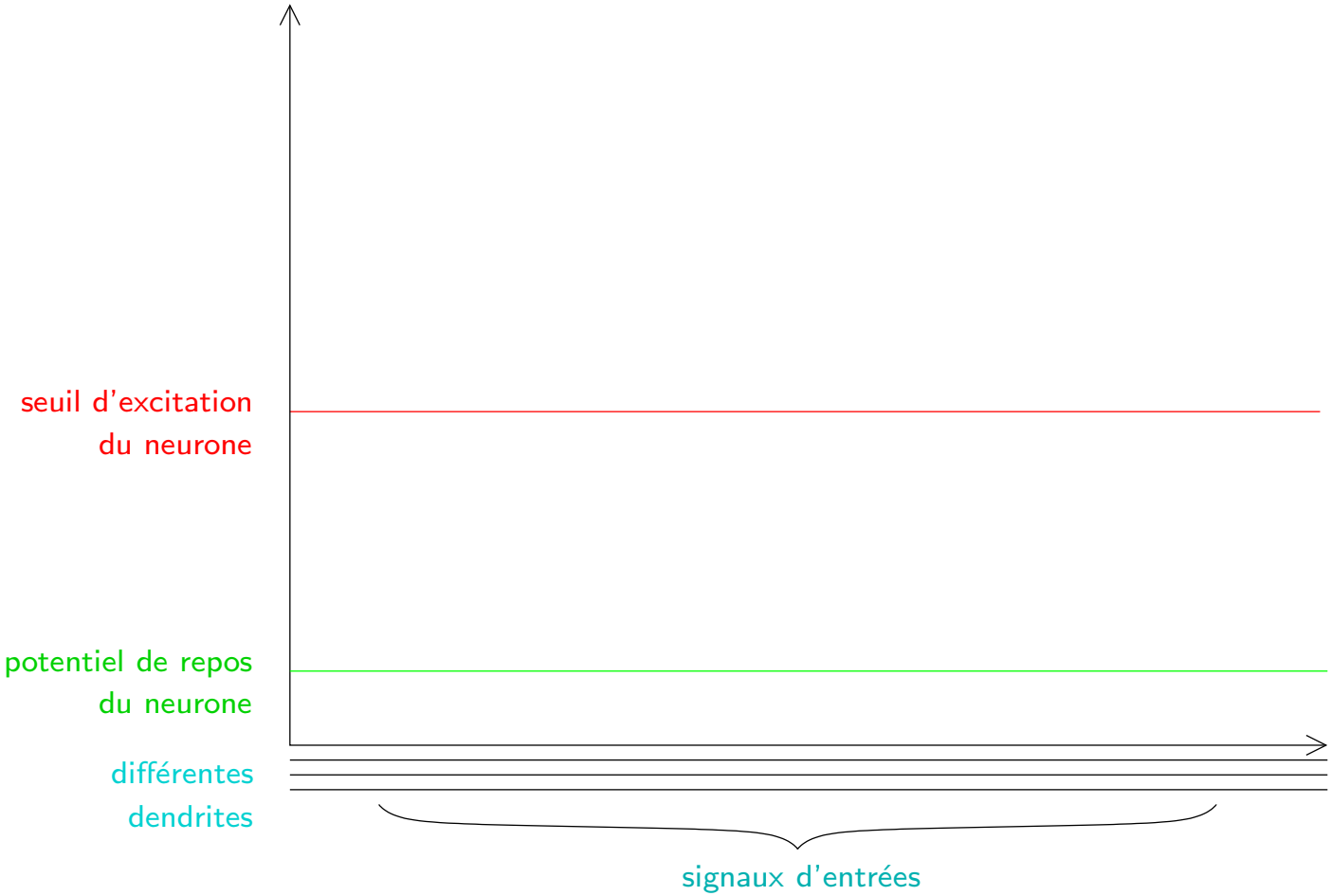
(a) One neuron



(b) Connected neurons

Synaptic integration

without synchronization



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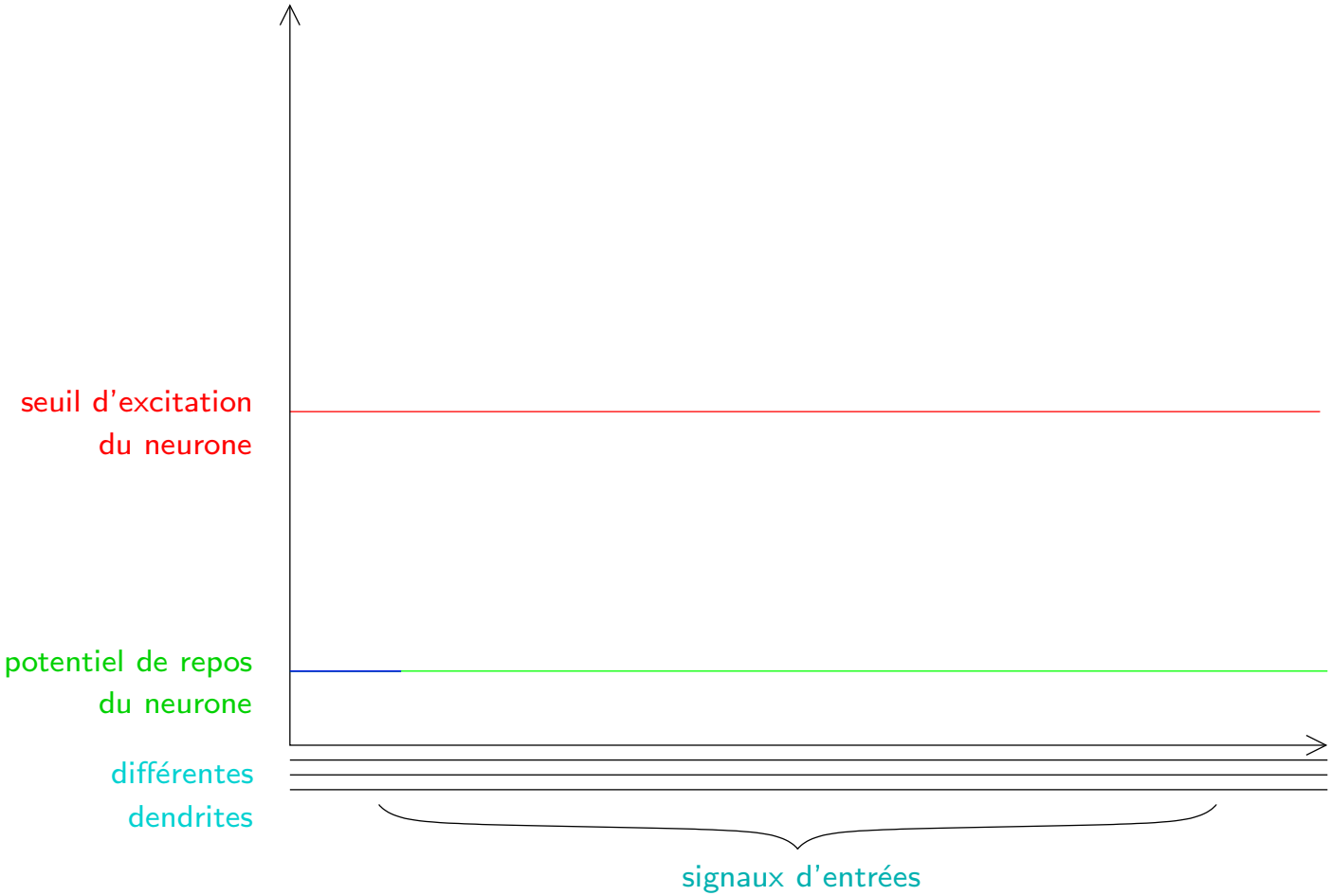

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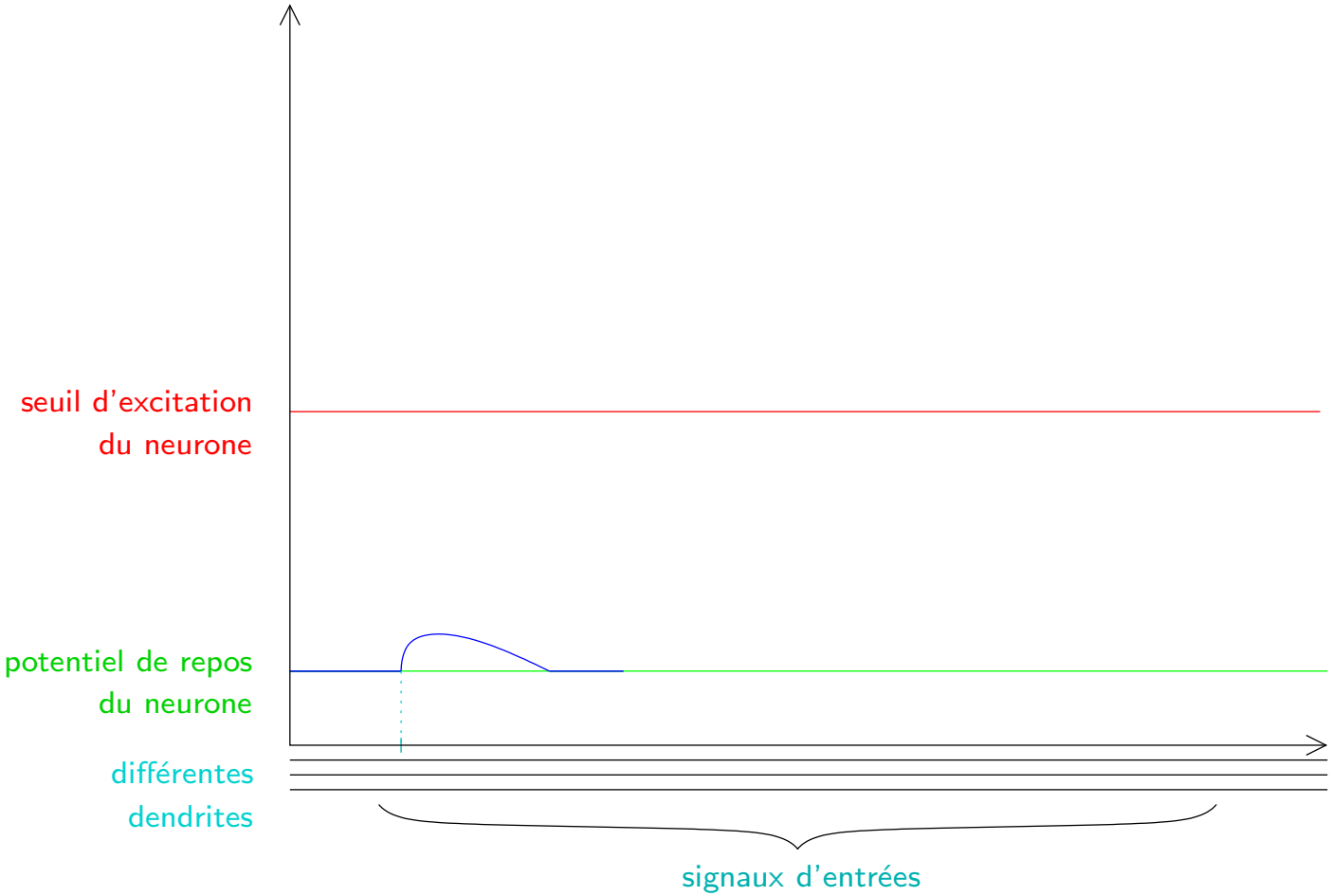

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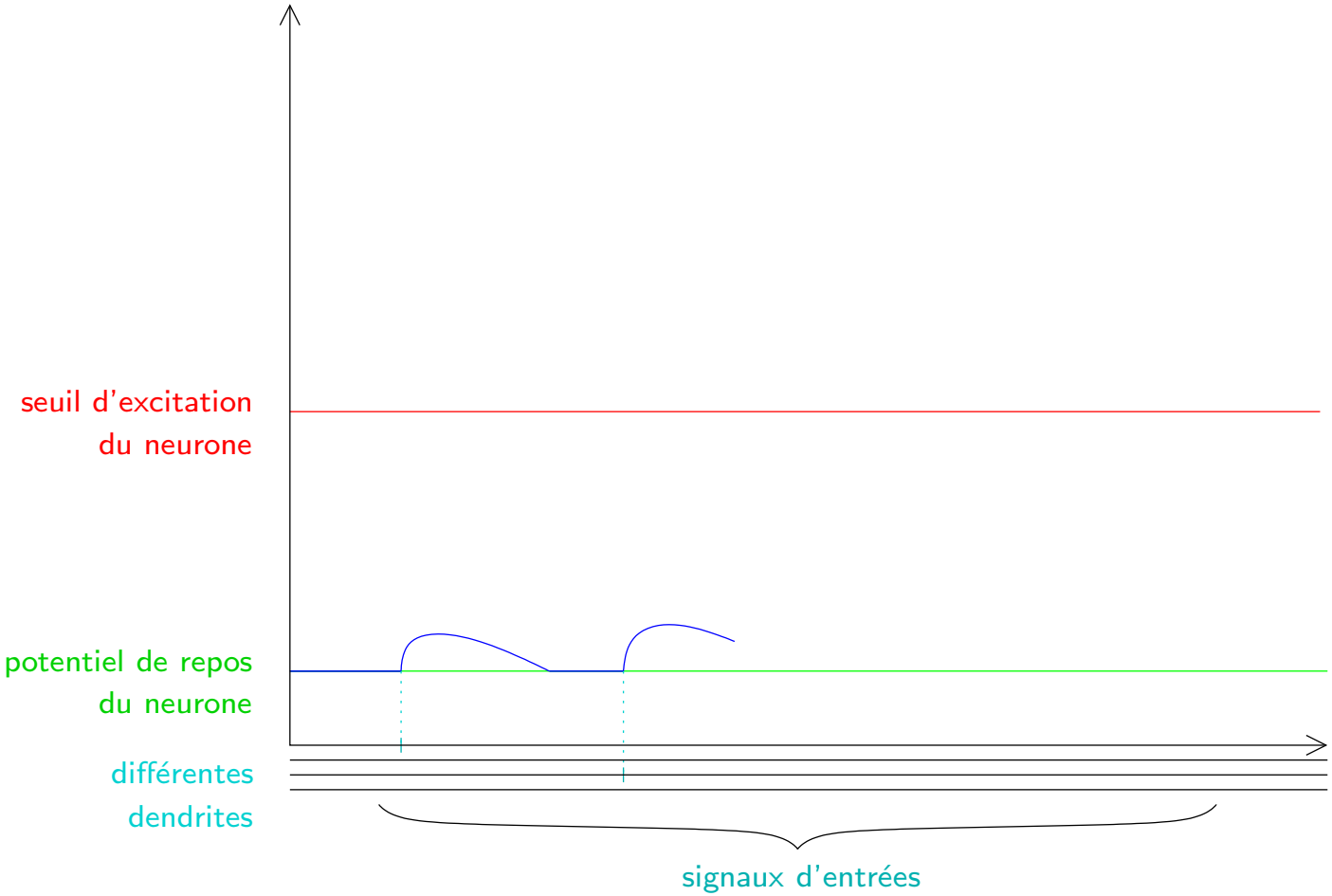

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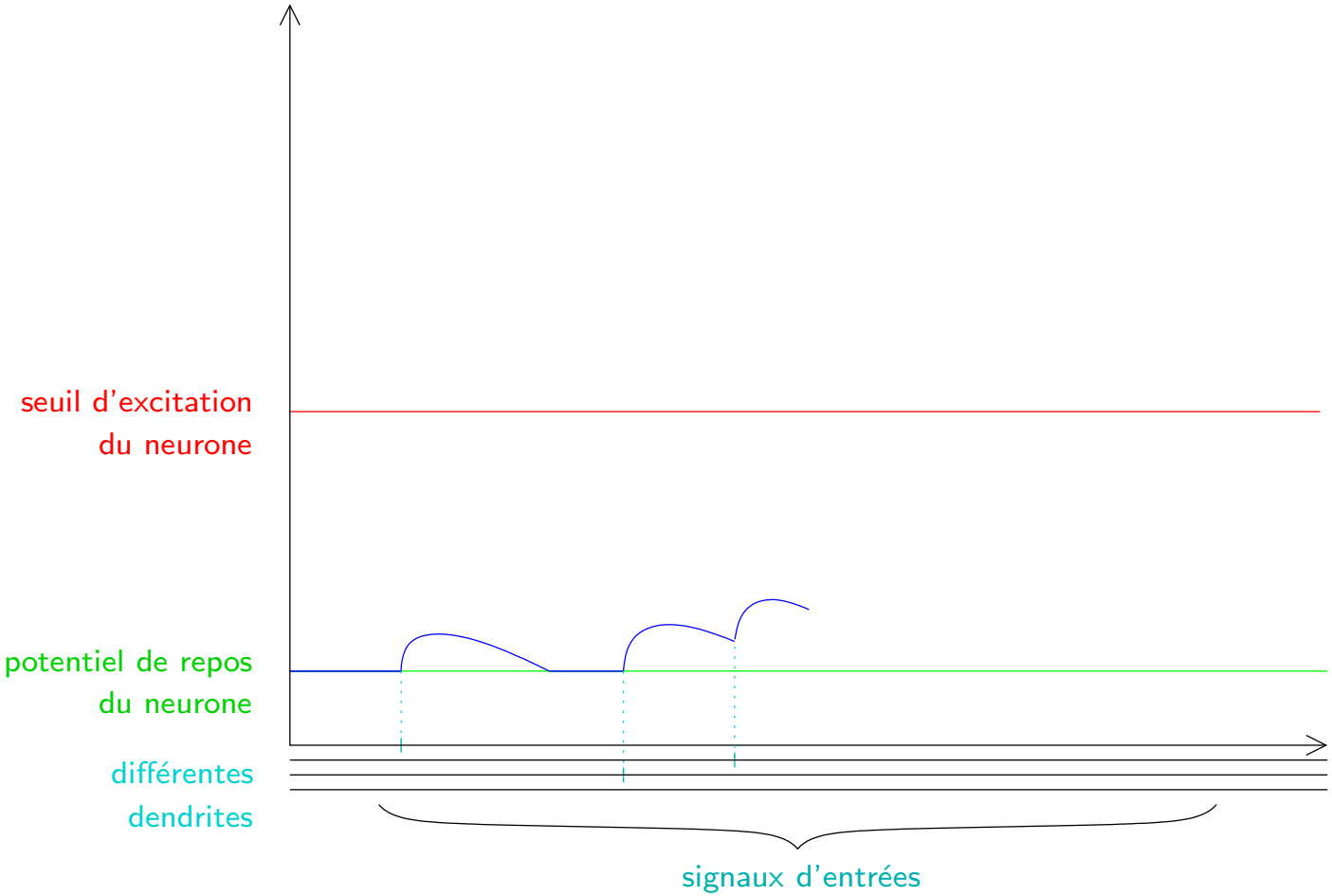

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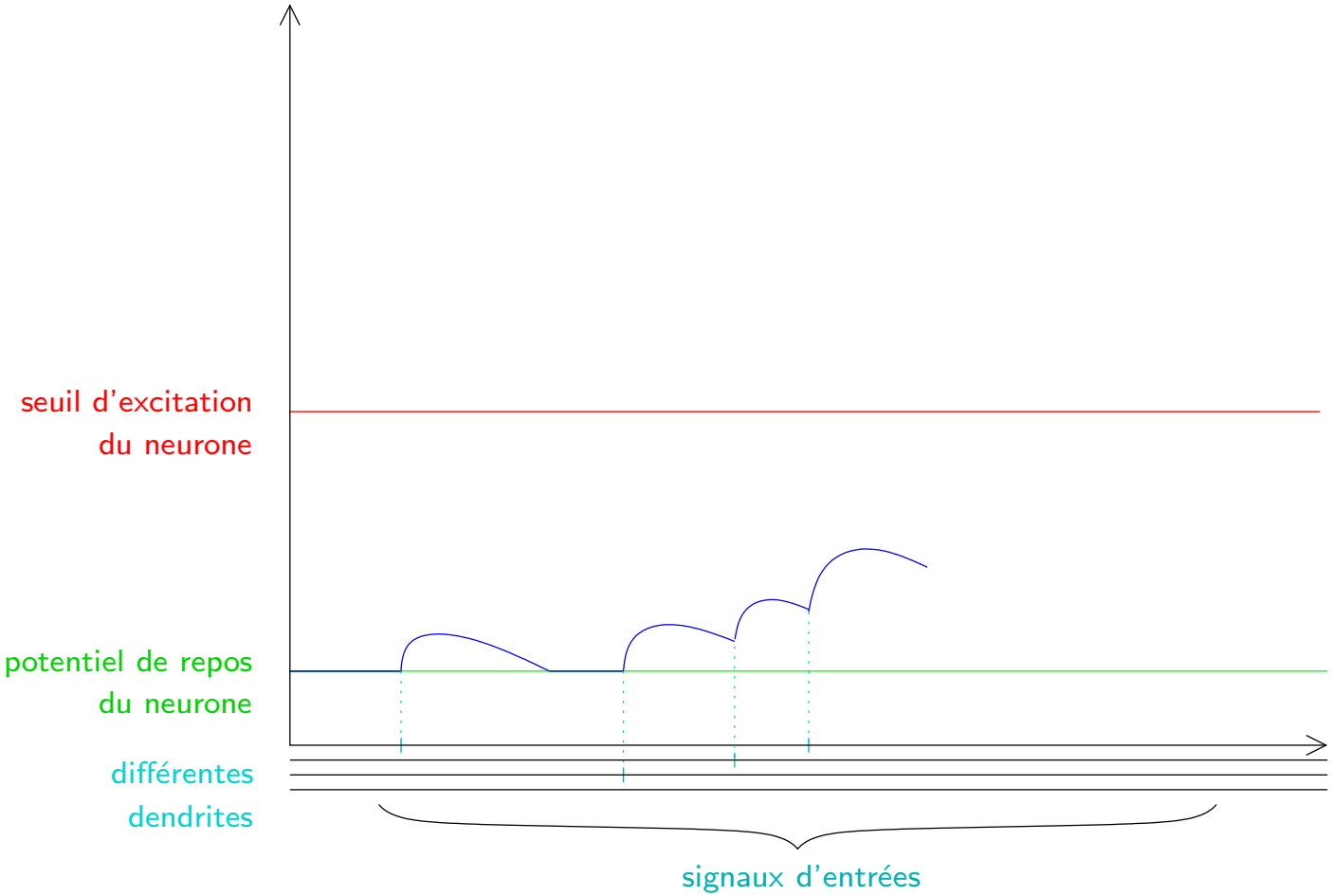

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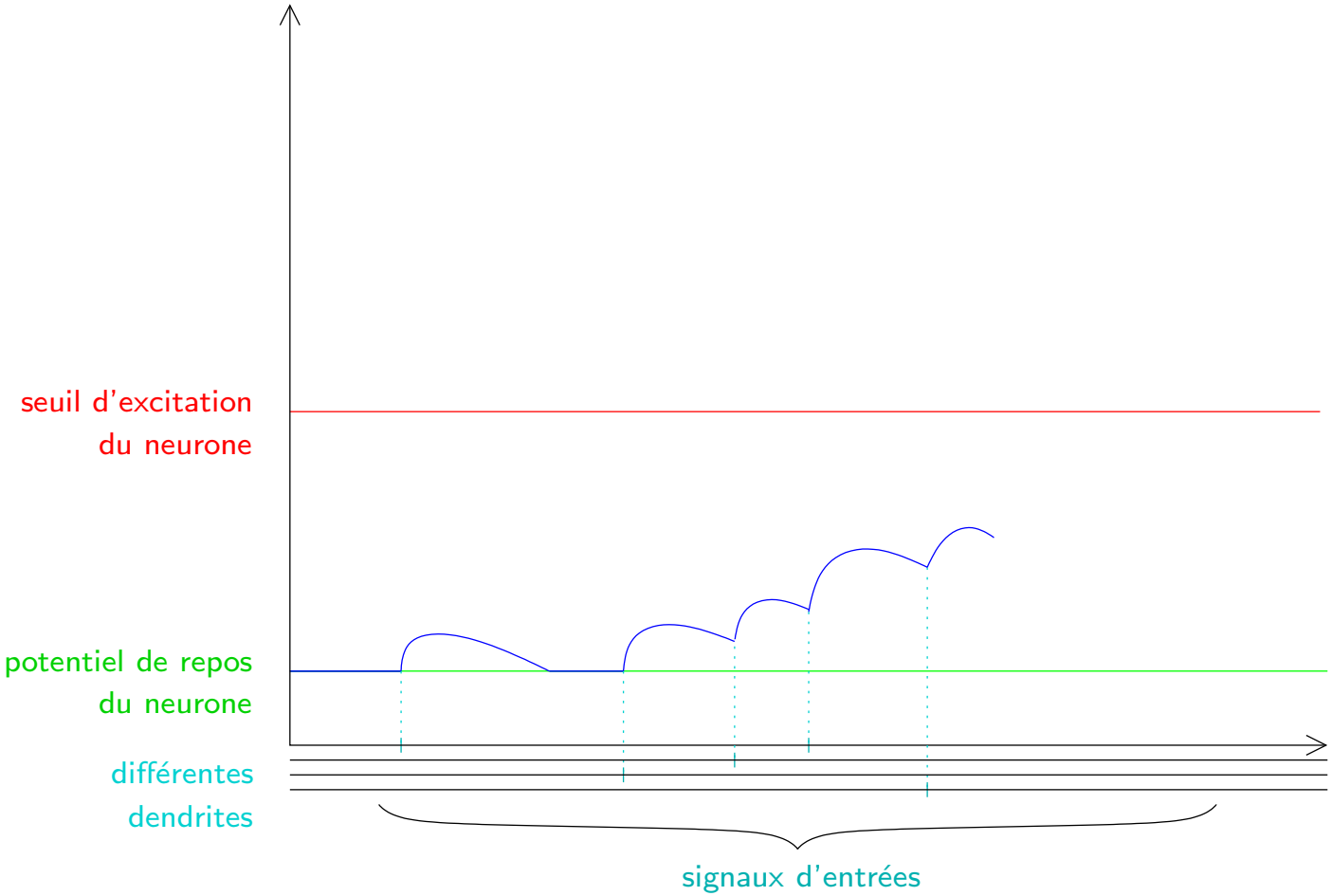

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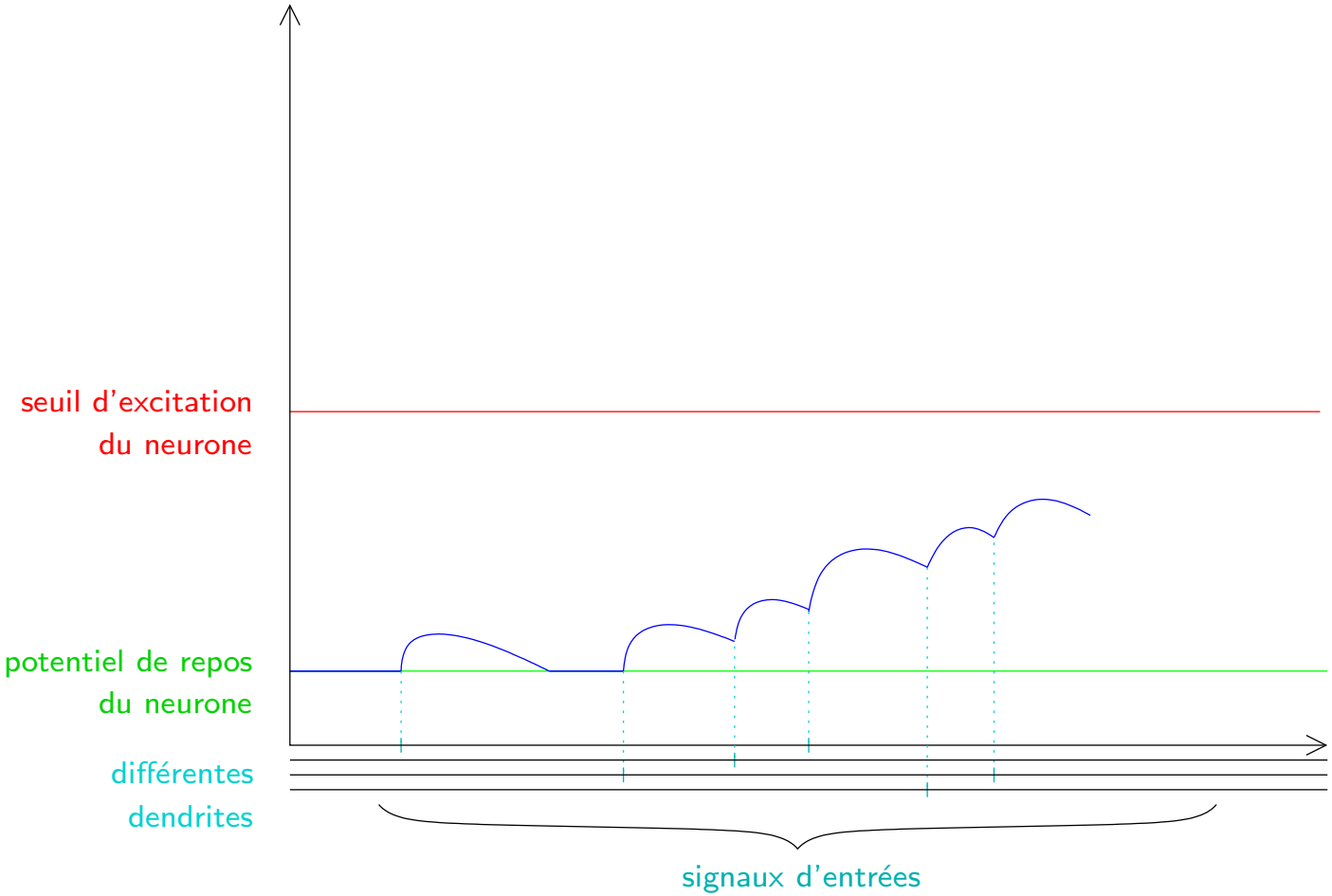

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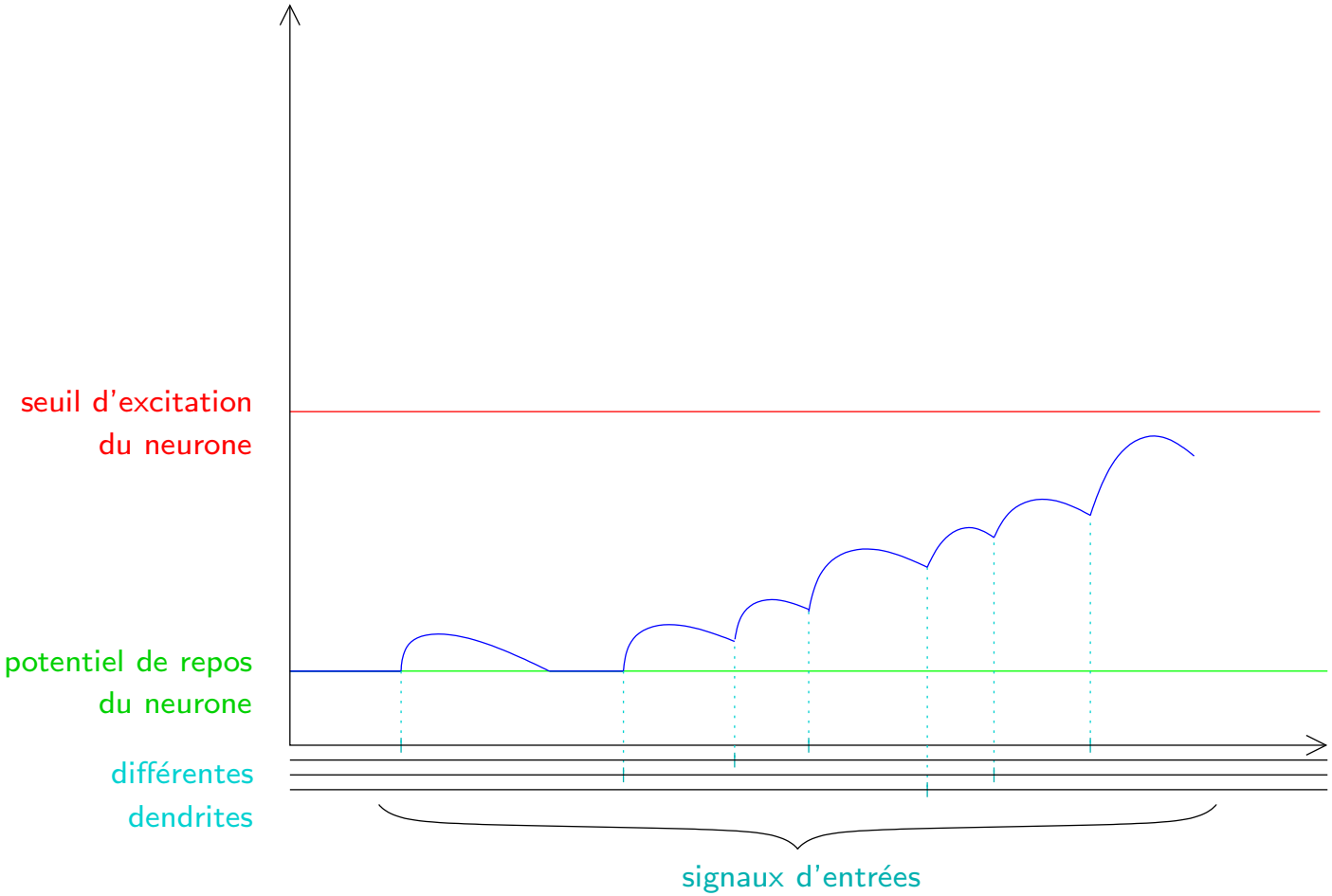

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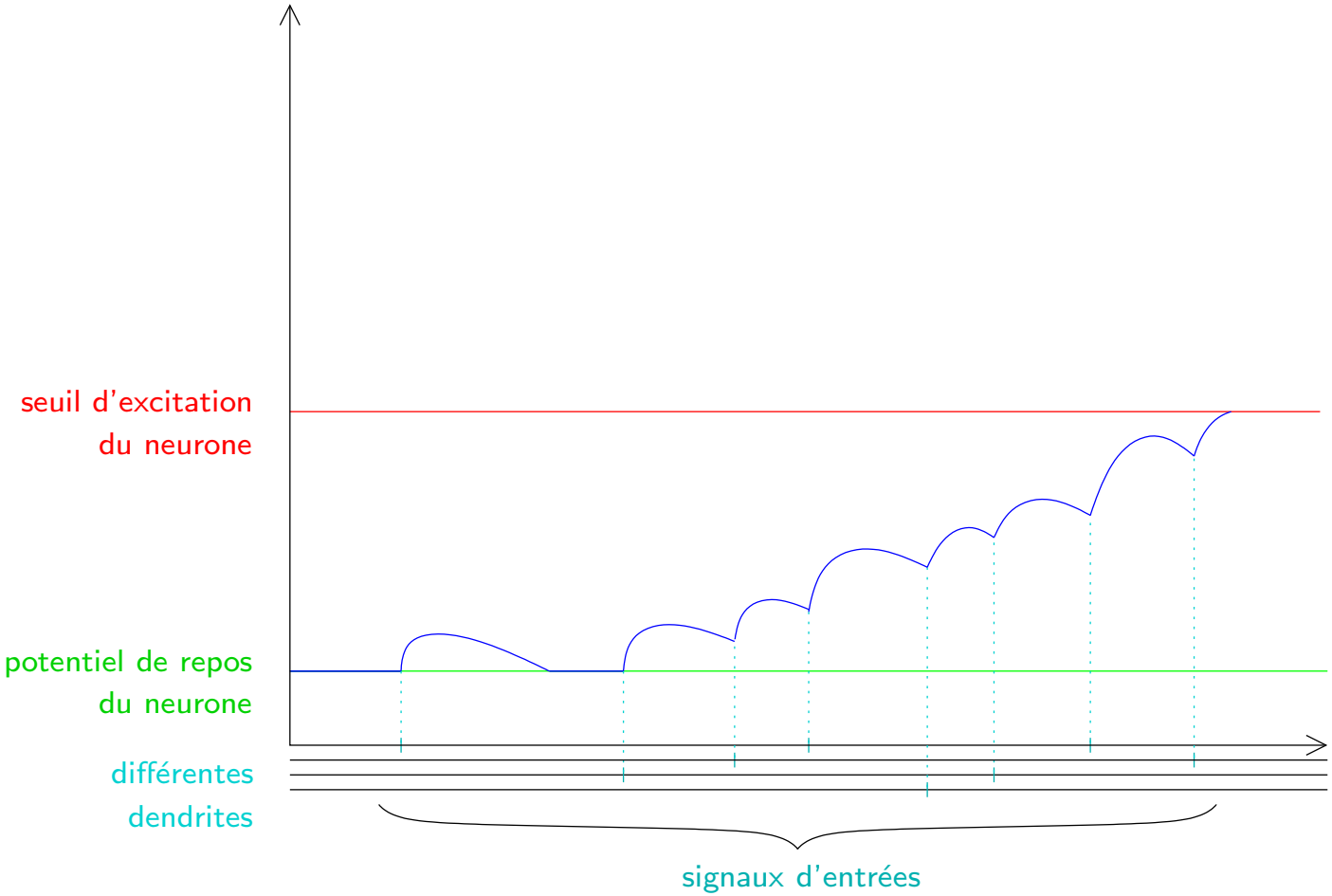

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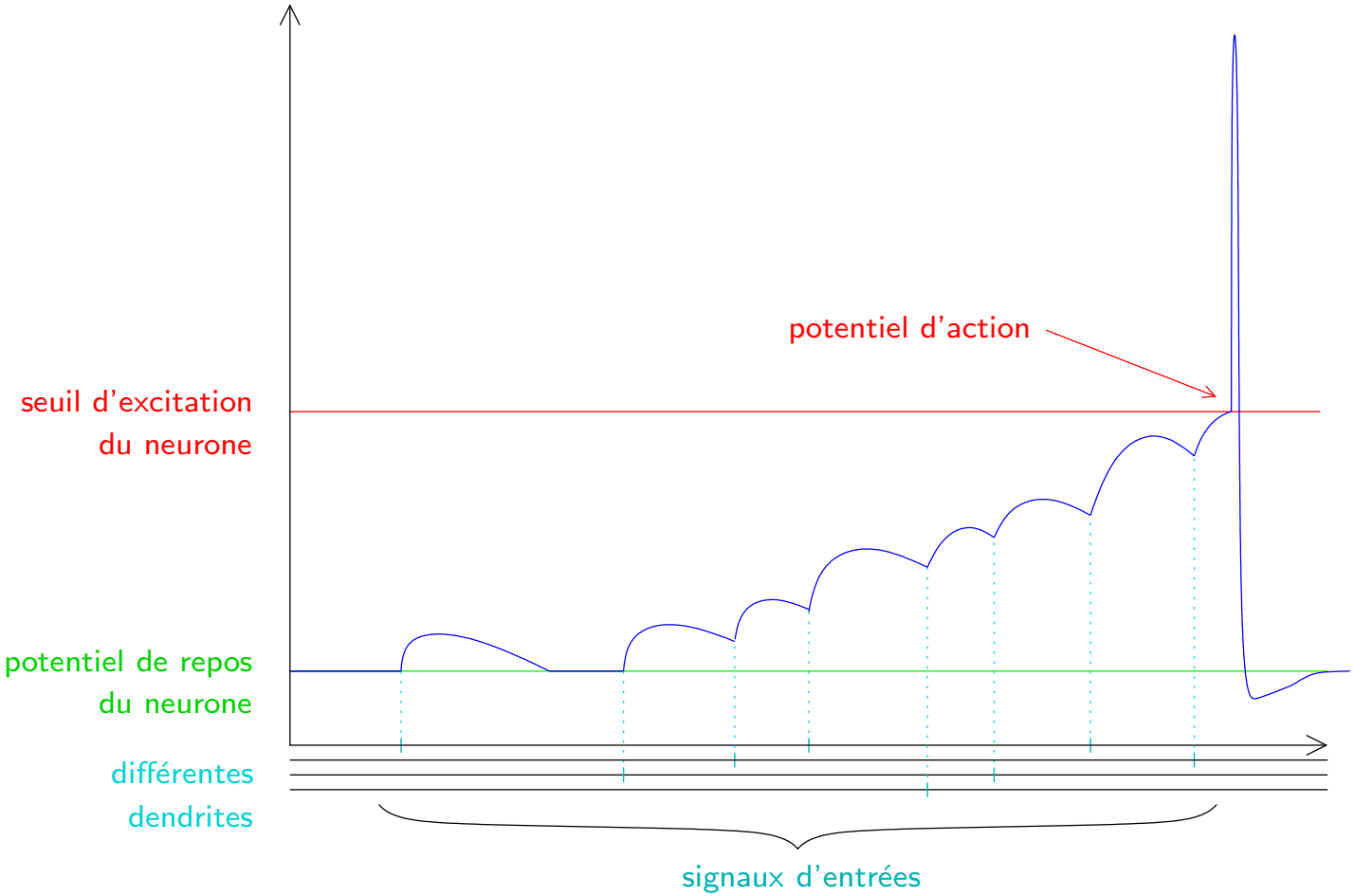

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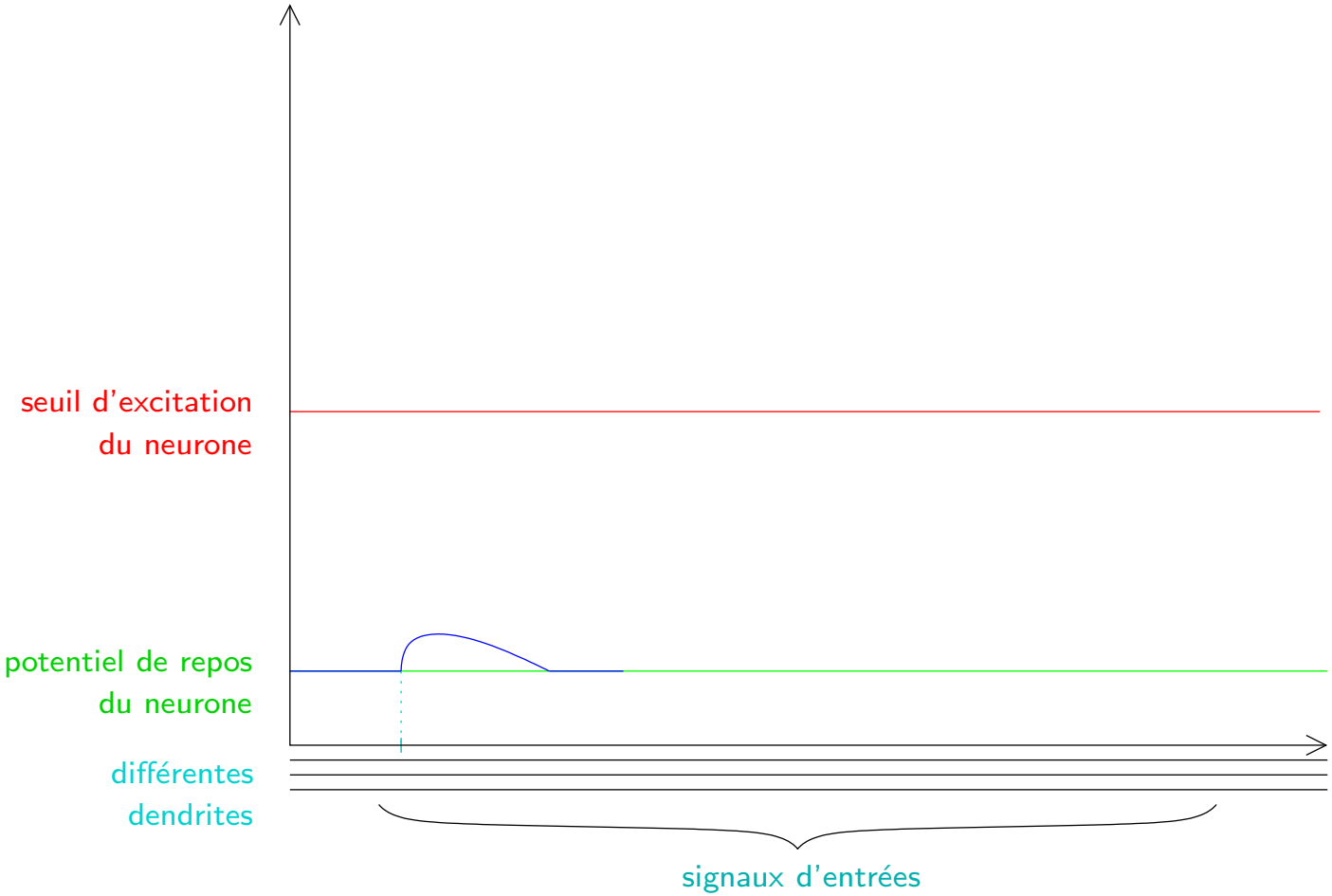

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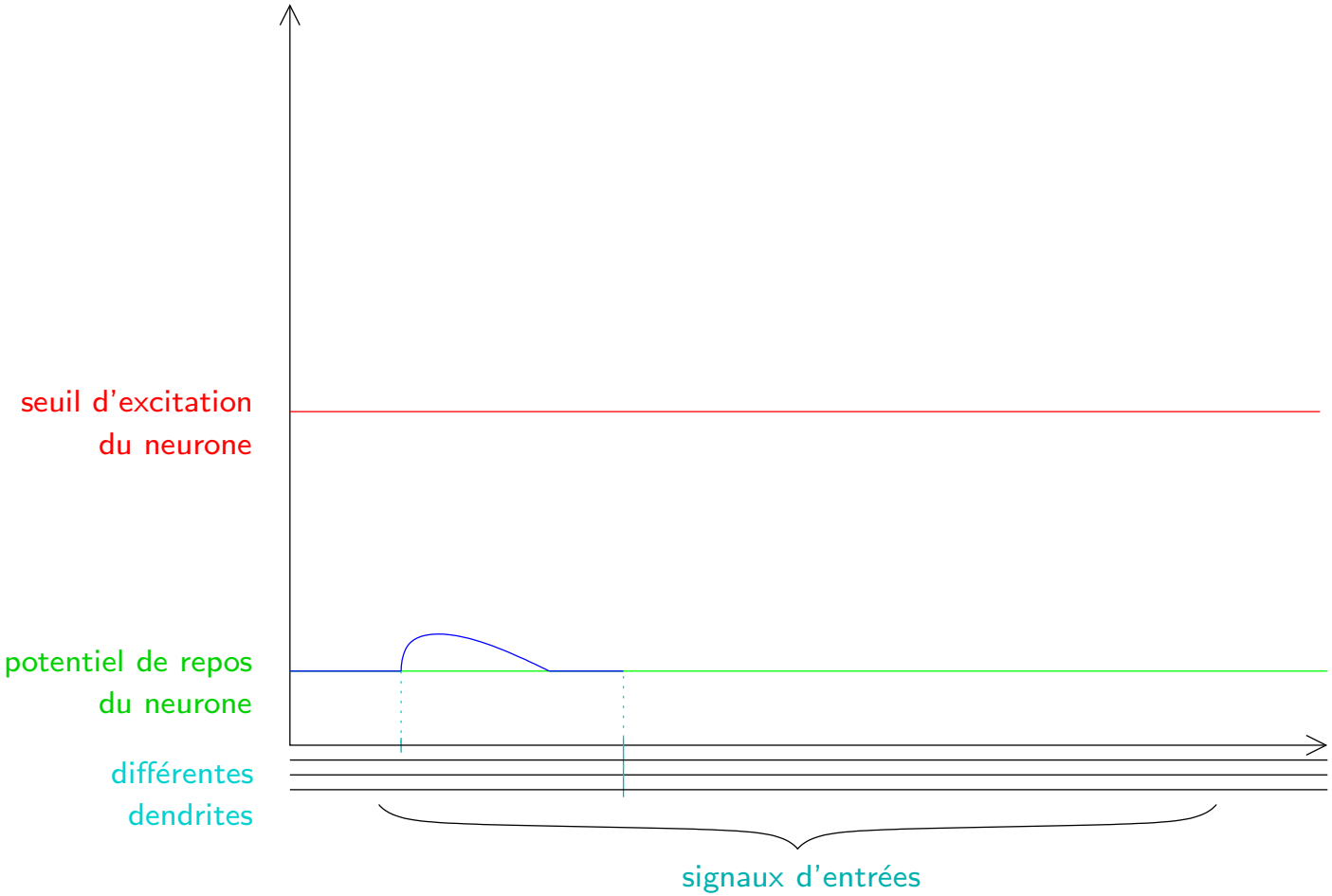

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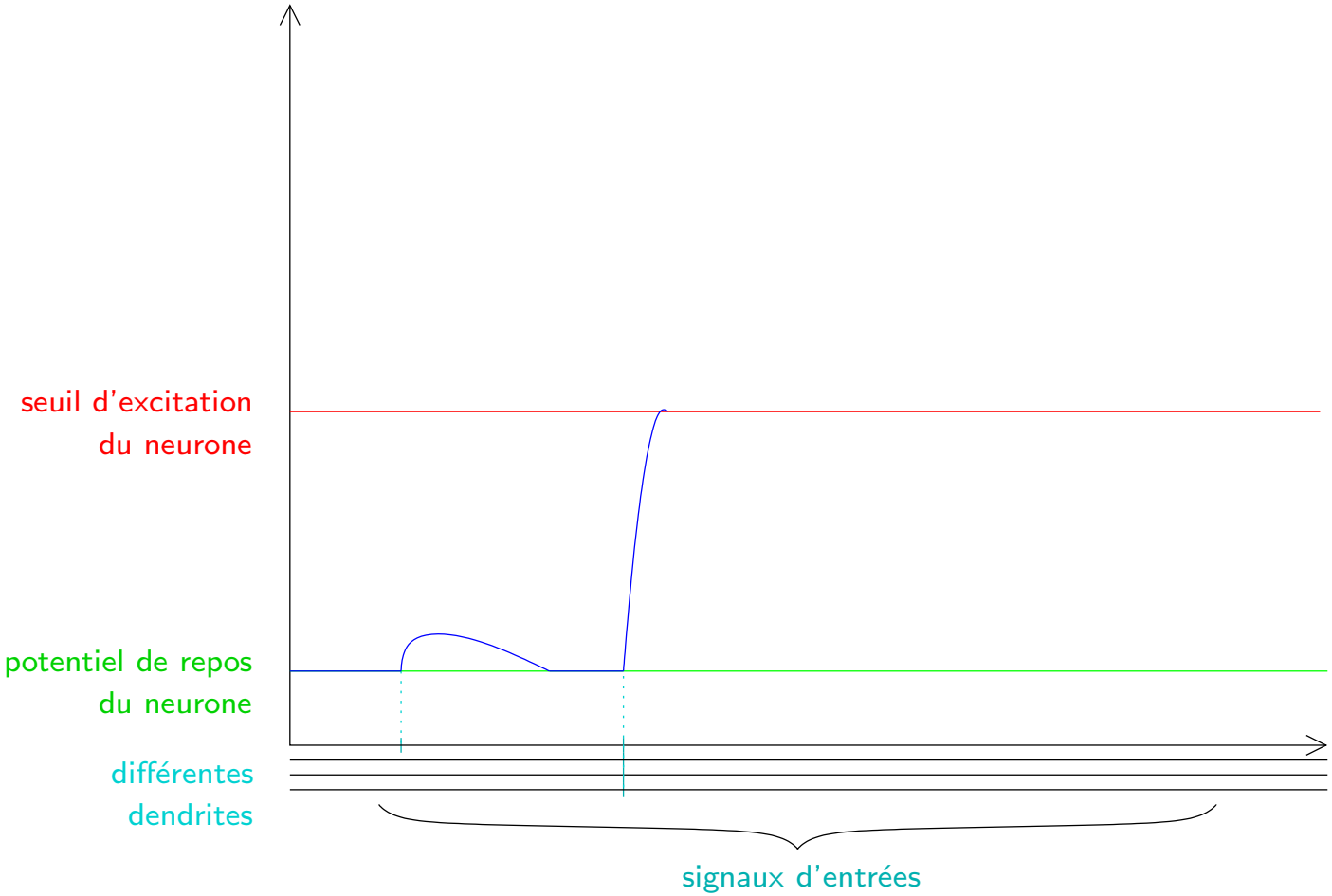

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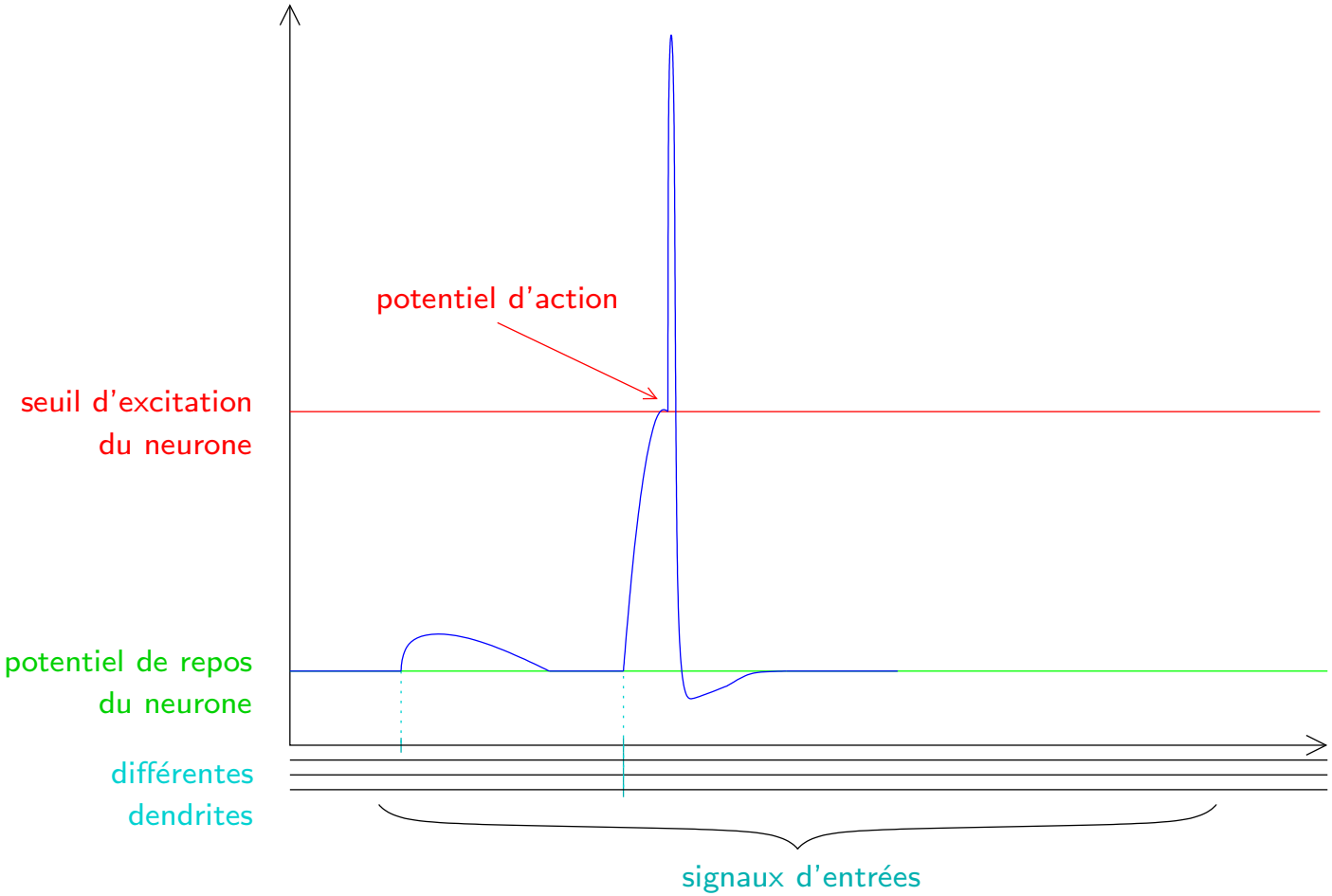

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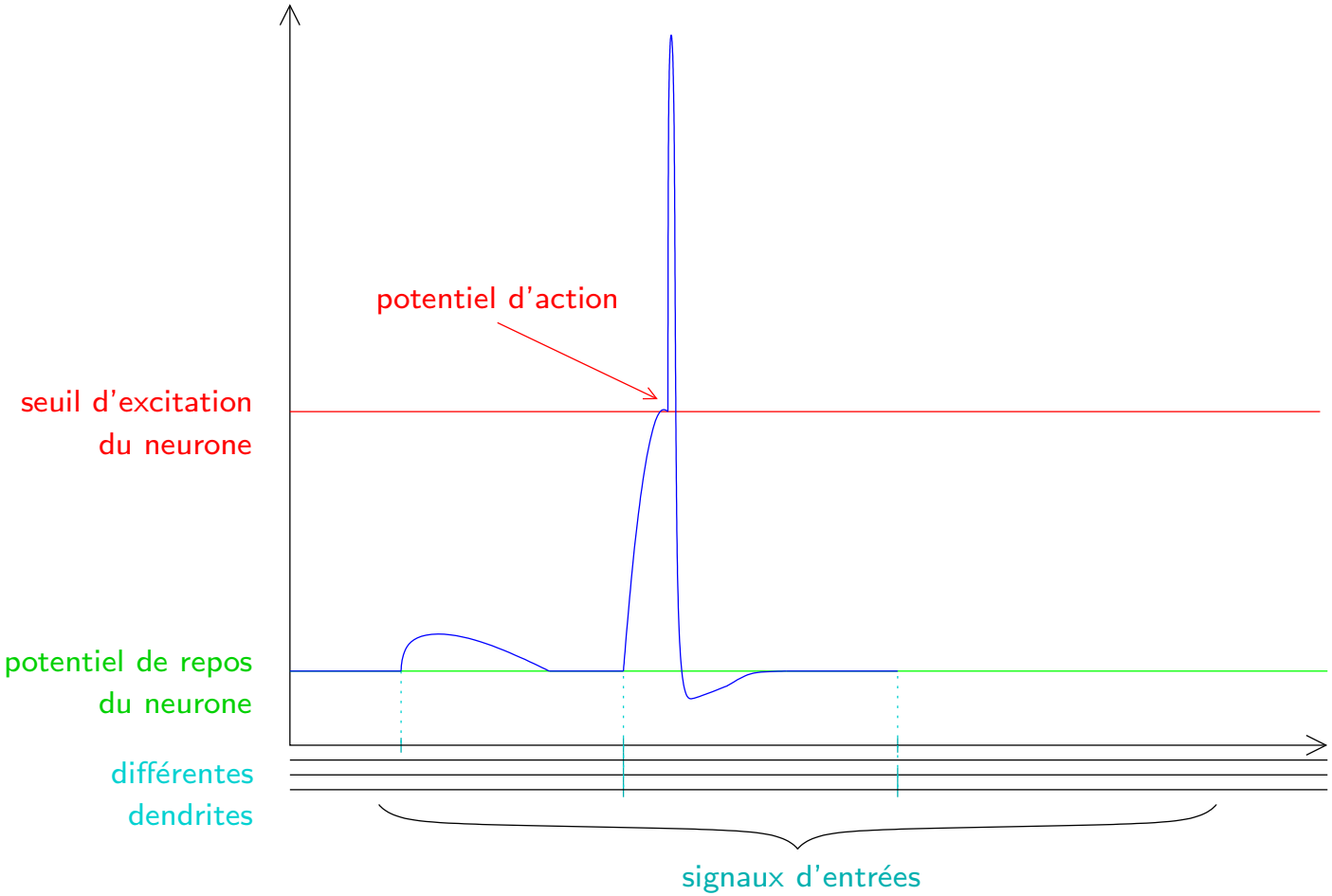

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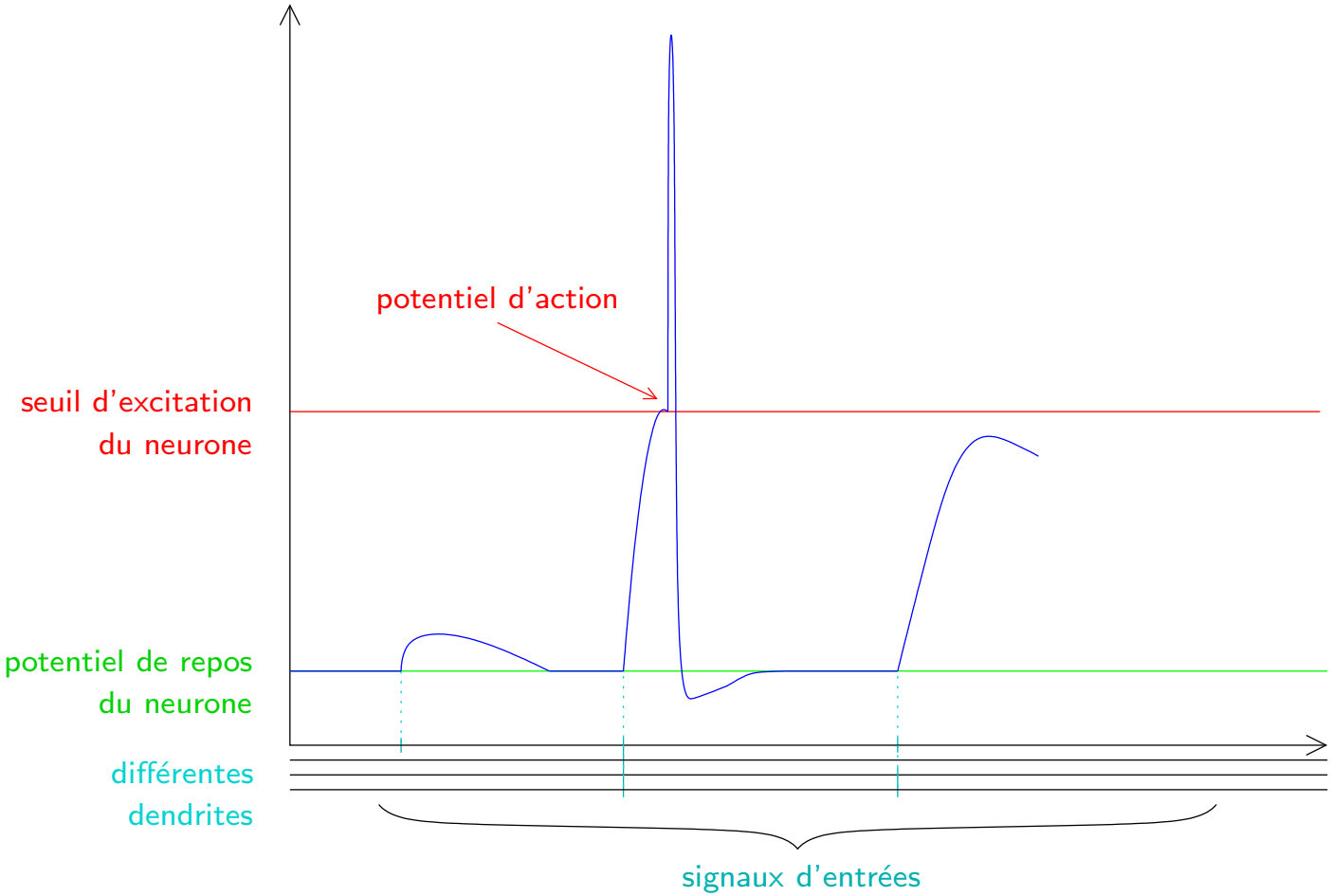

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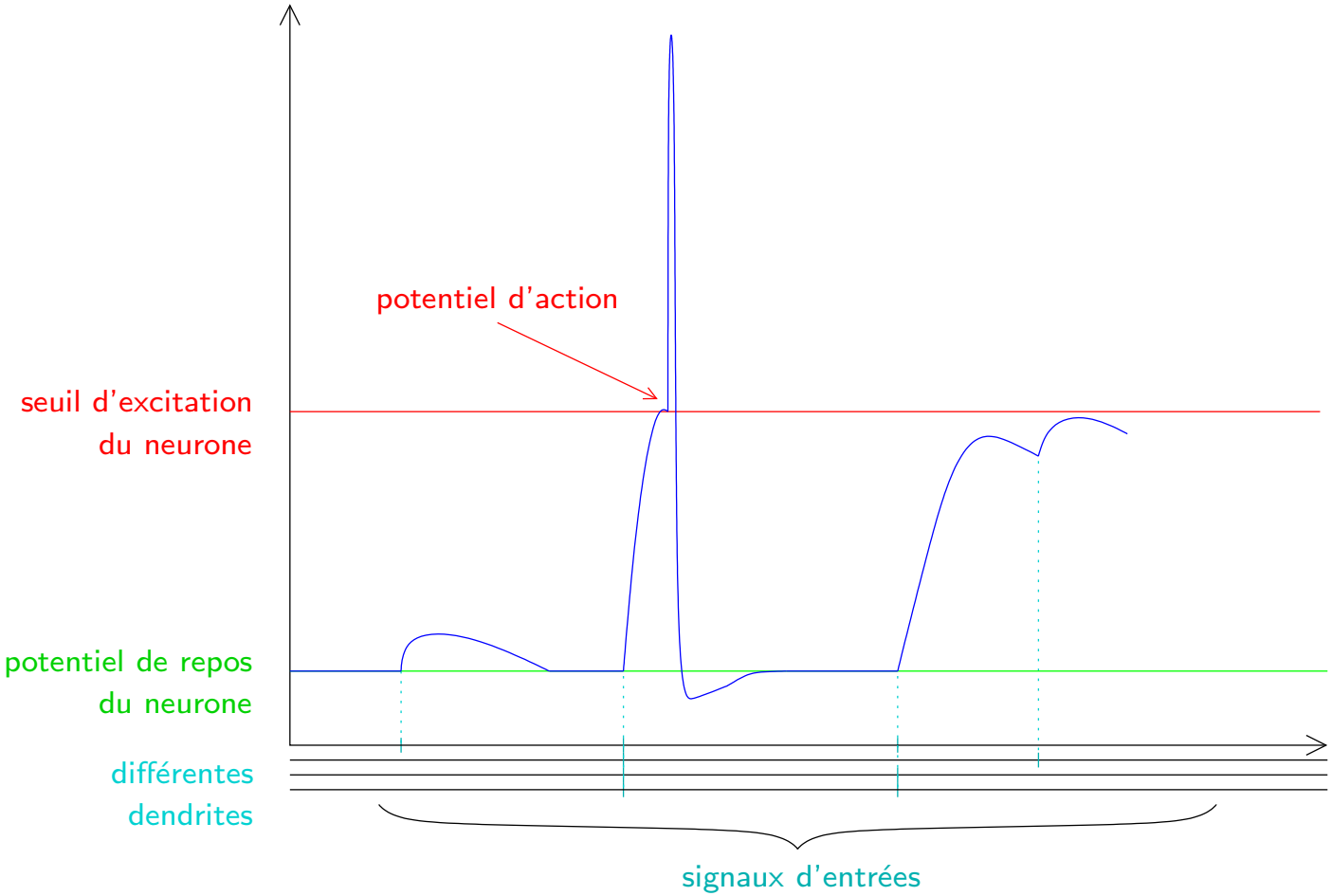

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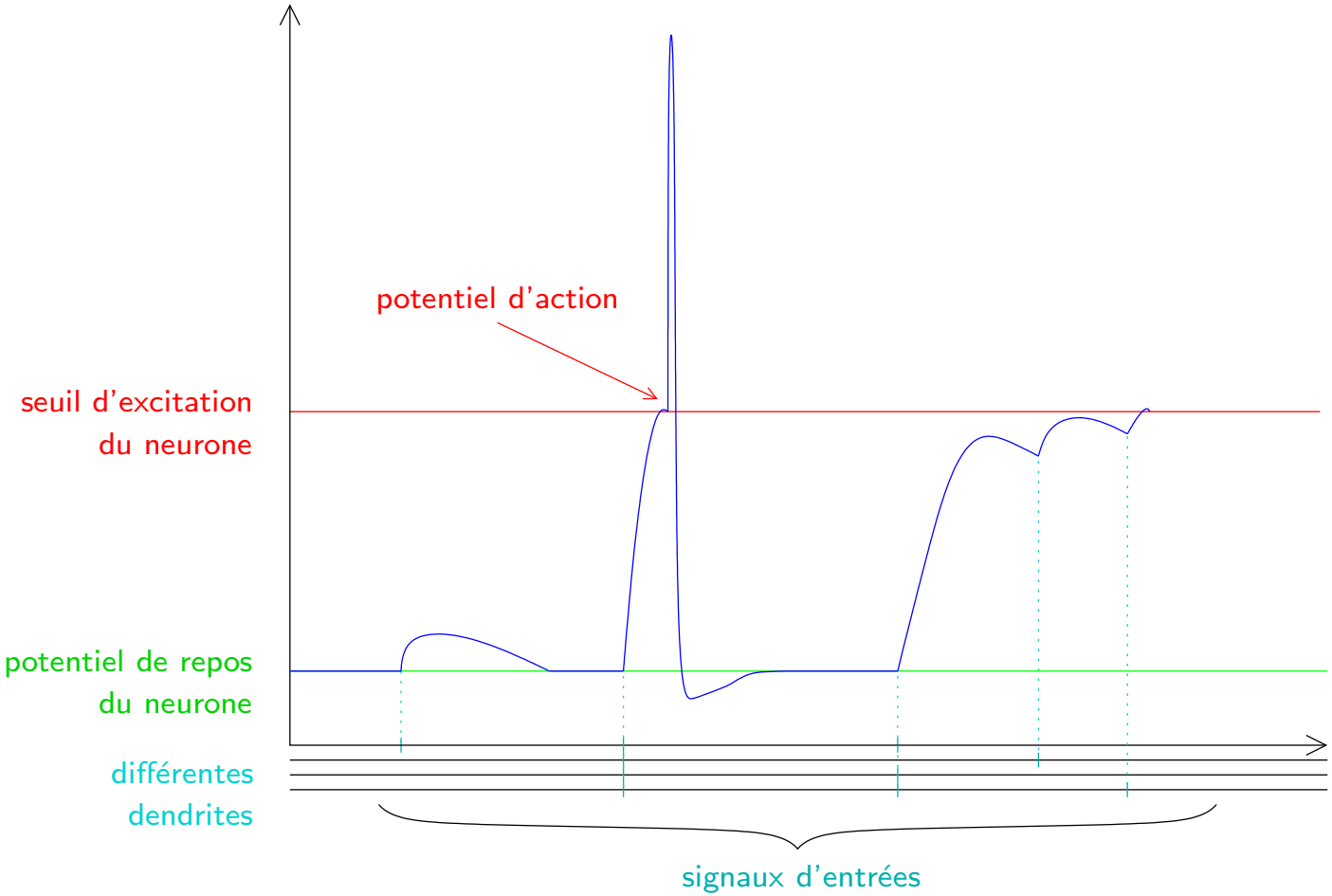

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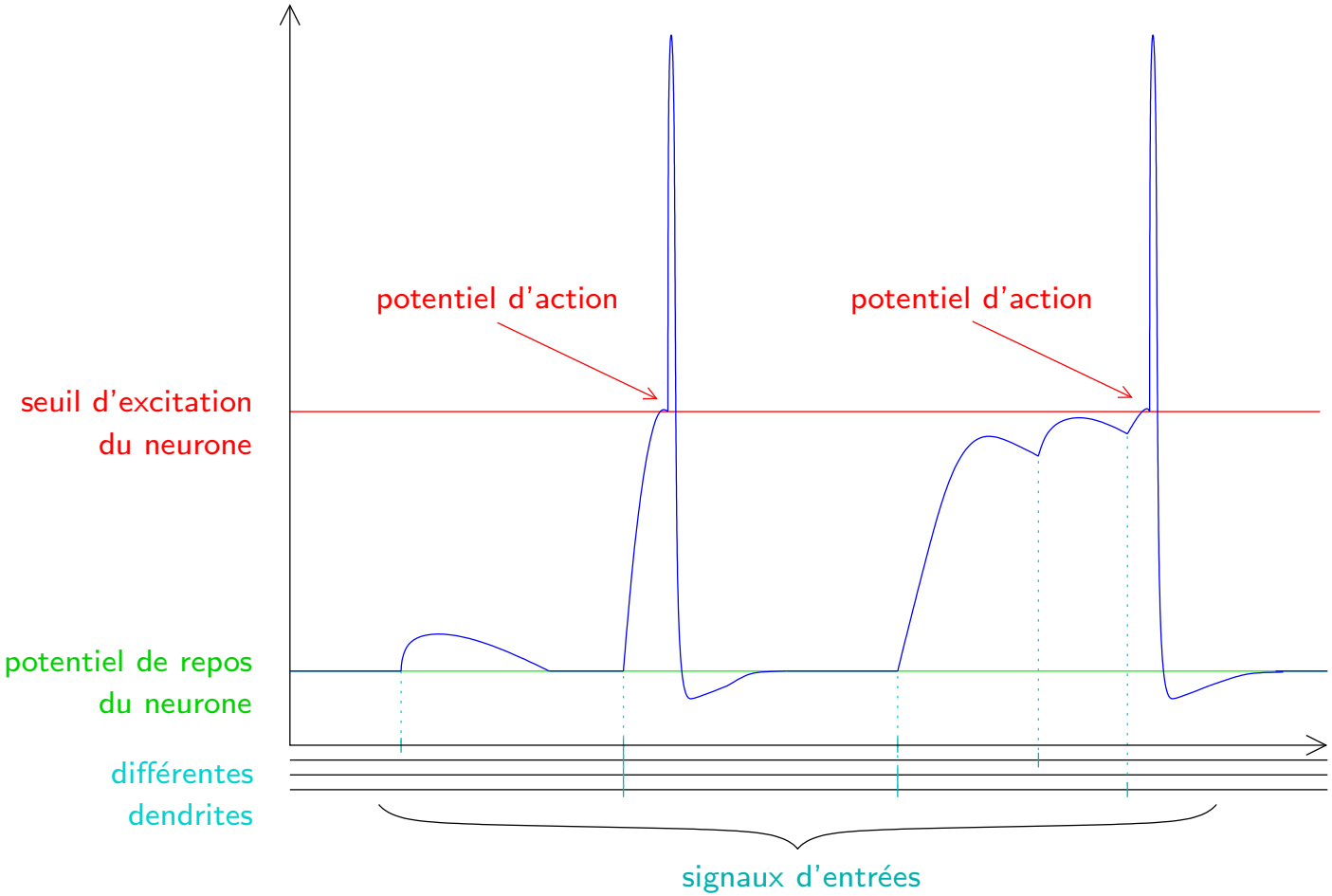

Point process and genomics

There are several "events" of different types on the DNA that may "work" together in synergy.

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Why real multivariate point processes in genomics ?

TRE

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- If the real structure $\rightarrow$ 3D point processes on graphs... (? ?)

Why just DNA ? RNA etc ...

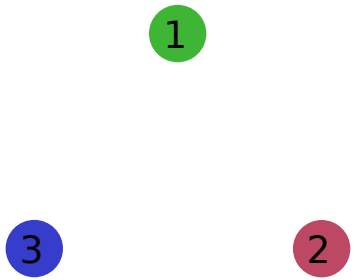
Point processes and conditional intensity

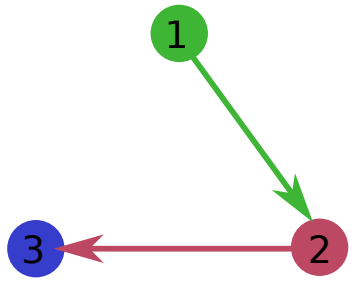
$$\underbrace{dN_t}_{\text{Nbr observed points in } [t,t+dt]} = \overbrace{\underbrace{\lambda(t)\,dt}_{\text{Expected Number given the past before } t}}^{\textbf{intensity}} + \underbrace{\text{noise}}_{\textit{Martingales differences}}$$

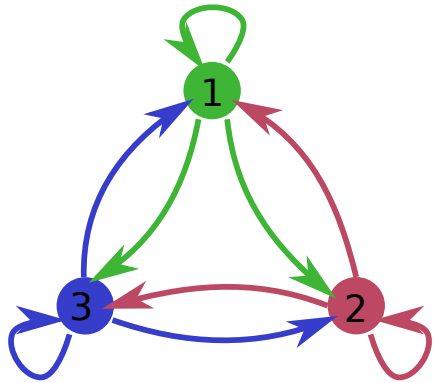
$$\lambda(t) = \text{instantaneous frequency} = \textbf{random}, \text{ depends on previous points}$$

Local independence graphs

(Didelez (2008))







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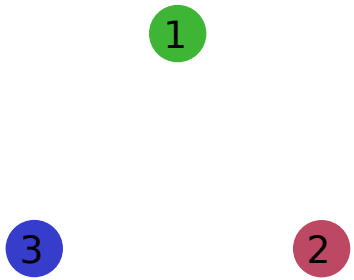
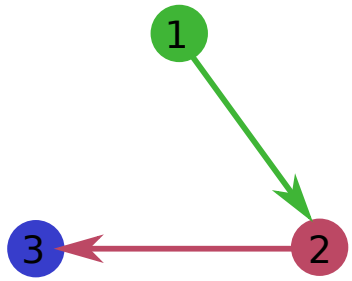
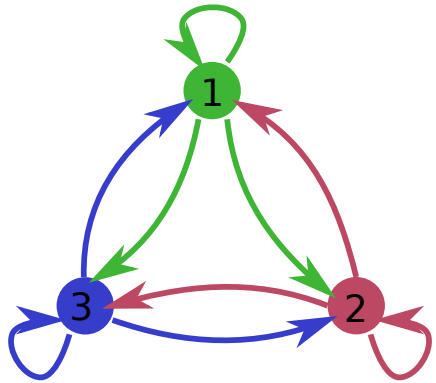
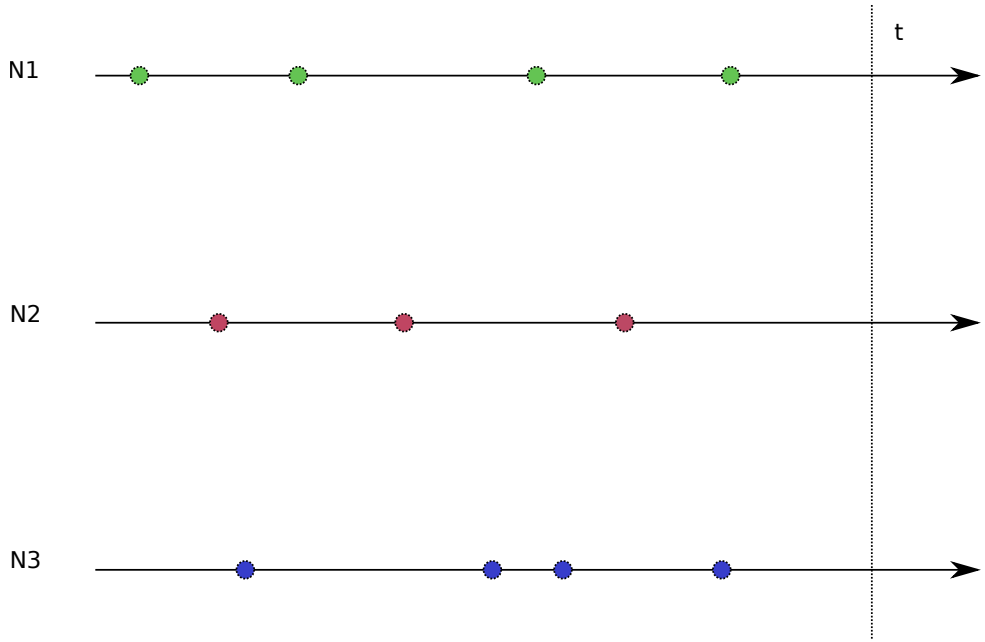






If one is able to infer a local independence graph, then we may have access to "functional connectivity".
$\hookrightarrow$ needs a "model" .

Multivariate Hawkes processes



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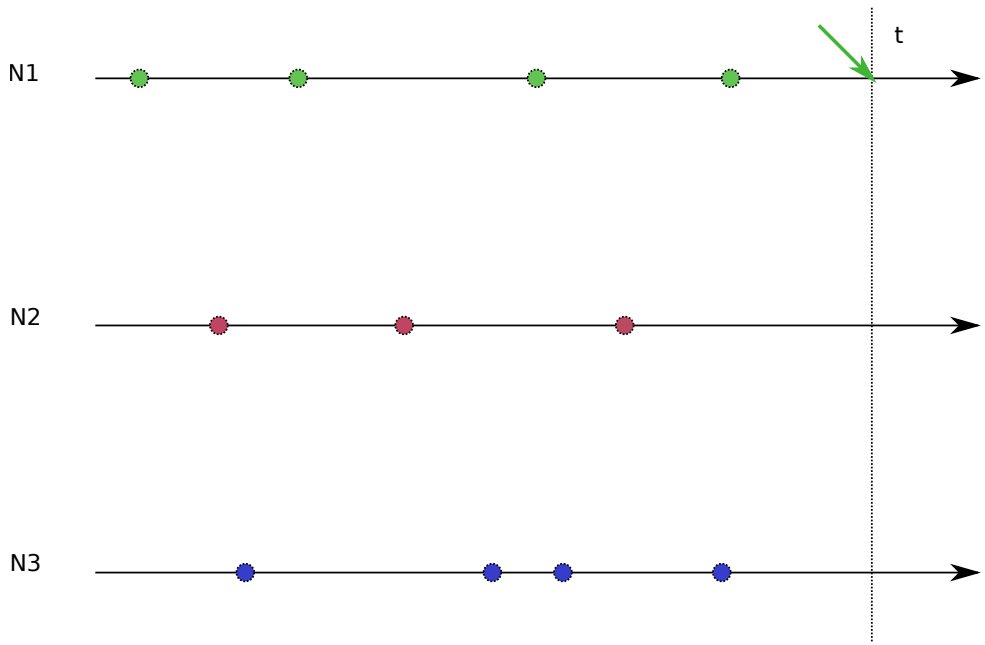

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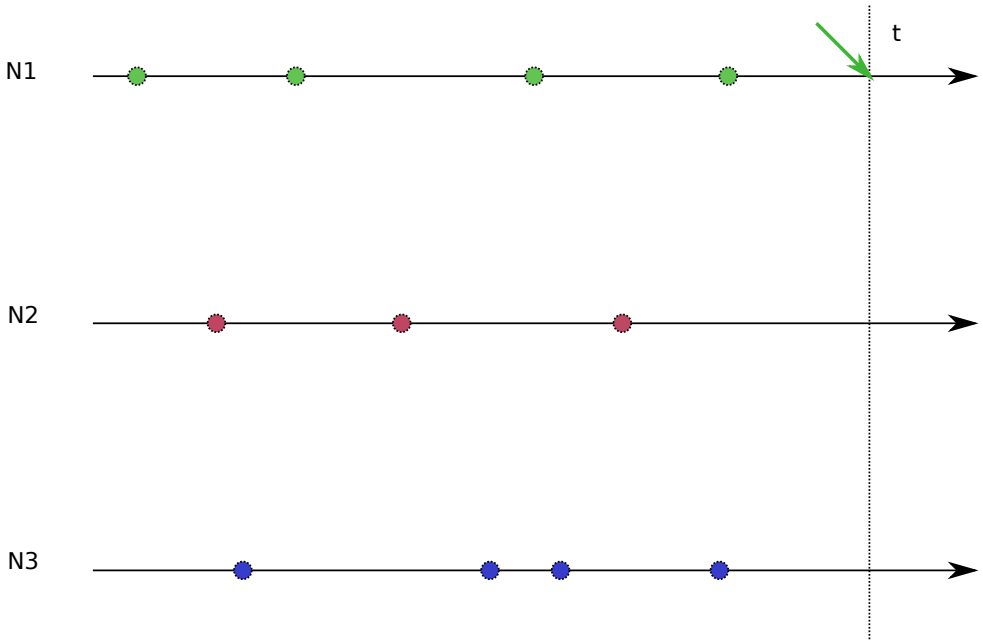

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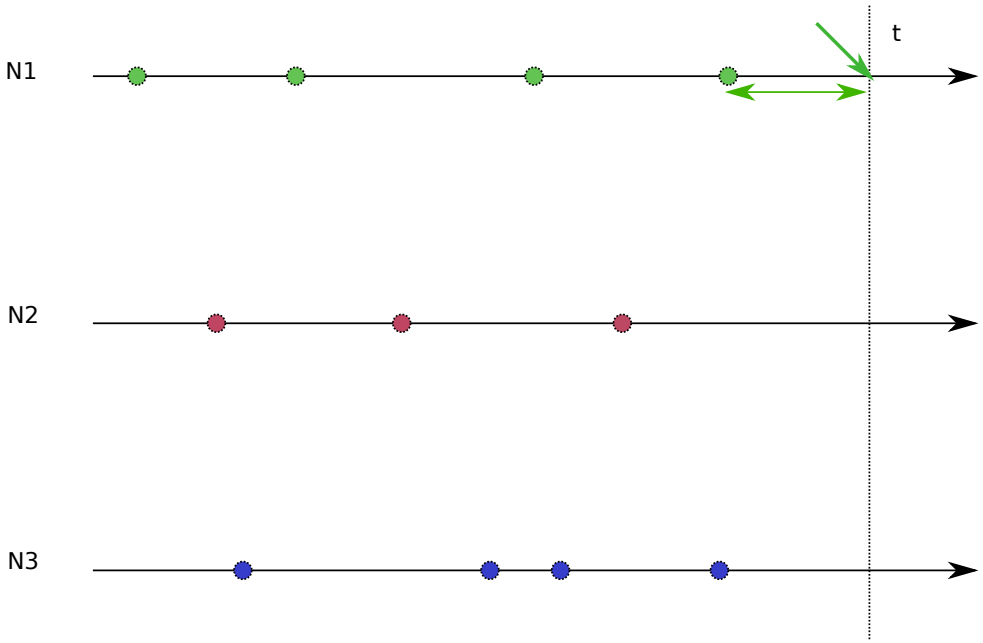

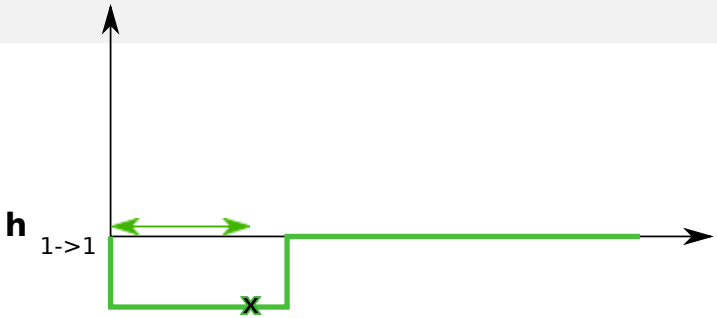

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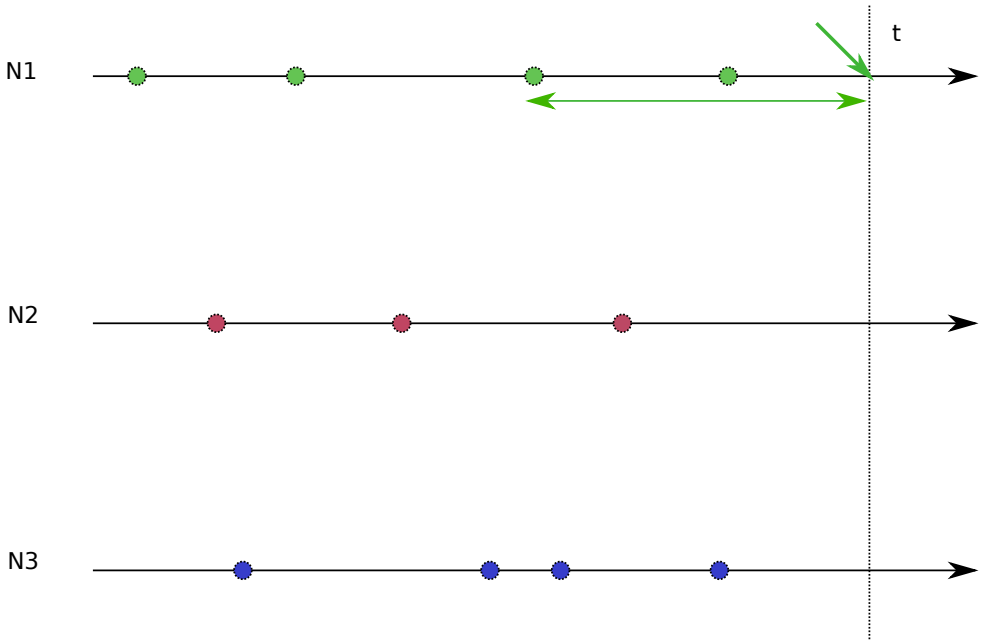

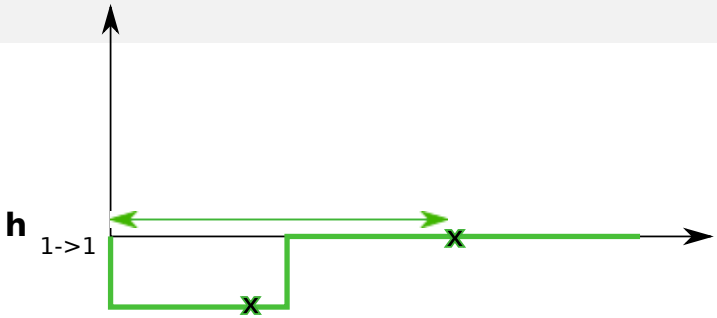

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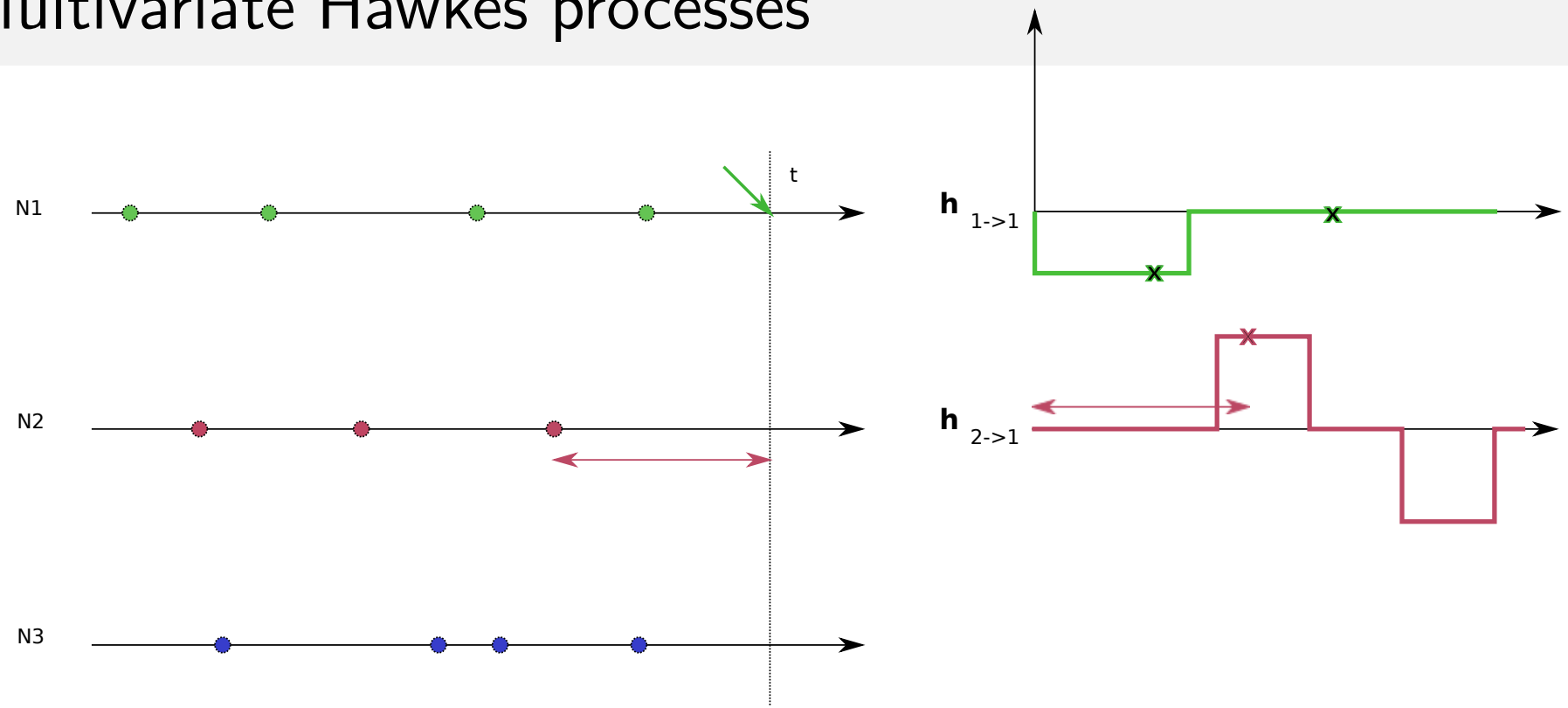

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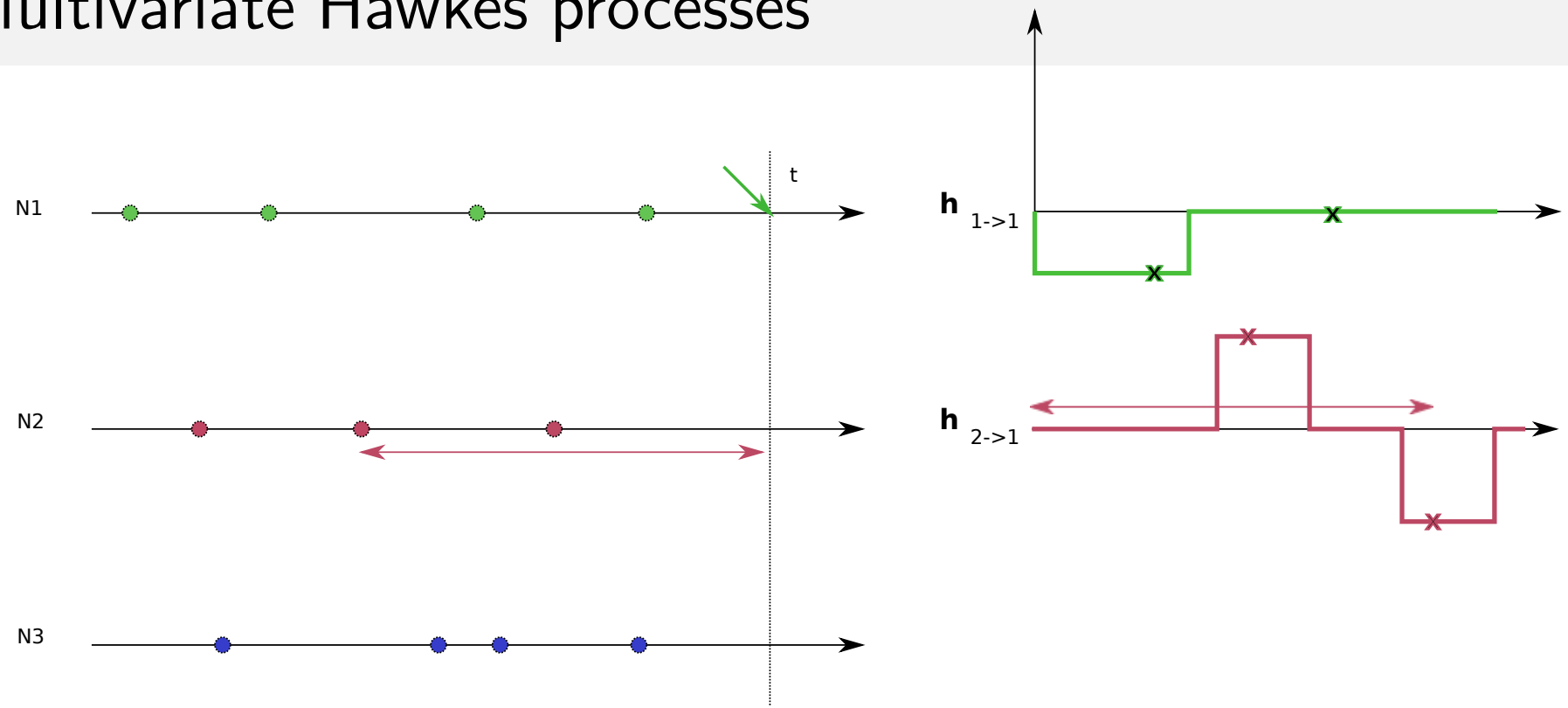

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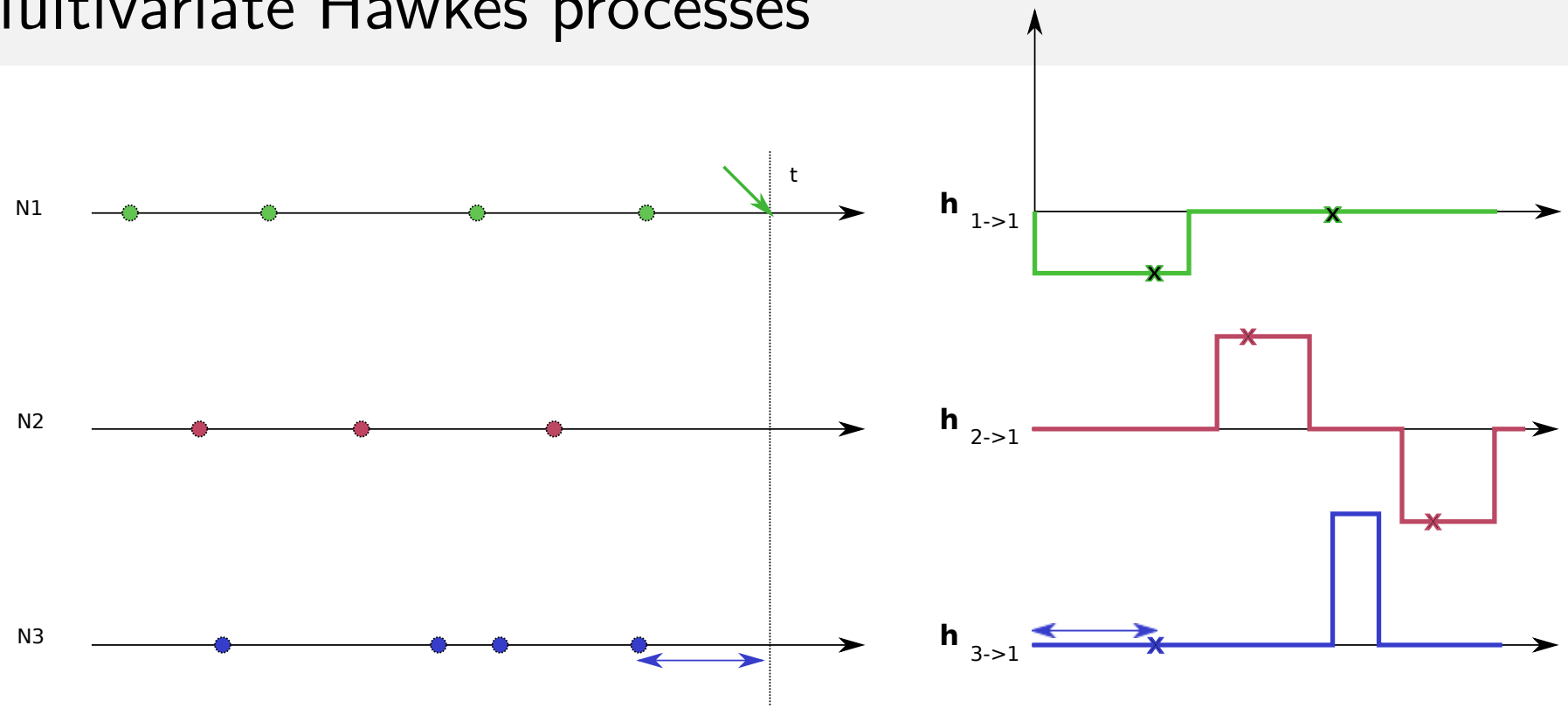

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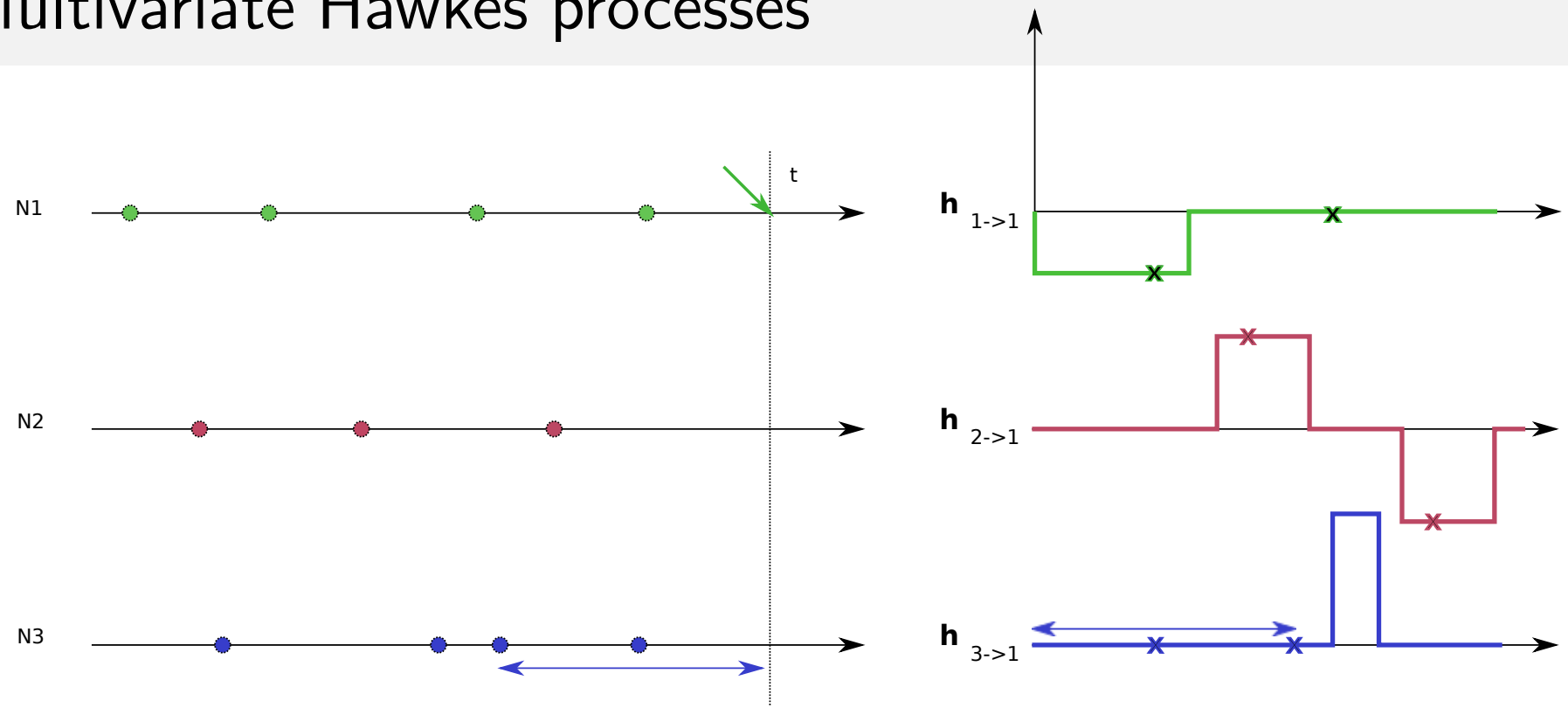

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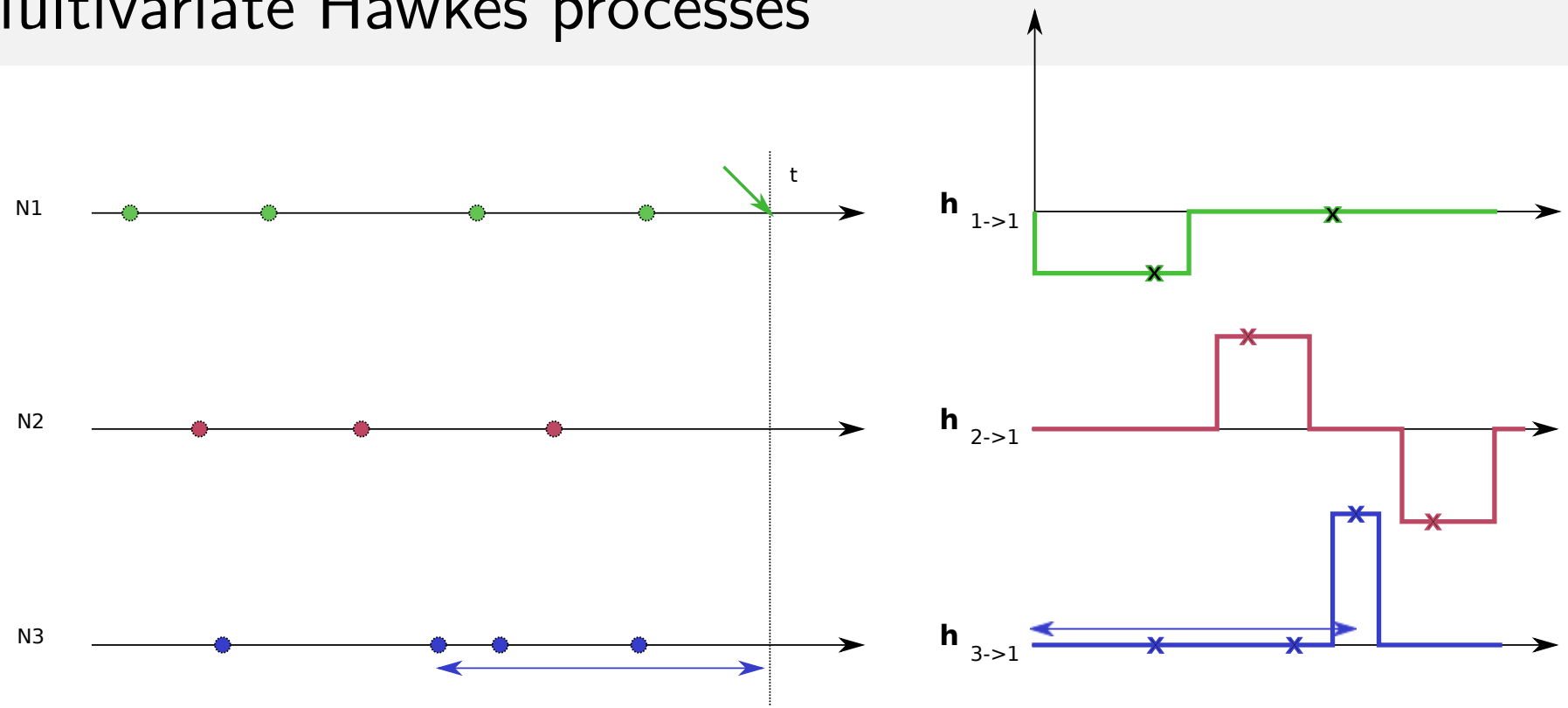

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More formally

- Only excitation (all the $h_\ell^{(r)}$ are positive) : for all r,

$$\lambda^{(r)}(t) = \nu_r + \sum_{\ell=1}^{M} \int_{-\infty}^{t-} h_\ell^{(r)}(t-u) dN_u^{(\ell)}.$$

Branching / Cluster representation, stationary process if the spectral radius of $\left(\int h_\ell^{(r)}(t) dt \right)$ is < 1.

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- Exponential (Multiplicative shape but no guarantee of a stationary version ...) $\lambda^{(r)}(t) = \exp\left(\nu_r + \sum_{\ell=1}^{M} \int_{-\infty}^{t-} h_\ell^{(r)}(t-u) dN_u^{(\ell)} \right)$.

Previous works

- Maximum likelihood estimates eventually (Ogata, Vere-Jones etc mainly for sismology, Chornoboy et al., for neuroscience, Gusto and Schbath for genomics)
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- Main problem not enough spikes ! so either over-fitting because d (number of parameters) too large or bad estimation if d too small (see later).
- Also MLE are not that easy to compute (EM algorithm etc)
- $\neq$ least-squares if linear parametric model.

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- Hence minimize

$$\gamma(\eta) = -2\int_0^T \eta(t)dN_t + \int_0^T \eta(t)^2 dt,$$

  for a model $\eta = \lambda_{\mathbf{a}}(t)$,
- If multivariate minimize $\sum_m \gamma_m(\eta^{(m)})$.

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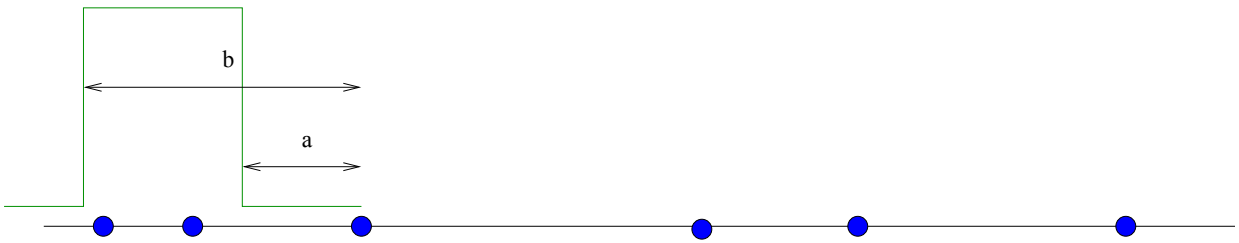

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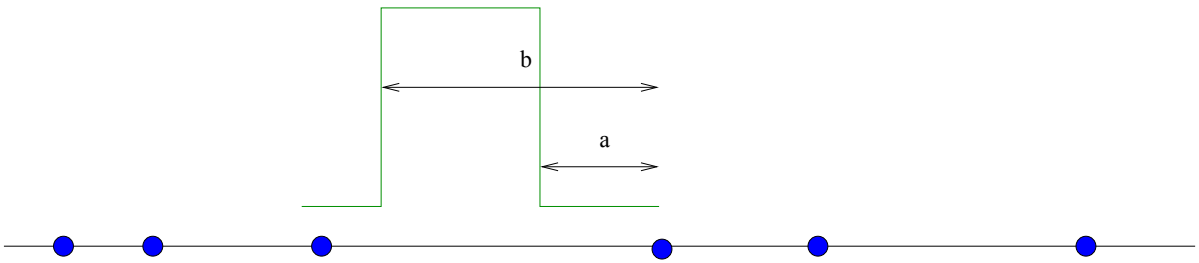

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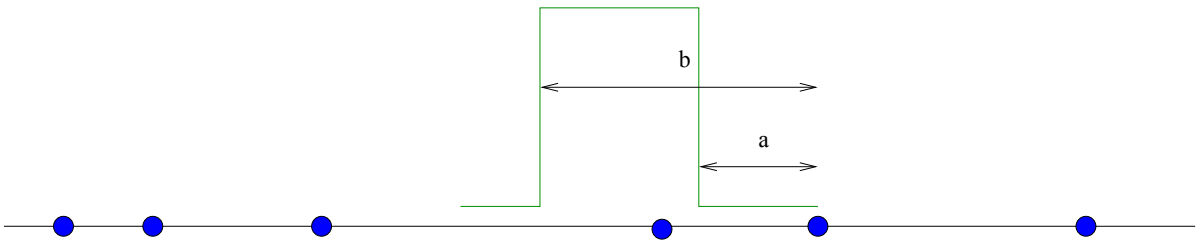

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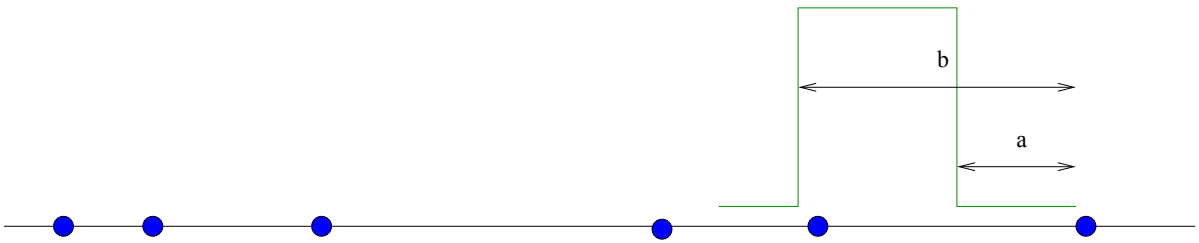

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The numerator is the number of pairs of points with delay in $[a, b]$

Full multivariate processes and linear parametrisation

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and with $\mathbf{c}_t^{(\ell)}$ being the vector of instantaneous count with delay of N_ℓ i.e.

$$(\mathbf{c}_t^{(\ell)})' = \left(N^{(\ell)}_{[t-\delta,t)}, \ldots, N^{(\ell)}_{[t-K\delta, t-(K-1)\delta)}\right).$$

An heuristic for the least-square estimator

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where in b lies again the number of couples with a certain delay (cross-correlogram).

Least-square estimate

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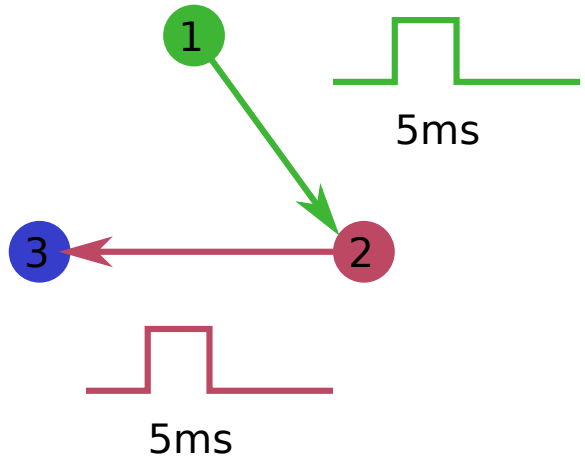
Least-square estimate

$$\hat{\mathbf{a}}^{(r)} = \mathbf{G}^{-1}\mathbf{b}^{(r)},$$

$\rightarrow$ simpler formula than Maximum Likelihood Estimators for similar properties (except efficiency).

What gain wrt cross correlogramm ?



only 2 non zero interaction functions over 9

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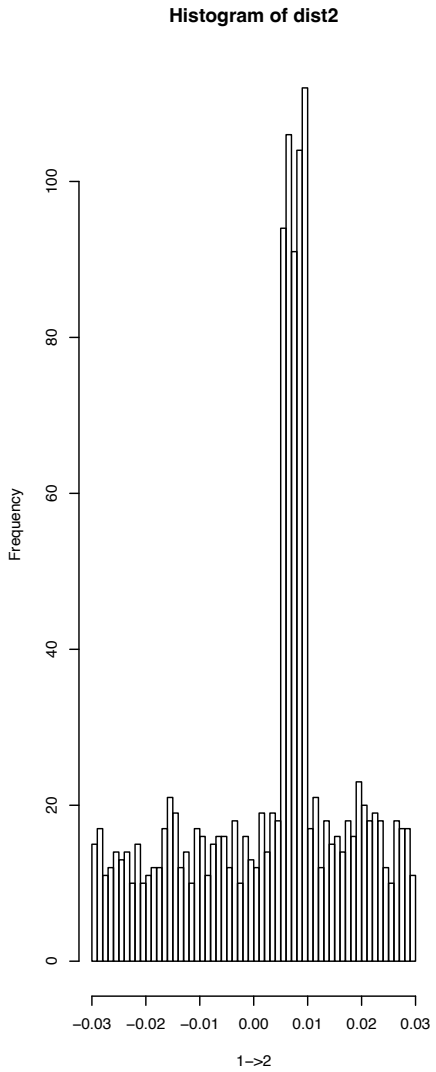

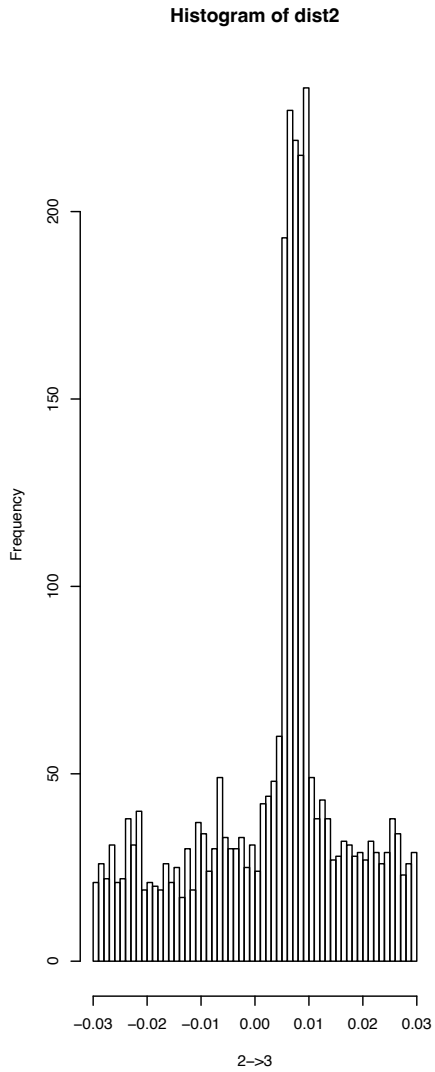

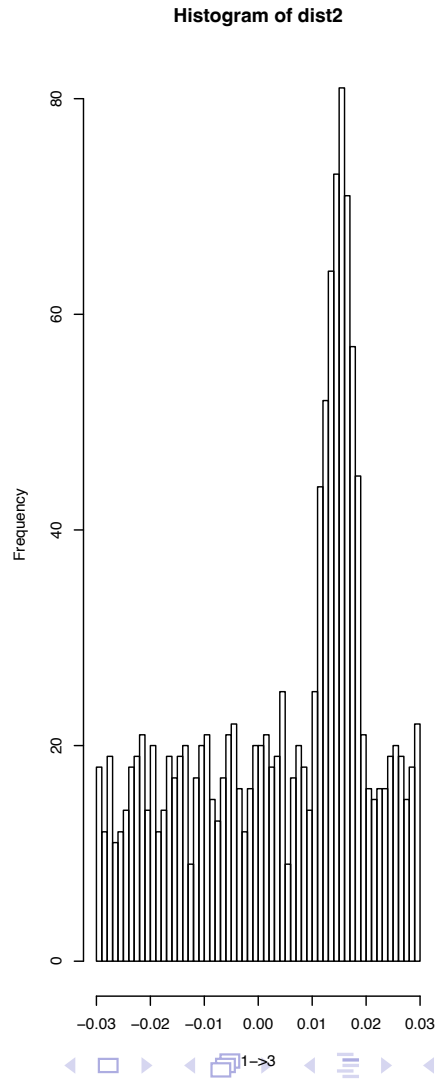

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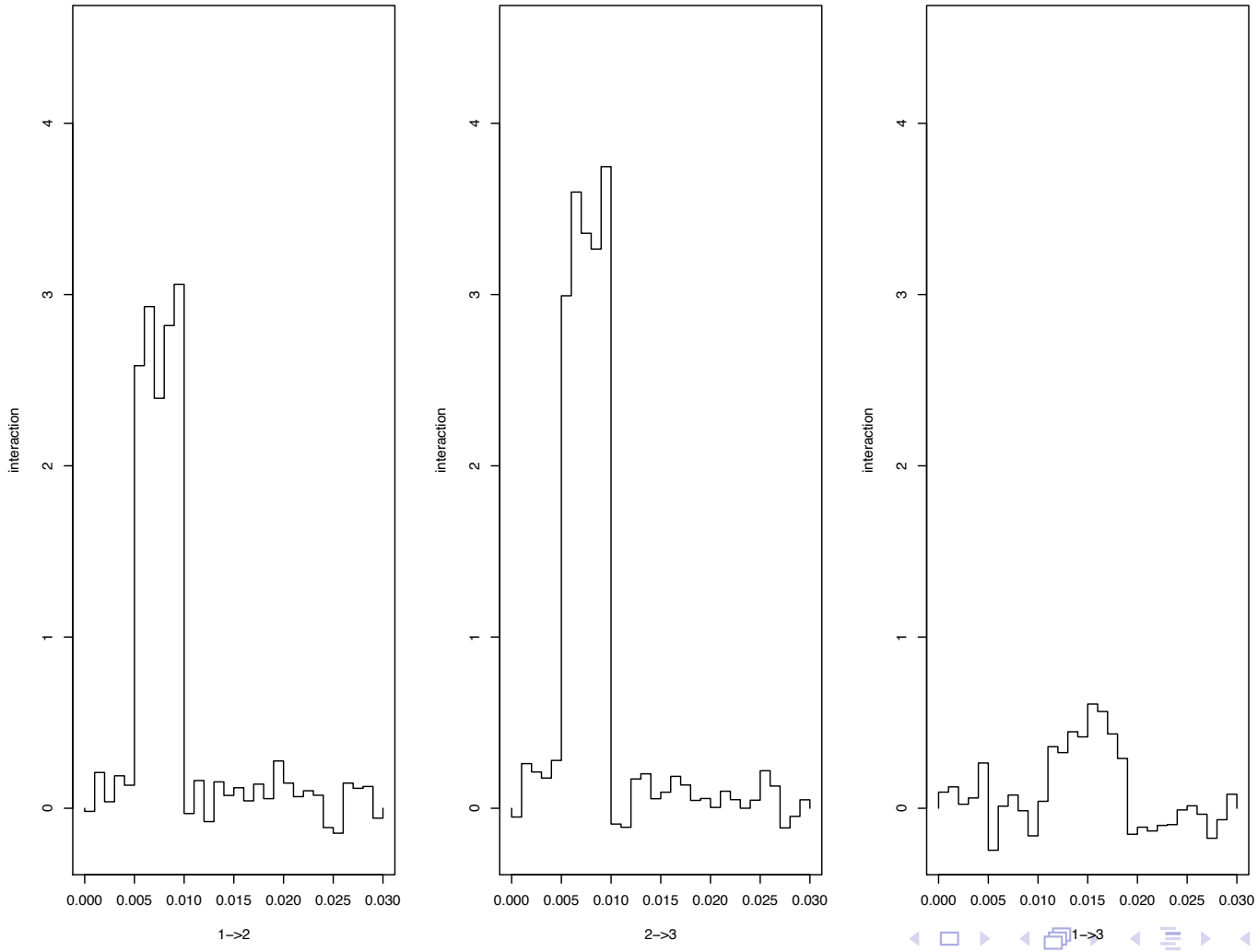

Adaptive statistics

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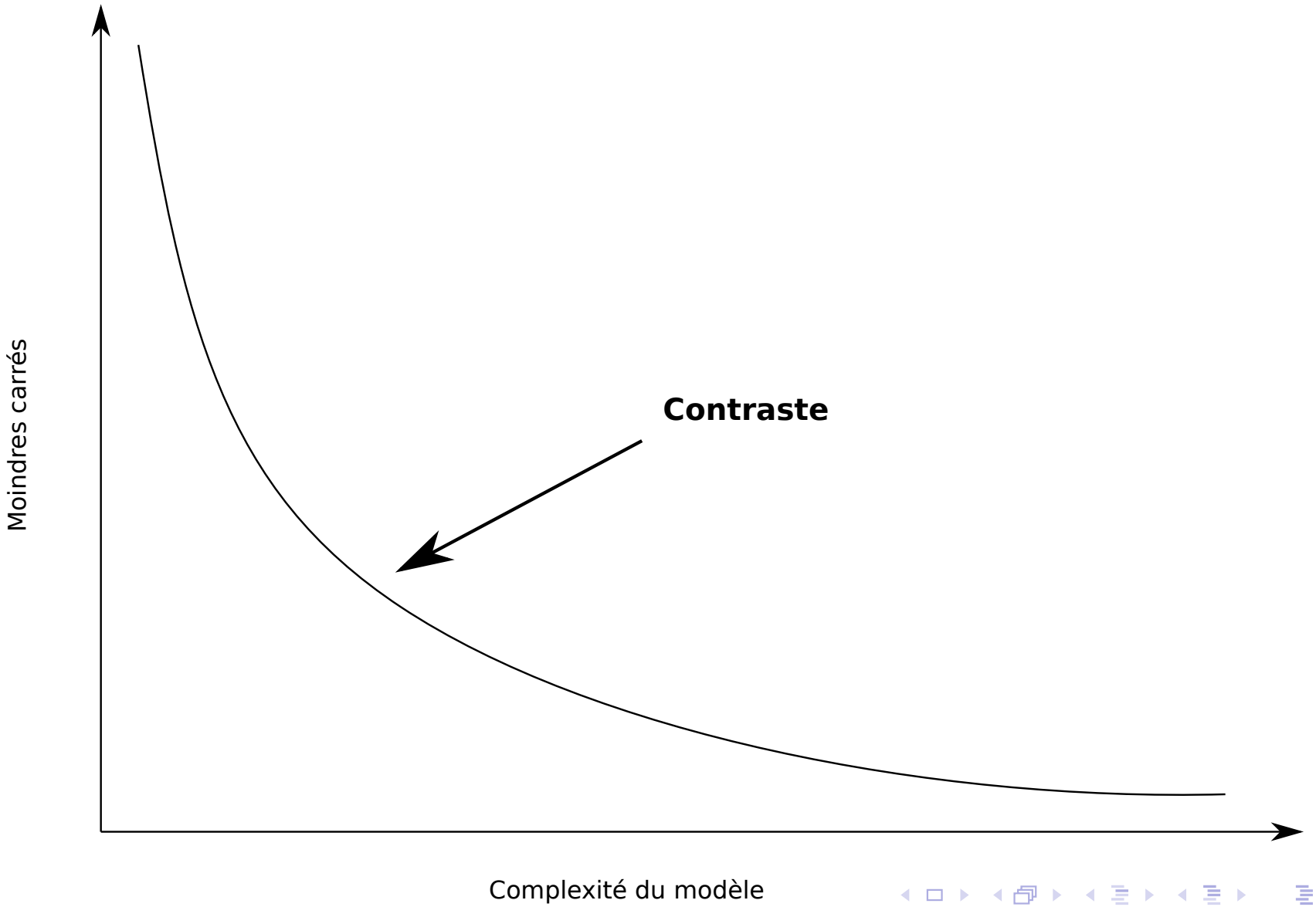
Over-fitting ! ! ! !

Penalty and parameter choices



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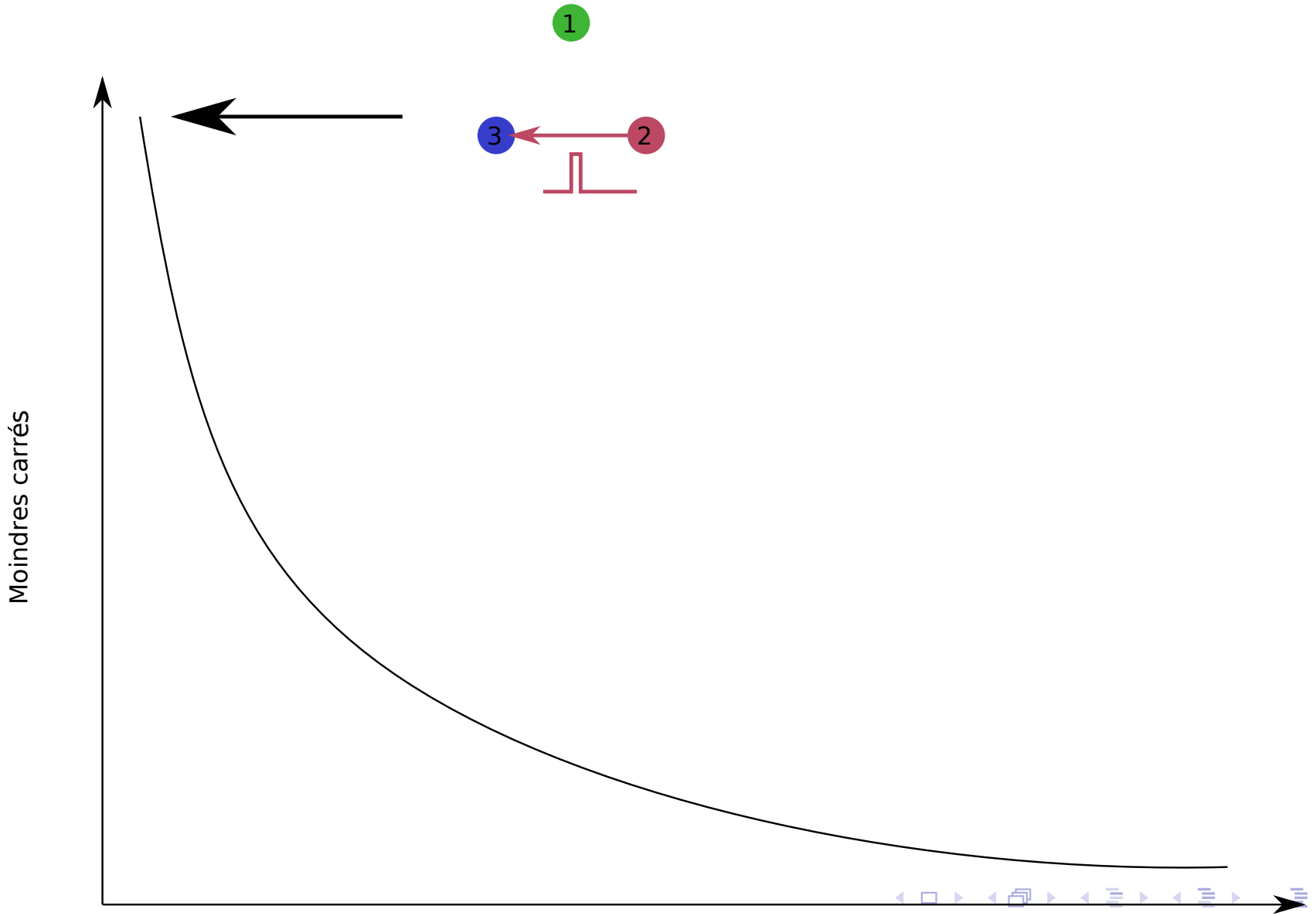

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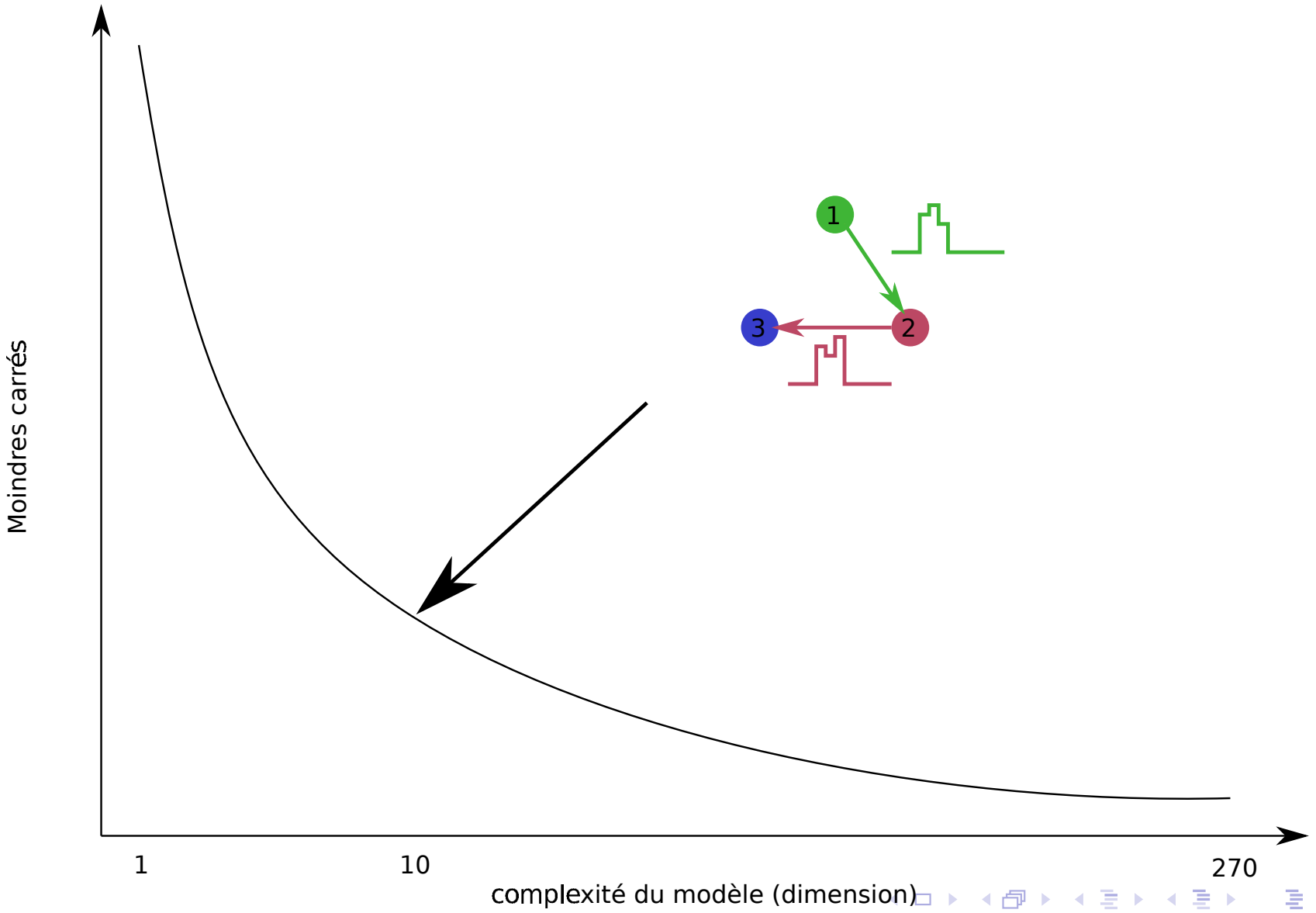

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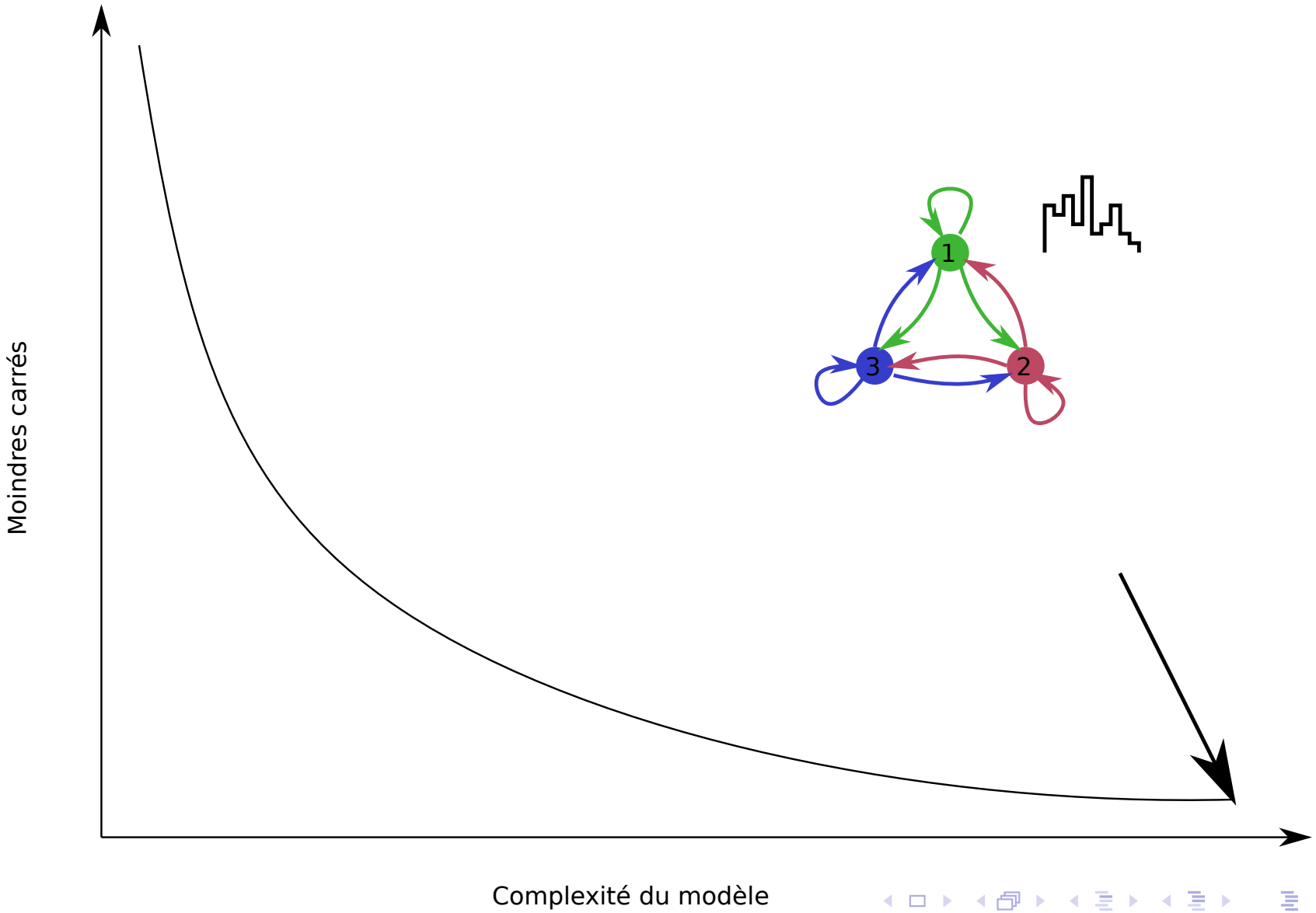

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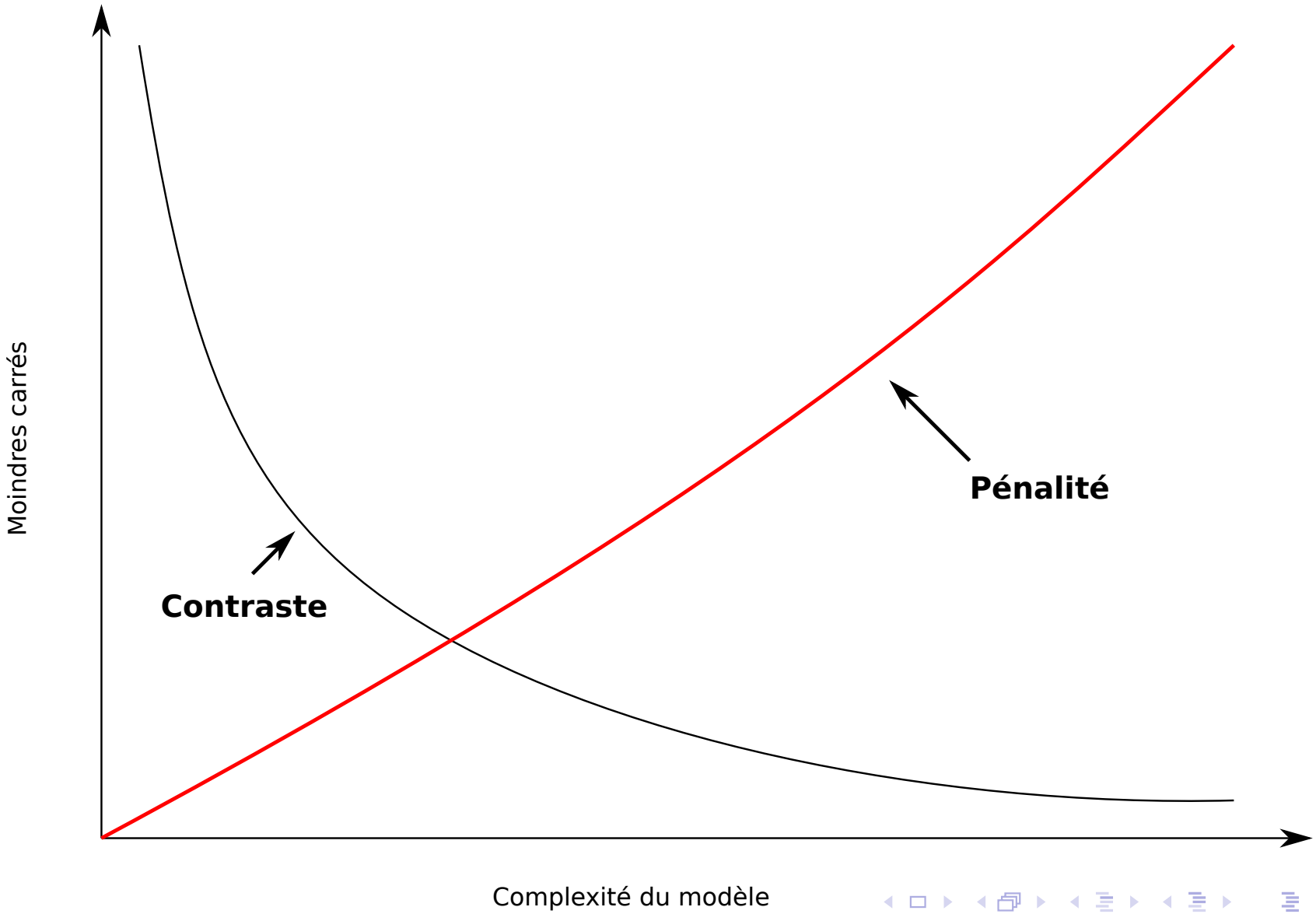

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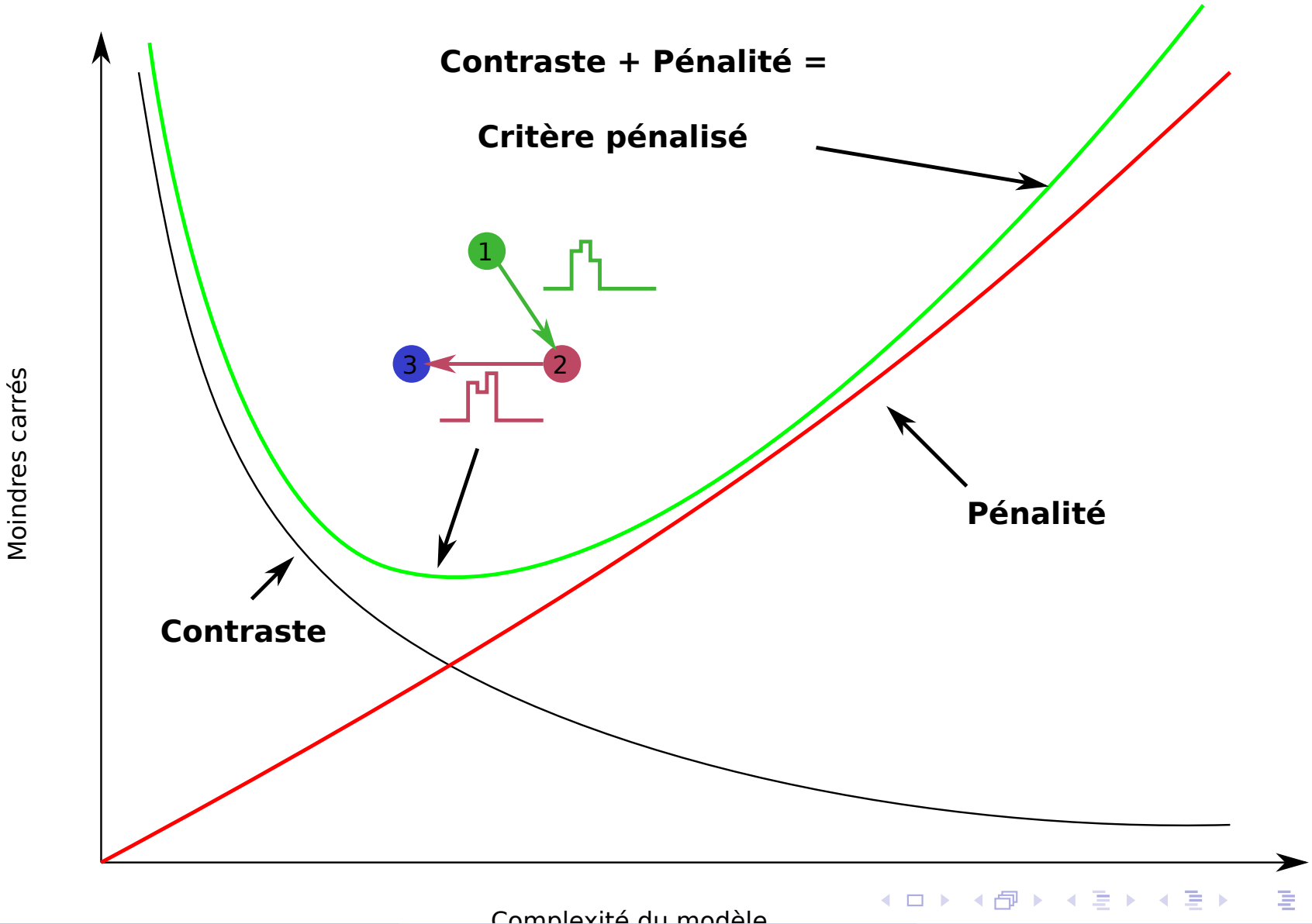

Previous works

- log-likelihood penalized by AIC (Akaike Criterion) (Ogata, Vere-Jones, Gusto and Schbath) : choose the model with d parameters such that

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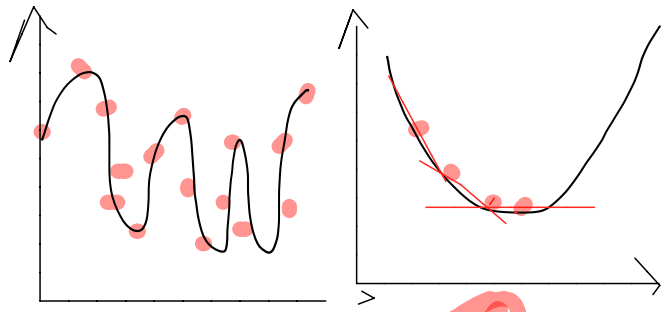

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- $\to$ convex criterion

Other works

- Maximum likelihood + exponential formula + ℓ_1 "group Lasso" penalty (Pillow et al. in neuroscience) but no mathematical proof
- Thresholding + tests for very particular bivariate models, oracle inequality (Sansonnet)

ℓ_1 penalty

with N.R. Hansen (Copenhagen), and V. Rivoirard (Dauphine) (2012)

The Lasso criterion can be expressed independently for each sub-process by :

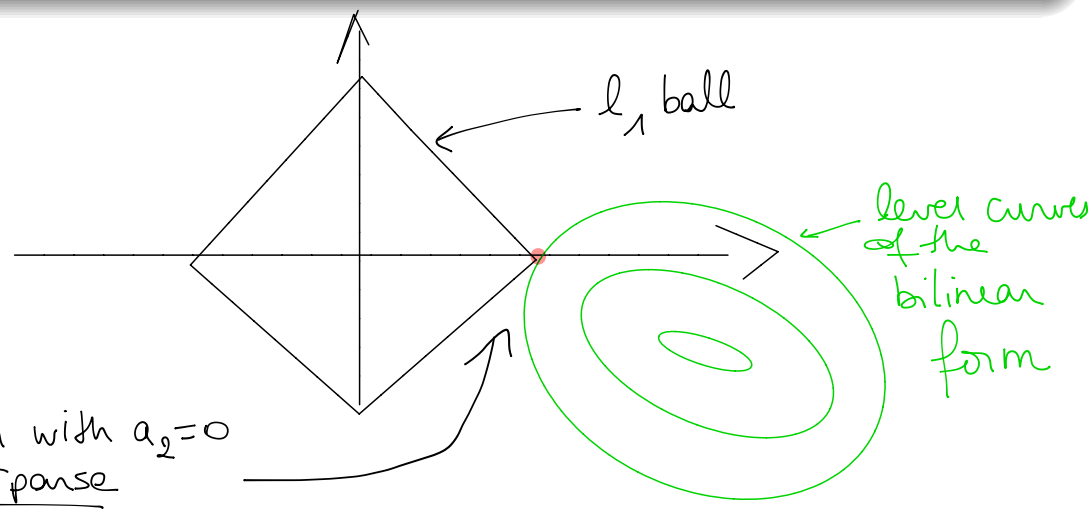
introduced by Tibshirani (1996) = Least-Absolute Shrinkage and selection Operator

Lasso criterion

$$\begin{aligned}\hat{\mathbf{a}}^{(r)} &= \text{argmin}_{\mathbf{a}}\{\gamma_T(\lambda_{\mathbf{a}}^{(r)}) + \text{pen}(\mathbf{a})\} \\ &= \text{argmin}_{\mathbf{a}}\{-2\mathbf{a}'\mathbf{b}_r + \mathbf{a}'\mathbf{G}\mathbf{a} + 2(\mathbf{d}^{(r)})'|\mathbf{a}|\}\end{aligned}$$

with $\mathbf{d}'|\mathbf{a}| = \sum_k d_k|a_k|$

Minimizing Lasso
$\Leftrightarrow$ minimizing bilinear form under constraint on l_1 norm.



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- The crucial choice is the $\mathbf{d}^{(r)}$, should be data-driven !
- The theoretical validation : be able to state that our choice is the best possible choice.
- The practical validation : on simulated Hawkes processes (done), on simulated neuronal networks, on real data (RNRP Paris 6 work in progress)...

Theoretical Validation

Recall that

$$\mathbf{b}^{(r)} = \int_0^T \mathbf{Rc}_t dN^{(r)}(t)$$

and

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Hansen,Rivoirard,RB

If $\mathbf{G} \geq cI$ with $c > 0$ and if

$$\left| \int_0^T \mathbf{Rc}_t \left(dN^{(r)}(t) - \lambda^{(r)}(t) dt \right) \right| \leq \mathbf{d}^{(r)}, \quad \forall r$$

then

$$\sum_r \|\lambda^{(r)} - \mathbf{Rc}_t \hat{\mathbf{a}}^{(r)}\|^2 \leq \square \inf_{\mathbf{a}} \left\{ \sum_r \|\lambda^{(r)} - \mathbf{Rc}_t \mathbf{a}\|^2 + \frac{1}{c} \sum_{i \in supp(\mathbf{a})} (d_i^{(r)})^2 \right\}.$$

Small proof (Univariate case)

Let us fix some $\mathbf{a}$ and let $\eta_{\mathbf{a}}(t) = \mathbf{Rc}_t'\mathbf{a}$, our candidate intensity.
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$$\leq \|\lambda - \eta_{\mathbf{a}}\|^2 + 2\mathbf{d}'(|\mathbf{a}| - |\hat{\mathbf{a}}|) + 2\int_0^T [\eta_{\hat{\mathbf{a}}}(t) - \eta_{\mathbf{a}}(t)](dN_t - \lambda(t)dt)$$

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But

$$\|\mathbf{a} - \hat{\mathbf{a}}\|^2 \leq \frac{1}{c}(\mathbf{a} - \hat{\mathbf{a}})'\mathbf{G}(\mathbf{a} - \hat{\mathbf{a}}) = \frac{1}{c}\|\eta_{\mathbf{a}} - \eta_{\hat{\mathbf{a}}}\|^2$$
$$\leq \frac{2}{c}\left[\|\eta_{\mathbf{a}} - \lambda\|^2 + \|\lambda - \eta_{\hat{\mathbf{a}}}\|^2\right]$$

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NB: $uv \leq \varepsilon u^2 + \frac{1}{\varepsilon} v^2 \quad \forall \varepsilon > 0$

$\Leftarrow \quad \left(\sqrt{\varepsilon} u + \frac{v}{\sqrt{\varepsilon}}\right)^2 = \varepsilon u^2 + \frac{v^2}{\varepsilon} + 2uv.$

$$\leq \|\lambda - \eta_{\mathbf{a}}\|^2 + 4\|\mathbf{a} - \hat{\mathbf{a}}\| \left(\sum_{i \in supp(a)} \mathbf{d}_i^2 \right)^{1/2}$$

Hence for all $\alpha > 0$

$$\|\lambda - \eta_{\hat{\mathbf{a}}}\|^2 \leq \|\lambda - \eta_{\mathbf{a}}\|^2 + \alpha \left[\|\eta_{\mathbf{a}} - \lambda\|^2 + \|\lambda - \eta_{\hat{\mathbf{a}}}\|^2 \right] + \frac{8\alpha}{c} \sum_{i \in supp(a)} \mathbf{d}_i^2$$

What did we proved ? (Univariate)

Recall that

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and

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Known support

If there is a true parameter $\mathbf{a}^*$, then $\lambda(t) = \mathbf{R}\mathbf{c}_t'\mathbf{a}^*$ and

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If support of $\mathbf{a}^*$, S, known, the least-square estimate on $S \to \hat{\mathbf{a}}_S$

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If there is a true parameter $\mathbf{a}^*$, then $\lambda(t) = \mathbf{R}\mathbf{c}_t'\mathbf{a}^*$ and

$$\bar{\mathbf{b}} = \int_0^T \mathbf{R}\mathbf{c}_t \lambda(t) dt = \mathbf{G}\mathbf{a}^*$$

If support of $\mathbf{a}^*$, S, known, the least-square estimate on $S \to \hat{\mathbf{a}}_S$

$$\begin{aligned} \|\lambda - \mathbf{R}\mathbf{c}_t \hat{\mathbf{a}}_S\|^2 &= (\mathbf{a}^* - \hat{\mathbf{a}}_S)'\mathbf{G}(\mathbf{a}^* - \hat{\mathbf{a}}_S) \\ &= (\bar{\mathbf{b}} - \mathbf{b})'\mathbf{G}^{-1}(\bar{\mathbf{b}} - \mathbf{b}) \\ &\leq \frac{1}{c}\|\bar{\mathbf{b}} - \mathbf{b}\|^2 \end{aligned}$$

Oracle inequality

Lasso criterion

$$\begin{aligned}\hat{\mathbf{a}}^{(r)} &= \operatorname{argmin}_{\mathbf{a}}\{\gamma_T(\lambda_{\mathbf{a}}^{(r)}) + \mathrm{pen}(\mathbf{a})\} \\ &= \operatorname{argmin}_{\mathbf{a}}\{-2\mathbf{a}'\mathbf{b}_r + \mathbf{a}'\mathbf{G}\mathbf{a} + 2(\mathbf{d}^{(r)})'|\mathbf{a}|\}\end{aligned}$$

Hansen,Rivoirard,RB

If $\mathbf{G} \geq cI$ with $c > 0$ and if

$$\left|\mathbf{b}^{(r)} - \bar{\mathbf{b}}^{(r)}\right| \leq \mathbf{d}^{(r)}, \quad \forall r$$

then

$$\sum_r \|\lambda^{(r)} - \mathbf{R}\mathbf{c}_t\hat{\mathbf{a}}^{(r)}\|^2 \leq \square \inf_{\mathbf{a}} \left\{ \sum_r \|\lambda^{(r)} - \mathbf{R}\mathbf{c}_t\mathbf{a}\|^2 + \frac{1}{c} \sum_{i \in supp(\mathbf{a})} (d_i^{(r)})^2 \right\}.$$

Comments

- If we can find $\mathbf{d}$ sharp and data-driven, then this is an oracle inequality ! ! :
  - Only an oracle could know the support in advance
  - Up to constant, we can do as well as the oracle

Comments

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- We do not pay anything here for the size, except **G** invertible
  - it's a "cheap" oracle inequality ... In fact could be done under more relaxed assumption...
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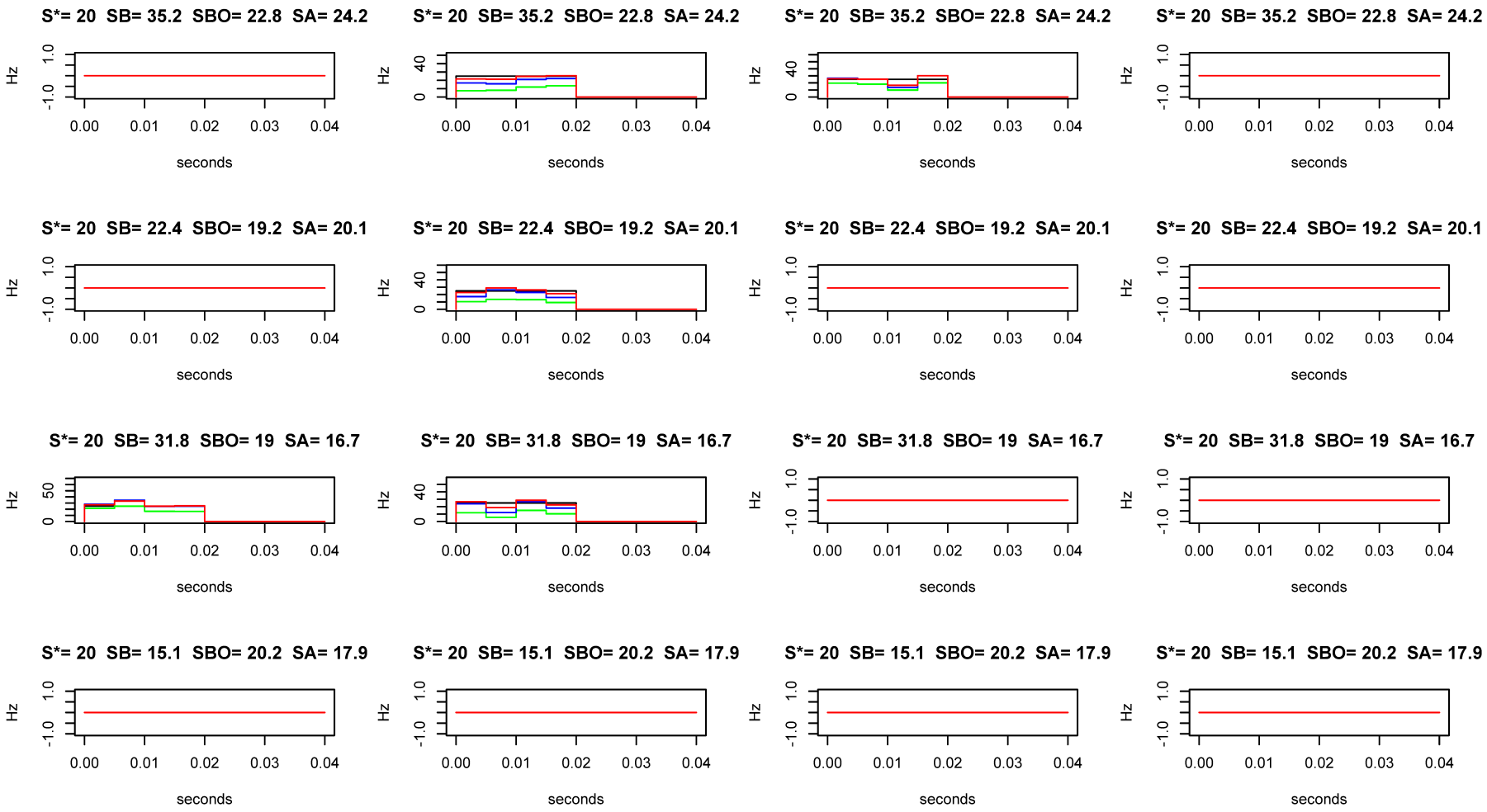
  - c large (and we can observe it !) $\rightarrow$ quite confident for good reconstruction =quality indicator.
- True even if the Hawkes linear model not true ! ! ! and we just pay "quality of approximation"
- quality of approximation + unavoidable price due to estimation.

Mathematical "debts"

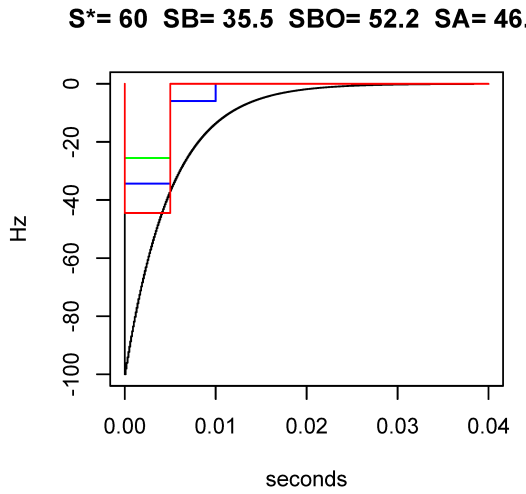
- works for other models (anything that is linear), works for other basis and dictionary ...
- Choice of **d** : $\rightarrow$ martingale calculus
- **G** invertible in some cases ? ? ? $\rightarrow$ branching structure

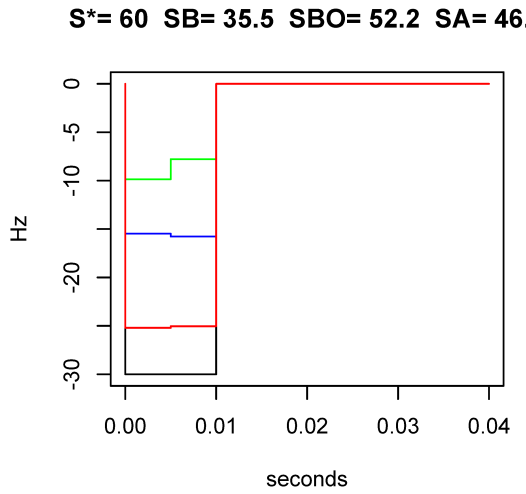
Simulation study - Estimation

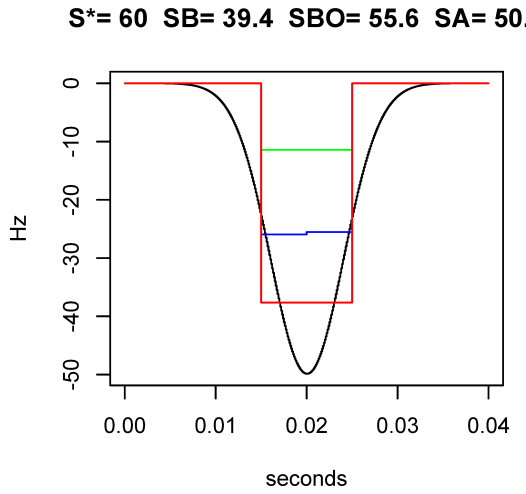


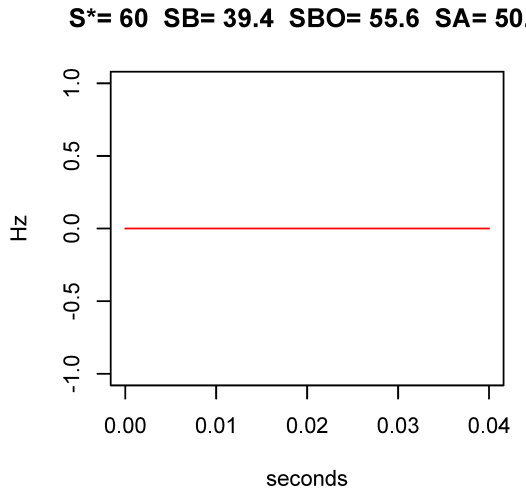
Interactions reconstructed with 'Adaptive Lasso', 'Bernstein Lasso' and 'Bernstein Lasso+OLS'. Above graphs, estimation of spontaneous rates

Another example with inhibition

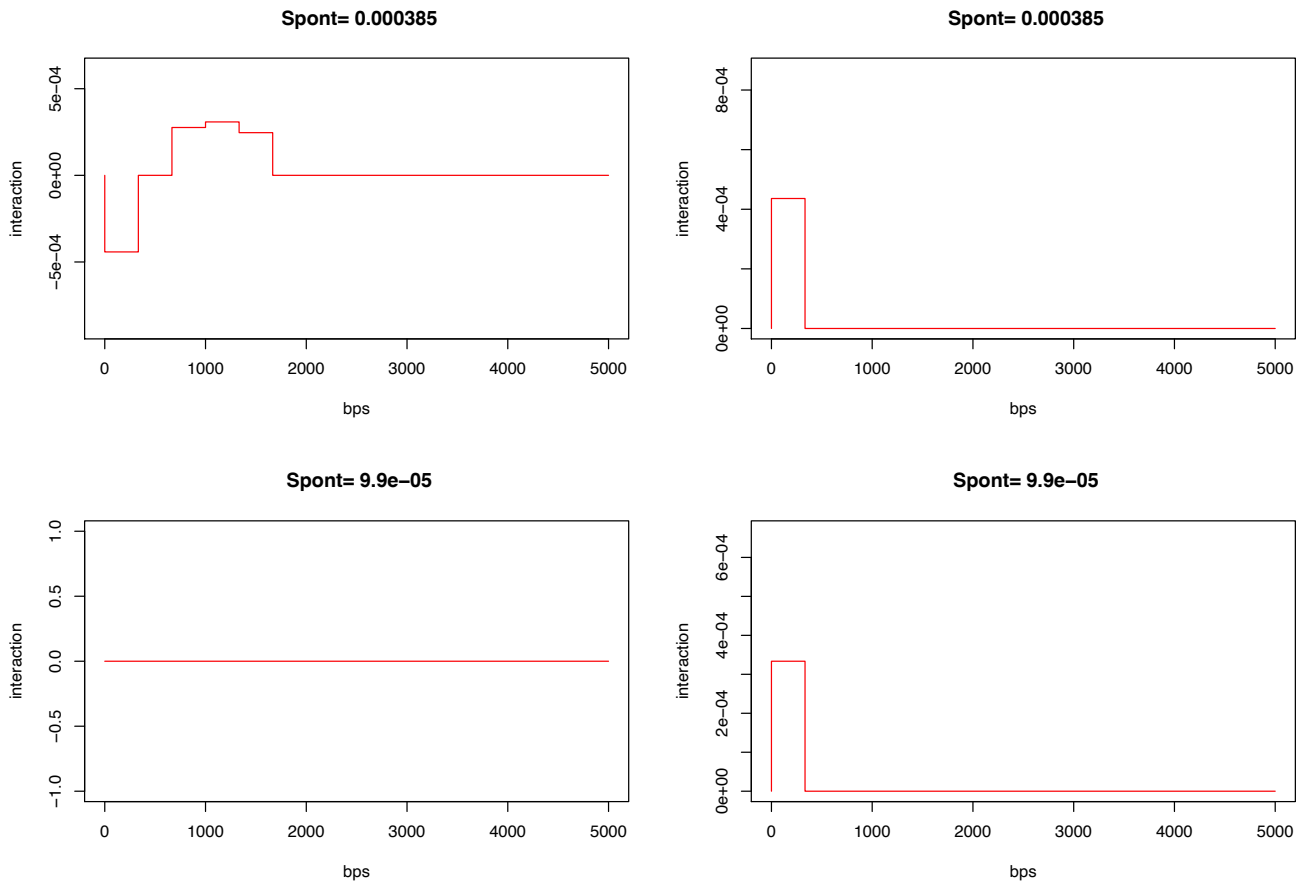








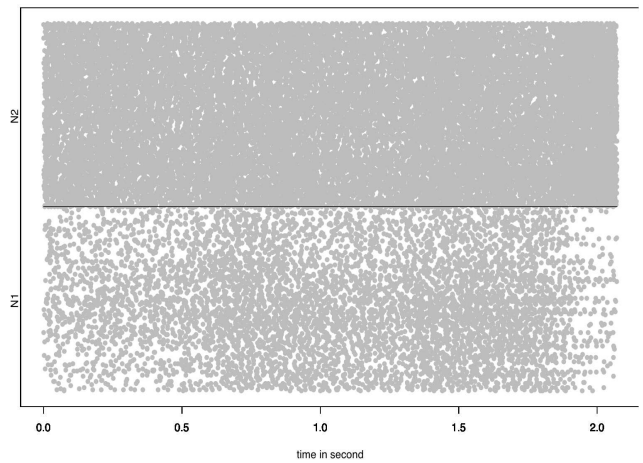
On real (genomic) data

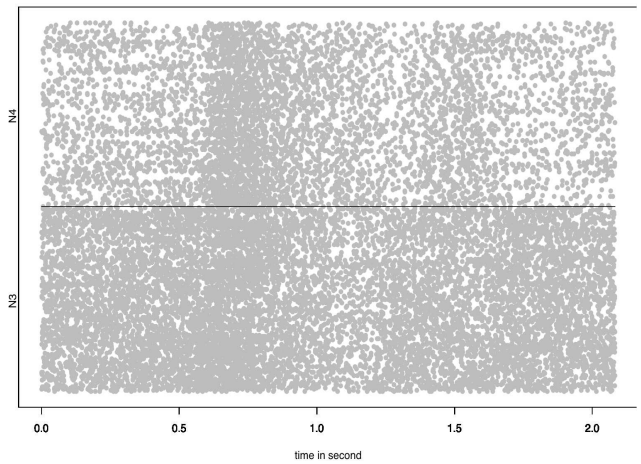


4290 genes and 1036 `tataat` of E. coli ($T = 9288442$, $A = 10000$)

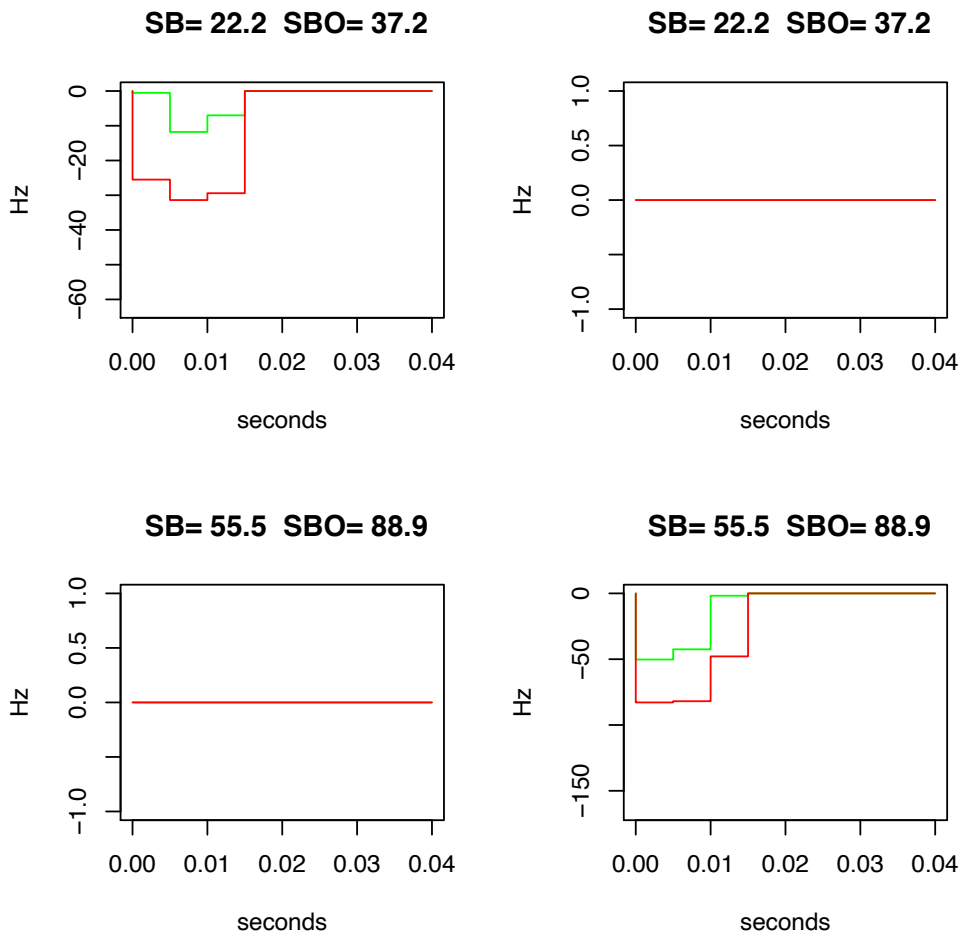
Sensory-motor task

(F. Grammont (Nice), A. Riehle (Marseille))





On neuronal data (sensorimotor task)



30 trials : monkey trained to touch the correct target when illuminated.

Accept the test of Hawkes hypothesis. Work with F. Grammont. V.

Tests

with C. Tuleau-Malot, F. Grammont (Nice) and V. Rivoirard (Dauphine) (2013)

- It is possible to test whether a process is a Hawkes process with prescribed interaction functions by using the time-rescaling theorem. (known since Ogata)

Tests

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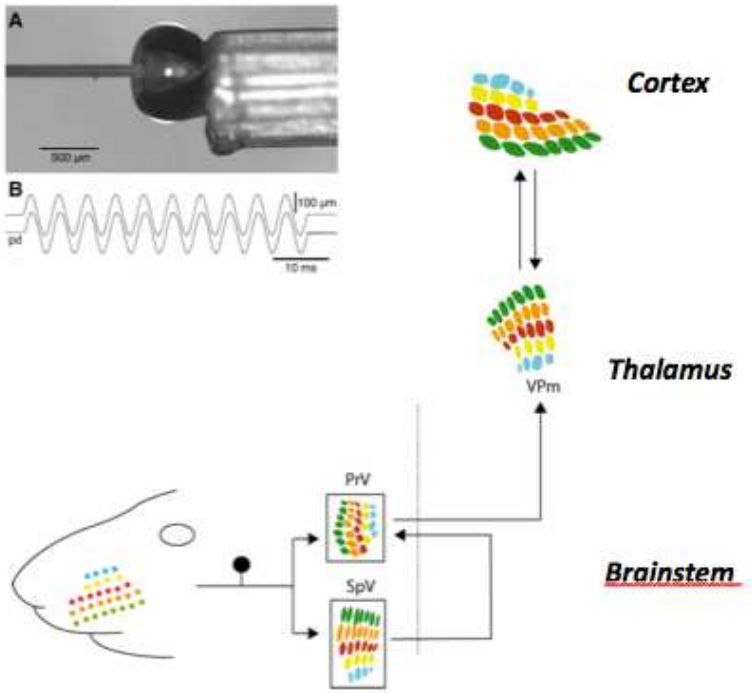
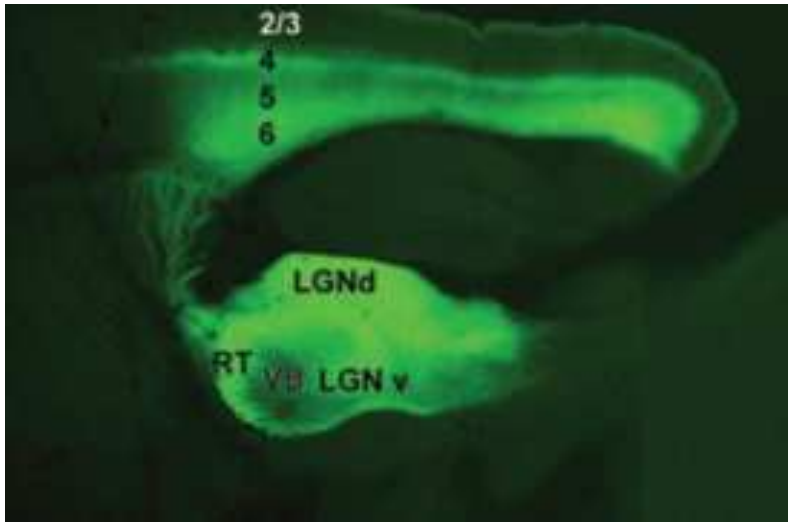
- It is possible to test whether a process is a Hawkes process with prescribed interaction functions by using the time-rescaling theorem. (known since Ogata)
- By using subsampling, it is possible to plug an estimate of the functions and still to test with controlled asymptotic level.
- On both previous data sets, the Hawkes hypothesis is accepted (p-values depends on the sub-sample, usually between 20 and 80 %), whereas the homogeneous Poisson hypothesis (i.e. no interactions) is rejected (p-values in $10^{-4}, 10^{-16}$ for the neuronal data)

Functional Connectivity ?

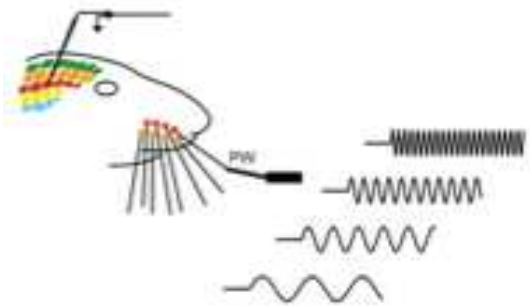
(Equipe RNRP de Paris 6)

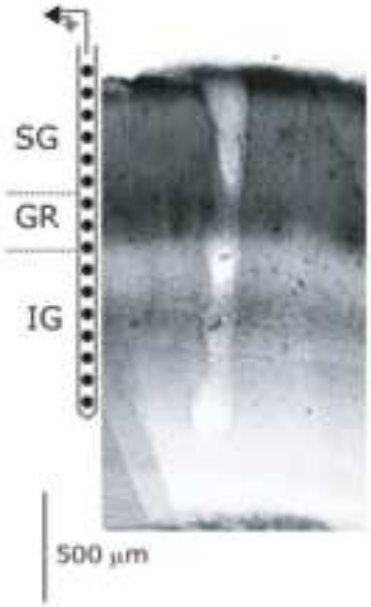


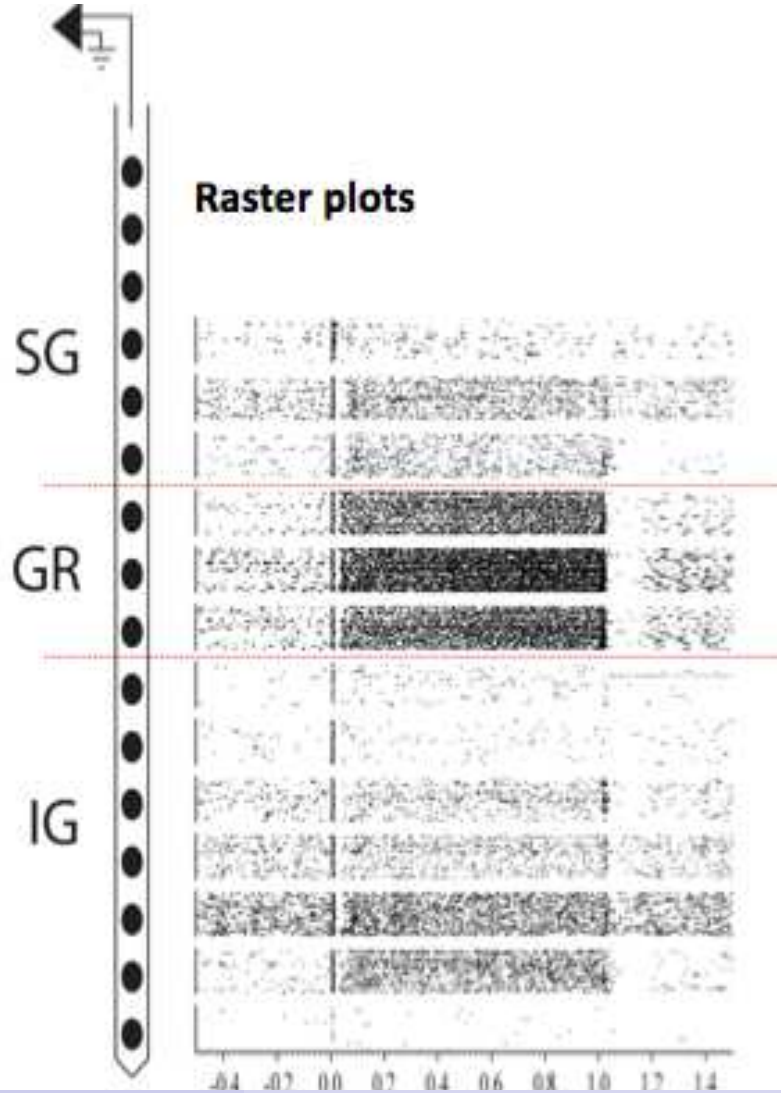


Tetrode data on the rat

(Equipe RNRP de Paris 6)



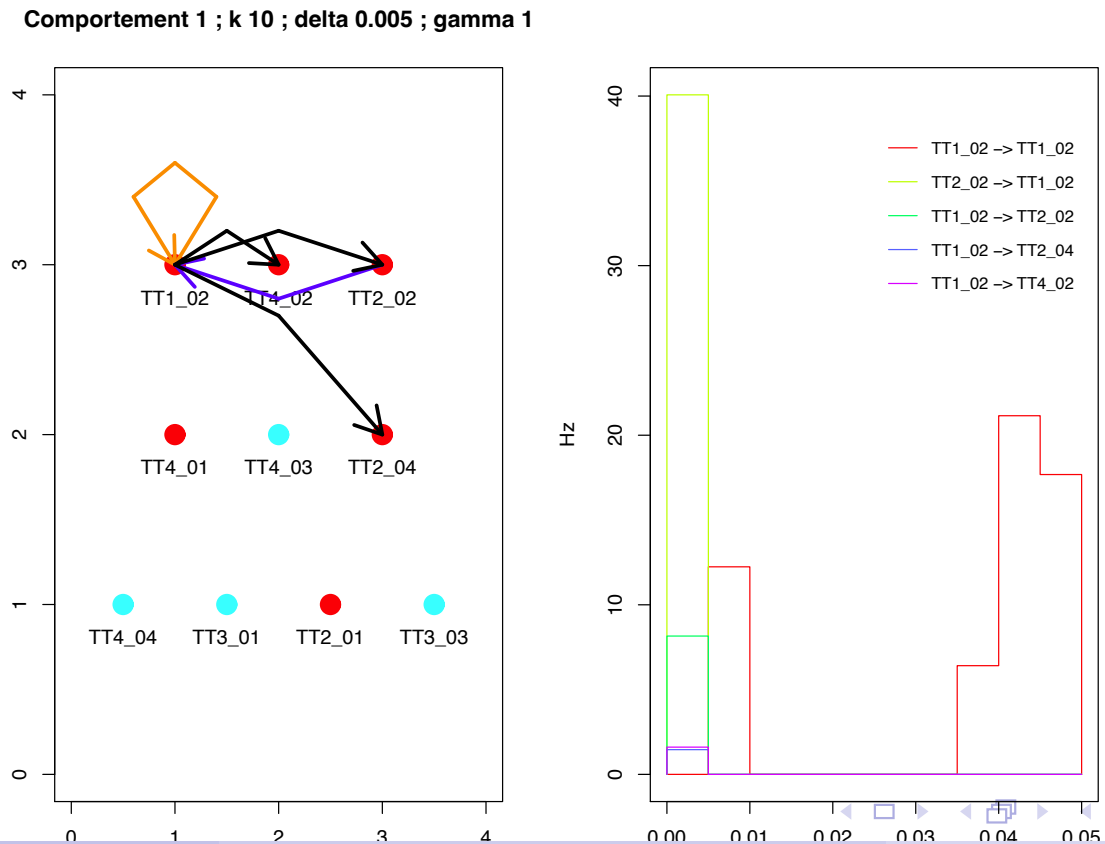




On neuronal data (vibrissa excitation)

Joint work with RNRP (Paris 6). Behavior : vibrissa excitation at low frequency. $T = 90.5$, $M = 10$

Neuron	$TT1_{02}$	$TT2_{01}$	$TT2_{02}$	$TT2_{04}$	$TT3_{01}$	$TT3_{03}$	$TT4_{01}$	$TT4_{02}$	$TT4_{03}$	$TT4_{04}$
Spikes	9191	99	544	149	15	18	136	282	8	6



Application on real data

Data :

Neuron	$TT1_{02}$	$TT2_{01}$	$TT2_{02}$	$TT2_{04}$	$TT3_{01}$	$TT3_{03}$	$TT4_{01}$	$TT4_{02}$	$TT4_{03}$	$TT4_{04}$
Spikes	9191	99	544	149	15	18	136	282	8	6

Simulation :

Neuron	$TT1_{02}$	$TT2_{01}$	$TT2_{02}$	$TT2_{04}$	$TT3_{01}$	$TT3_{03}$	$TT4_{01}$	$TT4_{02}$	$TT4_{03}$	$TT4_{04}$
Spikes	9327	92	585	148	13	23	133	271	8	3

Application on real data

Data :

Neuron	$TT1_{02}$	$TT2_{01}$	$TT2_{02}$	$TT2_{04}$	$TT3_{01}$	$TT3_{03}$	$TT4_{01}$	$TT4_{02}$	$TT4_{03}$	$TT4_{04}$
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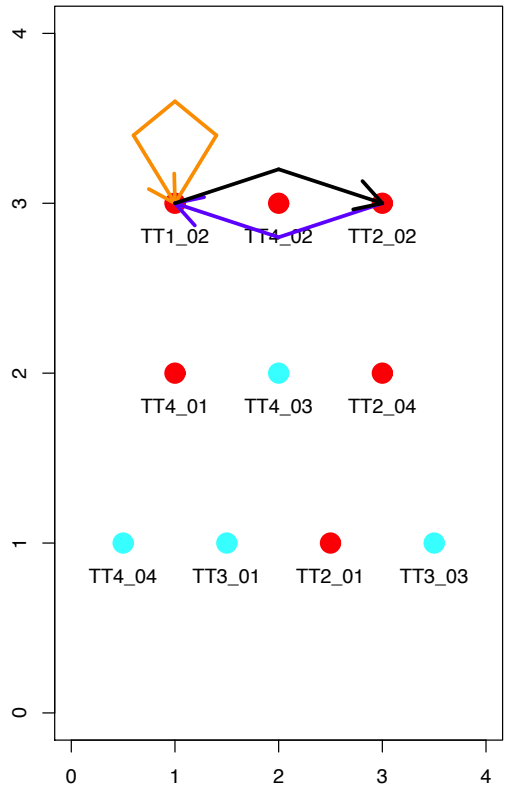
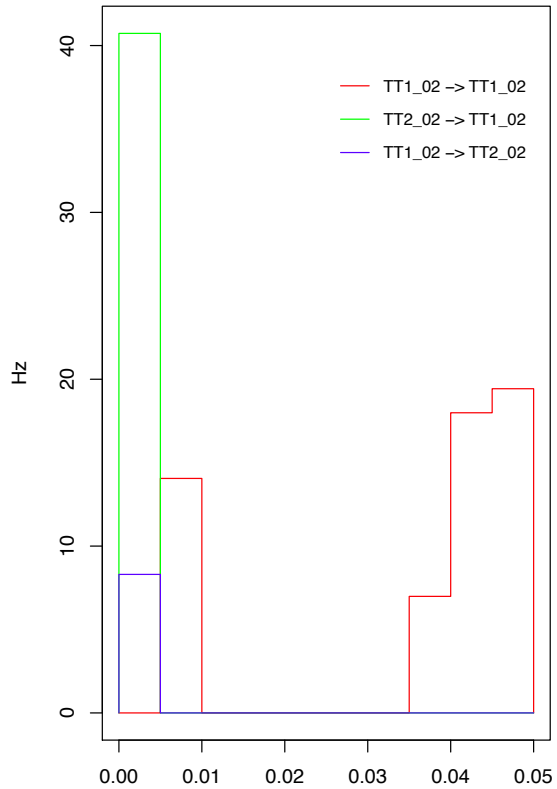
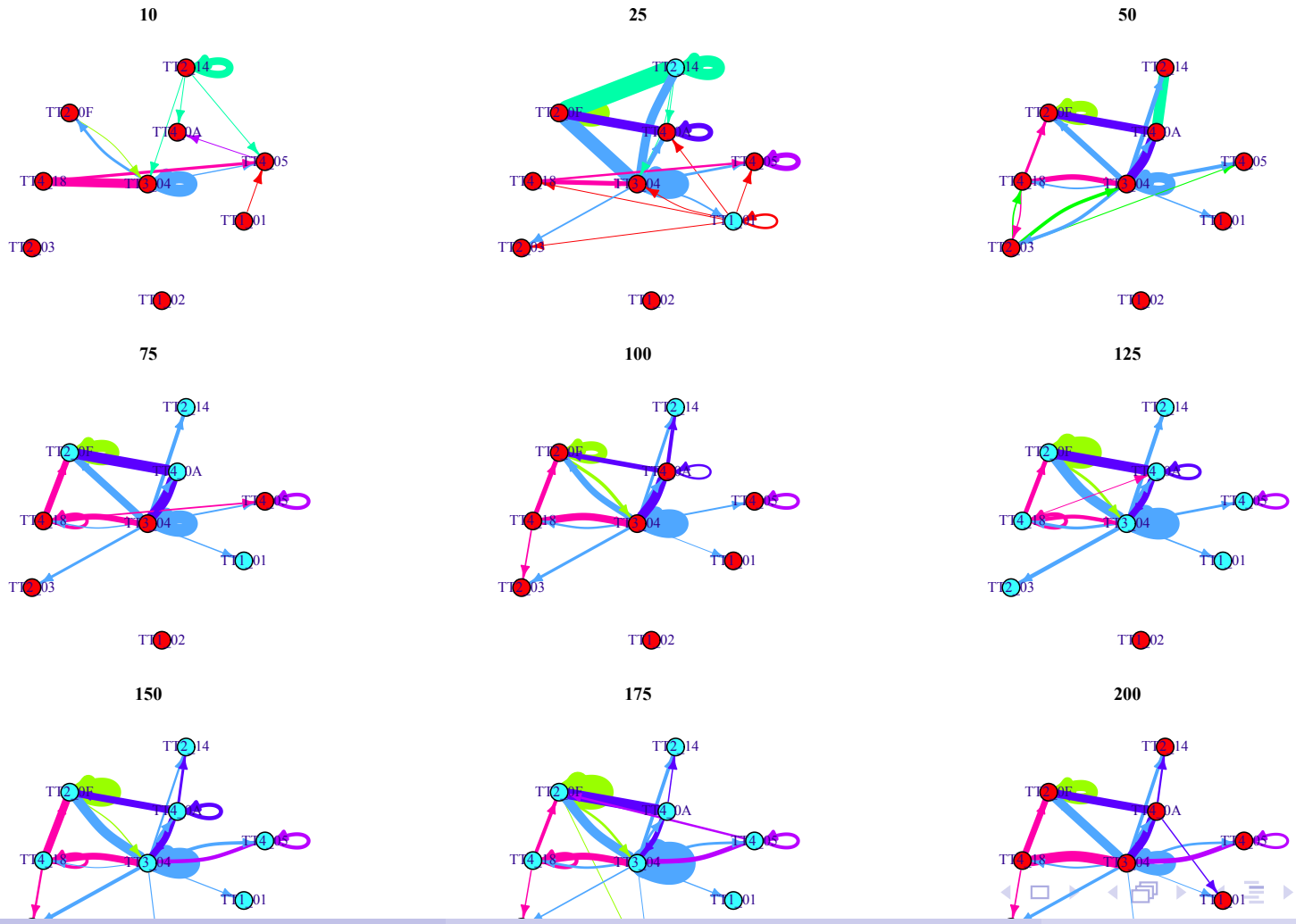




Evolution of the dependance graph as a fonction of the vibrissa excitation



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Probability generating functional and Laplace transform

p.g.fl

For any point process N, the functional which associates to any positive function h

$$\mathbb{E}(\prod_{x \in N} h(x))$$

Laplace functional

For any point process N, the functional which associates to any function f

$$\mathbb{E}\left(\exp\left[\int f(x) dN_x\right]\right).$$

equivalence with $f = \log(h)$.

Campbell's theorem

Let N be a Poisson process with mean measure μ,

Poisson processes

- for all integer n, for all $A_1, \ldots, A_n$ disjoint measurable subsets of $\mathbb{X}$, $N_{A_1}, \ldots, N_{A_n}$ are independent random variables.
- for all measurable subset A of $\mathbb{X}$, N_A obeys a Poisson law with parameter depending on A and denoted $\mu(A)$.

Campbell's theorem

Let N be a Poisson process with mean measure μ,
typically, $d\mu_t = \lambda(t)dt$, with λ deterministic,

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Campbell's theorem

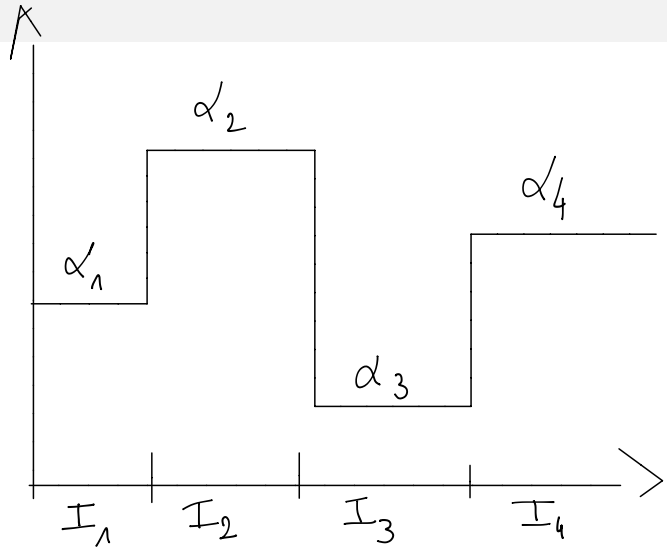
If f is such that $\int \min(|f(x)|, 1) d\mu_x < \infty$, then

$$\mathbb{E}\left(\exp\left[\int f(x) dN_x\right]\right) = \exp\left(\int \left[e^{f(x)} - 1\right] d\mu_x\right).$$

Small proof

Take f piecewise constant,

$$f = \sum_I \alpha_I \mathbf{1}_I.$$



$$\begin{aligned}
\mathbb{E}\left(\exp\left[\int f(x) dN_x\right]\right) &= \mathbb{E}\left(\prod_I e^{\alpha_I N_I}\right) \\
&= \prod_I \mathbb{E}\left(e^{\alpha_I N_I}\right) \text{ (by independence)} \\
&= \prod_I \exp\left((e^{\alpha_I} - 1)\mu_I\right) \text{ (Laplace of a Poisson)} \\
&= \exp\left(\int (e^{f(x)} - 1) d\mu_x\right)
\end{aligned}$$

Exponential inequality for Poisson process

Aim

For all $\xi > 0$,

$$\mathbb{P}\left(\int f(x)[dN_x - \lambda(x)dx] \geq ???(y)\right) \leq e^{-y}$$

NB : easier this way for interpretation in statistics... (quantile)

Let f be some fixed function (deterministic) and let $\theta > 0$.
Apply Campbell's theorem to θf :

$$\mathbb{E}\left(\exp\left[\int \theta f(x)[dN_x - \lambda(x)dx]\right]\right) = \exp\left(\int (e^{\theta f(x)} - \theta f(x) - 1)d\mu_x\right).$$

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Hence if $E = \exp\left[\int \theta f(x)[dN_x - \lambda(x)dx] - \int (e^{\theta f(x)} - \theta f(x) - 1)d\mu_x\right]$,

$$\mathbb{E}(E) = 1.$$

Towards a (weak) Bernstein inequality...

Therefore for all $y > 0$

$$\mathbb{P}\left(\int \theta f(x)[dN_x - \lambda(x)dx] \geq \int (e^{\theta f(x)} - \theta f(x) - 1)d\mu_x + y\right) = \mathbb{P}(E \geq e^y)$$

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By Markov,

$$\mathbb{P}\left(\int \theta f(x)[dN_x - \lambda(x)dx] \geq \int (e^{\theta f(x)} - \theta f(x) - 1)d\mu_x + y\right) \leq \mathbb{E}(E)e^{-y}.$$

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Therefore for all $y > 0$

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Hence for all $\theta, y > 0$

$$\mathbb{P}\left(\int f(x)[dN_x - \lambda(x)dx] \geq \frac{1}{\theta}\int (e^{\theta f(x)} - \theta f(x) - 1)d\mu_x + y/\theta\right) \leq e^{-y}.$$

Towards a (weak) Bernstein inequality...

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But

$$\begin{aligned} e^{\theta f(x)} - \theta f(x) - 1 &\leq \sum_{k=2}^{\infty} \frac{\theta^k}{k!}\|f\|_\infty^{k-2} f(x)^2 \\ &\leq \frac{\theta^2 f(x)^2}{2\left(1 - \frac{\theta\|f\|_\infty}{3}\right)} \end{aligned}$$

Let $v = \int f(x)^2 d\mu_x$.

Towards a (weak) Bernstein inequality...(2)

Let $v = \int f(x)^2 \lambda(x) dx$.
Hence for all $\theta, y > 0$

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Towards a (weak) Bernstein inequality...(2)

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Optimum in $\theta = g(v, \|f\|_\infty, y)$ and

NB : si $X \sim \mathcal{N}(m, \sigma^2)$
$\mathbb{P}(X - m \geq t) \leq e^{-\frac{t^2}{2\sigma^2}}$
ie $\mathbb{P}(X - m \geq \sqrt{2\sigma^2 y}) \leq e^{-y}$

Exponential inequality "à la" Bernstein

For all $y > 0$,

$$\mathbb{P}\left(\int f(x)[dN_x - \lambda(x)dx] \geq \sqrt{2vy} + \frac{\|f\|_\infty y}{3} \right) \leq e^{-y}$$

NB : $v = \mathbb{E}\left(\left[\int f(x)[dN_x - \lambda(x)dx] \right]^2 \right) = Var\left(\int f(x) dN_x \right)$. Hence first order Gaussian tail with variance v

Aim

- same kind of exponential inequality for Hawkes (or other general counting process)
- $f(.)$ deterministic has to be replaced by H_t (ex : $\mathbf{Rc}_t$) predictable
- The "variance" term v should follow $\int H_s dN_s \to$ expectation given the past.

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- $f(.)$ deterministic has to be replaced by H_t (ex : $\mathbf{Rc}_t$) predictable
- The "variance" term v should follow $\int H_s dN_s \to$ expectation given the past.
- optional : v still depends on λ, unknown $\to$ as to be replaced by observable quantity.

Martingale

Let N counting process with (predictable) intensity λ. Let H_s be any predictable process and $t > u$, then

$$\mathbb{E}(H_t dN_t \Big| \text{ past at t}) = H_t \mathbb{E}(dN_t | \text{ past at t}) = H_t \lambda(t) dt$$

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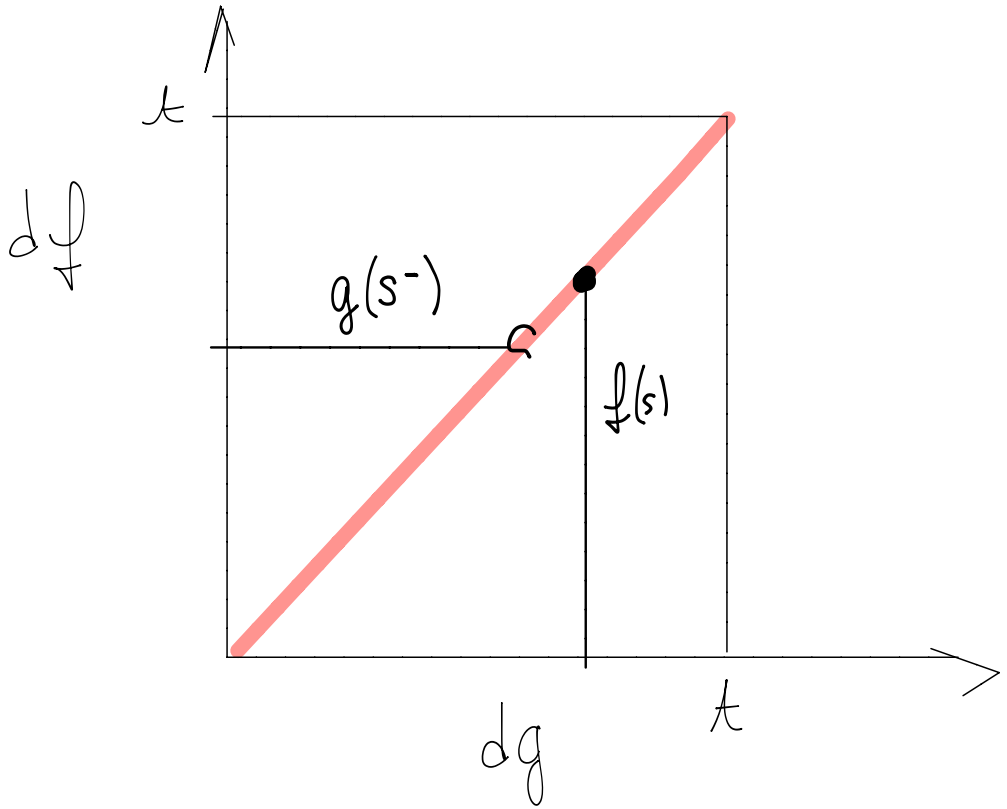
$$\mathbb{E}\left(\int_0^t H_s [dN_s - \lambda(s) ds] \Big| \text{ past at u} \right) = \int_0^u H_s [dN_s - \lambda(s) ds]$$

$\rightarrow$ martingale.

Bracket

A Stieljes integration by part : whatever the bounded variation càd-làg functions (i.e. works for dN_t, $\lambda(t)dt$ or $dN_t - \lambda((t)dt)$

$$f(t)g(t) = f(0)g(0) + \int_0^t f(s)dg_s + \int_0^t g(s^-)df_s$$



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$$= \int_0^t H_s^2 dN_s + \text{ martingale}$$

$\rightarrow$ compensator ($\simeq$ variance given the past) $V_t = \int_0^t H_s^2 \lambda(s)ds$

predictable and $\left(\int_0^t H_s[dN_s - \lambda(s)ds]\right)^2 - V_t$ martingale

Exponential martingale

Aim : the martingale equivalent of Campbell's theorem.

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$$E_t = \exp\left(\int_0^t H_s[dN_s - \lambda(s)ds] - \int_0^t (e^{H_s} - H_s - 1)\lambda(s)ds\right)$$

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Hence martingale and $\mathbb{E}(E_t) = E_0 = 1$.
NB : eventually, integrability problems, so $\mathbb{E}(E_t) \leq 1$...

Exponential inequality à la Bernstein

$H_s \to \theta H_s$ and the corresponding $E_t \to E$ in the Poisson Bernstein proof :

for all $\theta, y > 0$

$$\mathbb{P}\left(\int_0^t H_s [dN_s - \lambda(s)ds] \geq \frac{\theta V_t}{2\left(1 - \frac{\theta \|H\|_\infty}{3}\right)} + y/\theta \right) \leq e^{-y}$$

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- Assuming $\|H\|_\infty \leq b$ deterministic and known, not a big deal since second order term,
- but assuming $V_t \leq v$ and replacing V_t by v deterministic is not sharp at all !
- However optimising in θ needs a θ deterministic (because ultimately E_t depends on θ)

Slicing

Let $\mathcal{S} = \{w \leq V_t \leq v = (1+\epsilon)^K w\}$ and consider the slice

$$\mathcal{S}_k = \{(1+\epsilon)^k w \leq V_t \leq (1+\epsilon)^{k+1} w)\}$$

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Here choose $\theta = g((1+\epsilon)^{k+1} w, b, y)$ optimising ...

$$\mathbb{P}\left(\int_0^t H_s[dN_s - \lambda(s)ds] \geq \sqrt{2(1+\epsilon)^{k+1} wy} + \frac{by}{3} \text{ and } \mathcal{S}_k\right) \leq e^{-y}$$

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$$\mathbb{P}\left(\int_0^t H_s[dN_s - \lambda(s)ds] \geq \sqrt{2(1+\epsilon)V_t y} + \frac{by}{3} \text{ and } \mathcal{S}\right) \leq Ke^{-y}$$

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But $V_t = \int_0^t H_s^2 \lambda(s)ds$, still depends on the unknown λ
$\rightarrow$ one more turn using the fact that

$$\hat{V}_t = \int_0^t H_s^2 dN_s$$

is observable and

$$\hat{V}_t - V_t = \text{martingale}$$

Bernstein-type inequality for general counting process (Hansen, RB, Rivoirard)

Let $(H_s)_{s\geq 0}$ be a predictable process and $M_t = \int_0^t H_s(dN_s - \lambda(s)ds)$.
Let $b > 0$ and $v > w > 0$. For all $x, \mu > 0$ such that $\mu > \phi(\mu)$, let

$$\hat{V}_\tau^\mu = \frac{\mu}{\mu - \phi(\mu)} \int_0^\tau H_s^2 dN_s + \frac{b^2 x}{\mu - \phi(\mu)},$$

where $\phi(u) = \exp(u) - u - 1$.
Then for every stopping time τ and every $\varepsilon > 0$

$$\mathbb{P}\left(M_\tau \geq \sqrt{2(1+\varepsilon)\hat{V}_\tau^\mu x} + bx/3, \quad w \leq \hat{V}_\tau^\mu \leq v \text{ and } \sup_{s\in[0,\tau]} |H_s| \leq b \right) \leq 2\frac{\log(v/w)}{\log(1+\varepsilon)} e^{-x}.$$

$\rightsquigarrow$ a generic choice of $\mathbf{d}$ for the Lasso whatever the underlying process....

Back to Lasso for Hawkes

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Lasso criterion

$$\begin{aligned}\hat{\mathbf{a}}^{(r)} &= \operatorname{argmin}_{\mathbf{a}}\{\gamma_T(\lambda_{\mathbf{a}}^{(r)}) + \mathrm{pen}(\mathbf{a})\} \\ &= \operatorname{argmin}_{\mathbf{a}}\{-2\mathbf{a}'\mathbf{b}_r + \mathbf{a}'\mathbf{G}\mathbf{a} + 2(\mathbf{d}^{(r)})'|\mathbf{a}|\}\end{aligned}$$

M number of interacting processes, K number of bins, A maximal interaction delay.

Oracle inequality in probability

If $\mathbf{d} = \sqrt{2(1+\varepsilon)\hat{V}_T^{\mu} x} + bx/3$ then with probability larger than

$$1 - \square M^2 K \log(T)^2 e^{-x} - \mathbb{P}(\forall t < T, i, N^i_{[t-A,t]} > \square \log(T)^2) - \mathbb{P}(\mathbf{G} \not\geq cI)$$

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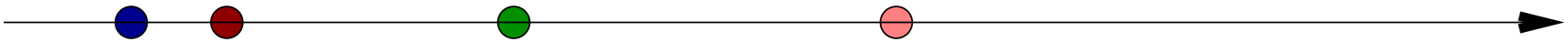
$$\sum_r \|\lambda^{(r)} - \mathbf{R}\mathbf{c}_t\hat{\mathbf{a}}^{(r)}\|^2 \leq \square \inf_{\mathbf{a}} \left\{ \sum_r \|\lambda^{(r)} - \mathbf{R}\mathbf{c}_t\mathbf{a}\|^2 + \frac{1}{c} \sum_{i \in supp(\mathbf{a})} (d_i^{(r)})^2 \right\}.$$

Mathematical debts

- Control of the number of points per interval
- Control of c the smallest eigenvalue of $\mathbf{G}$
- Choice of x.

Table of Contents

A branching representation of the linear Hawkes model (univariate)



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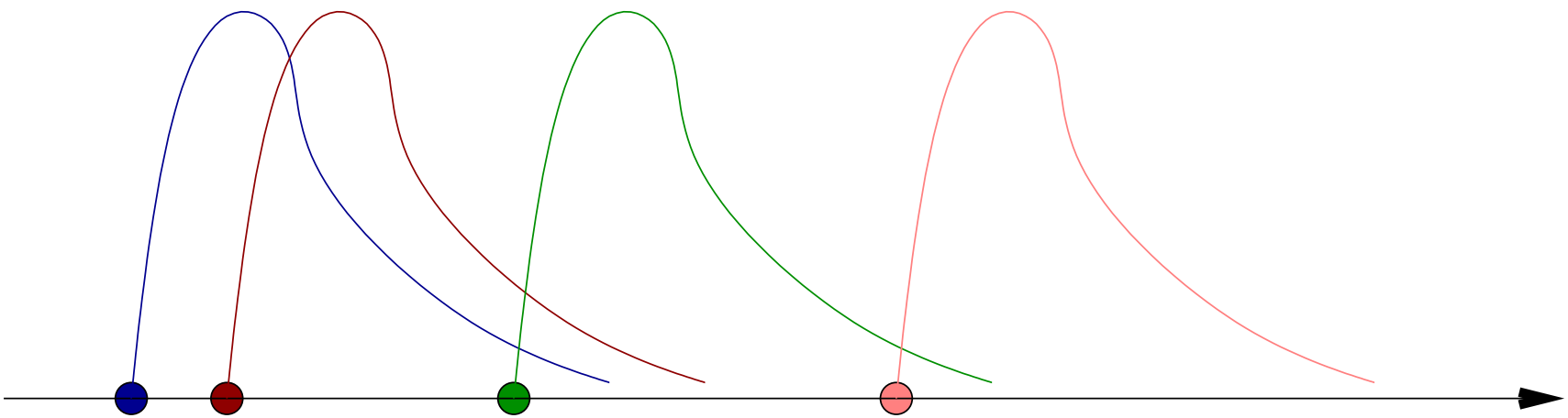

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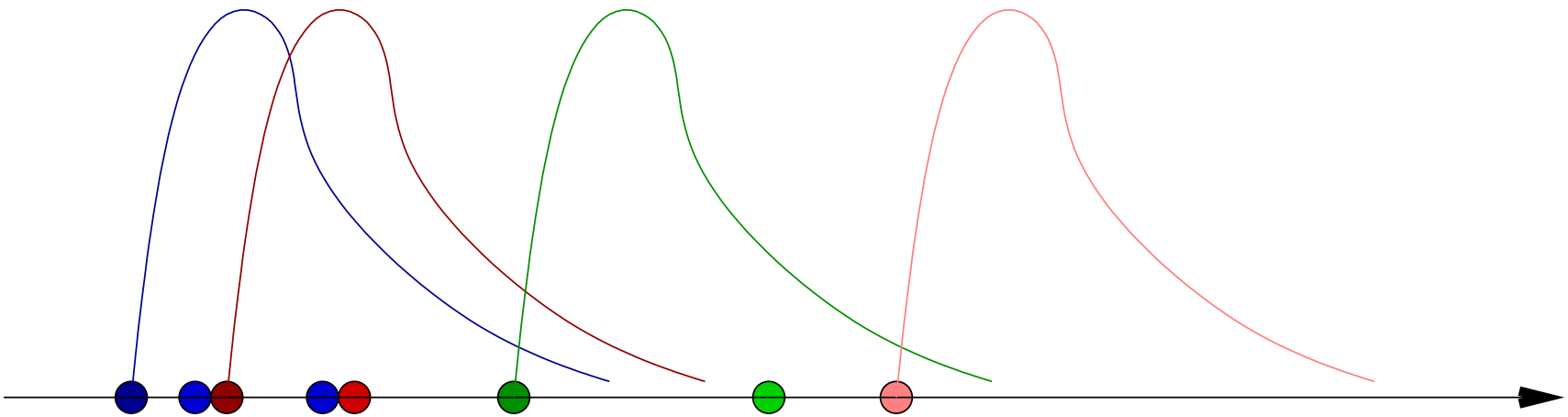

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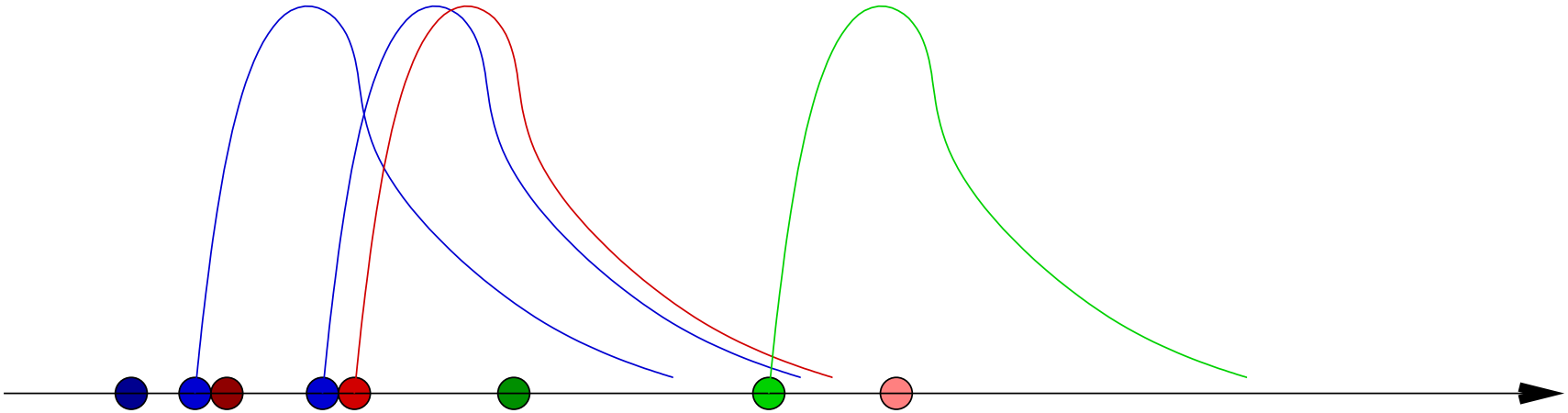

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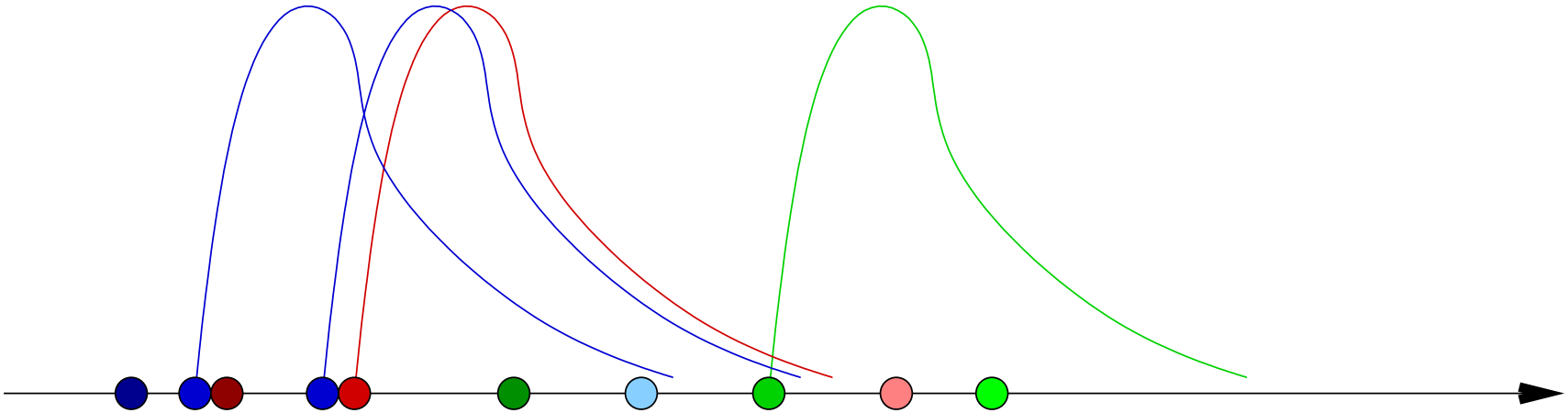

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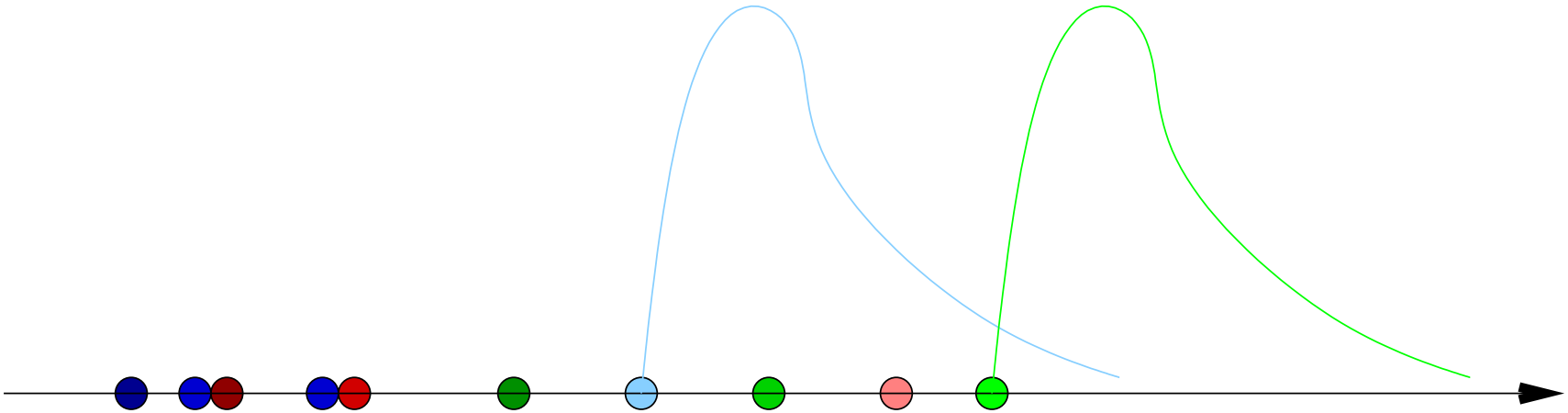

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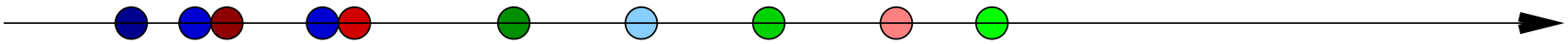

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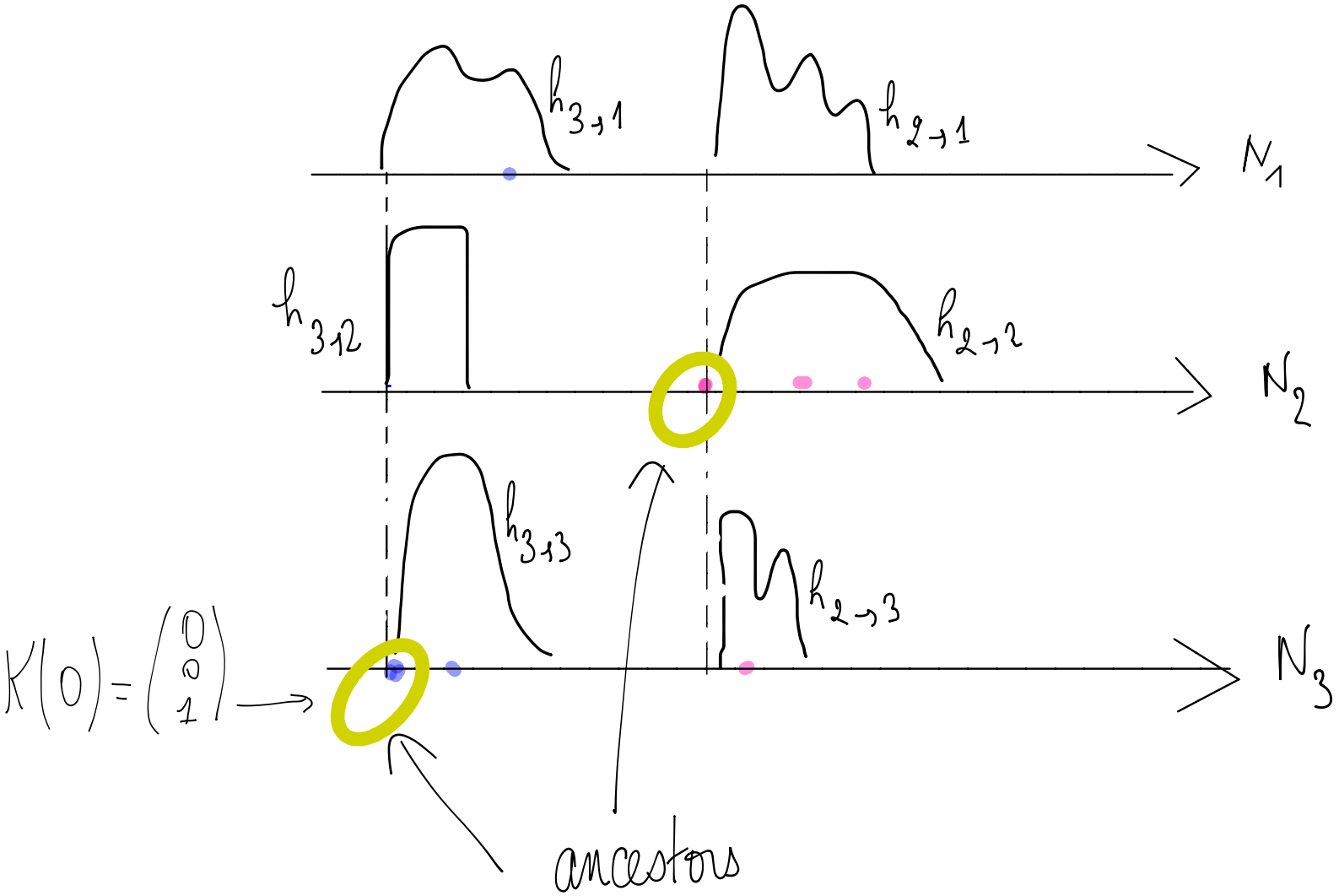

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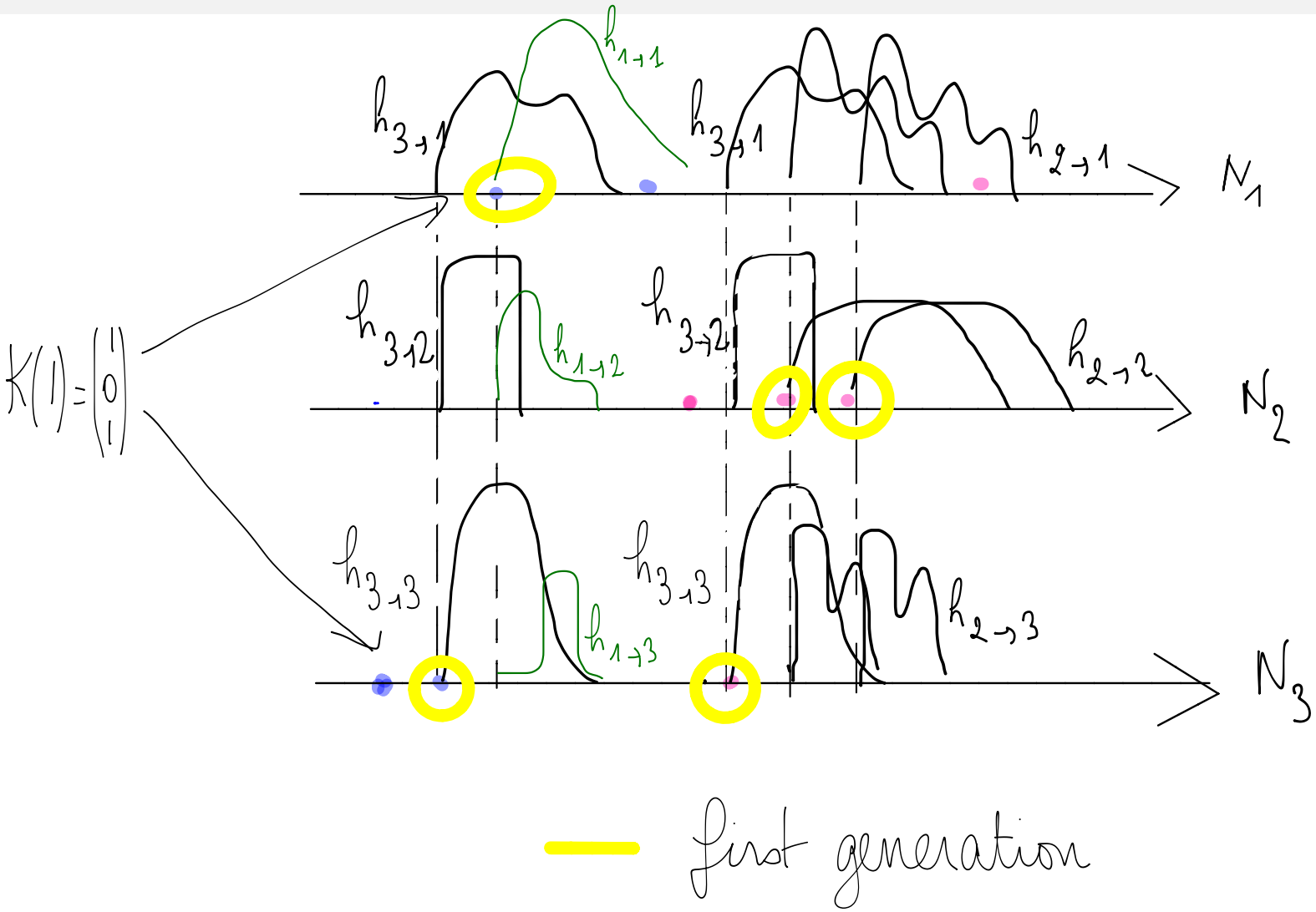

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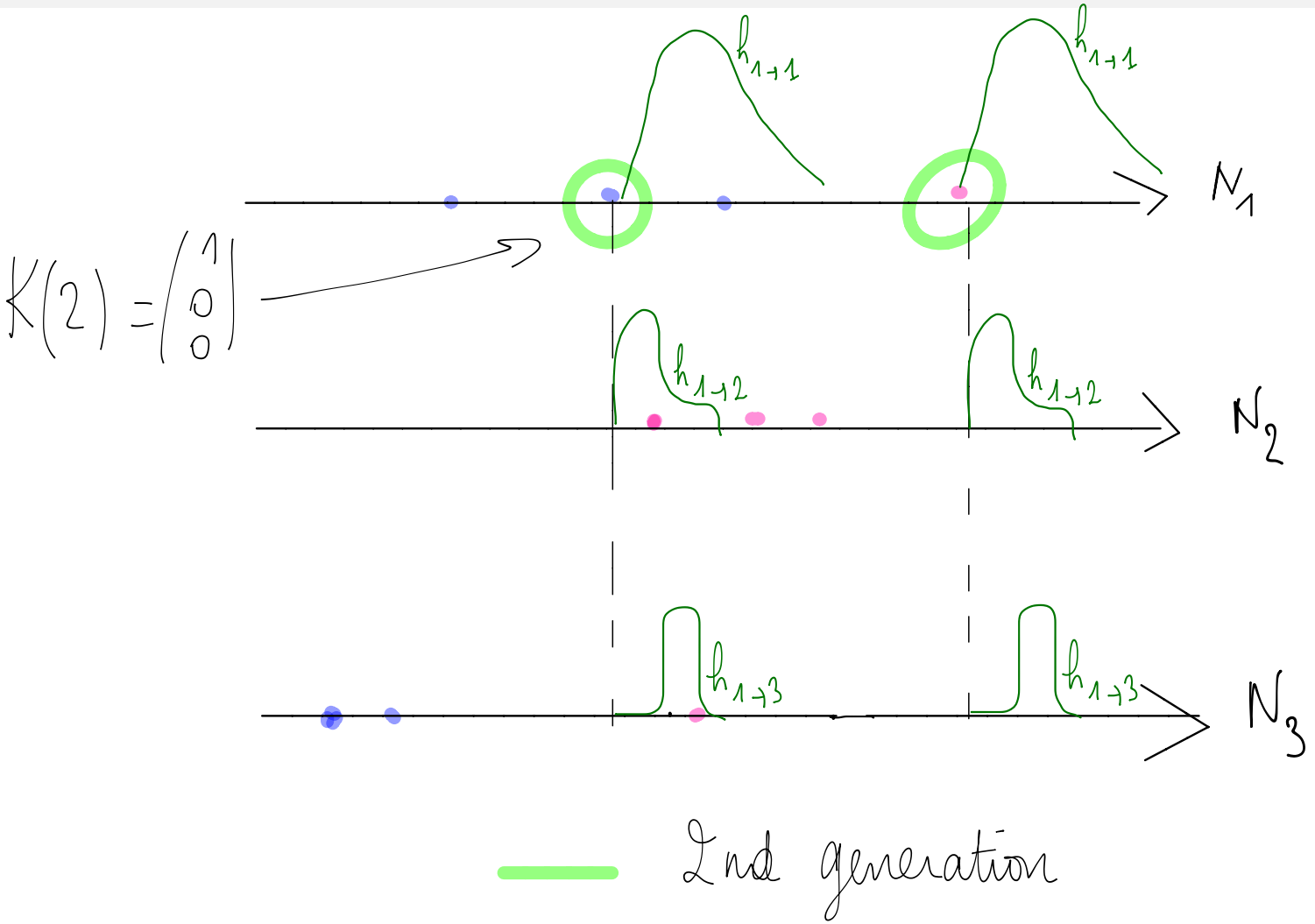

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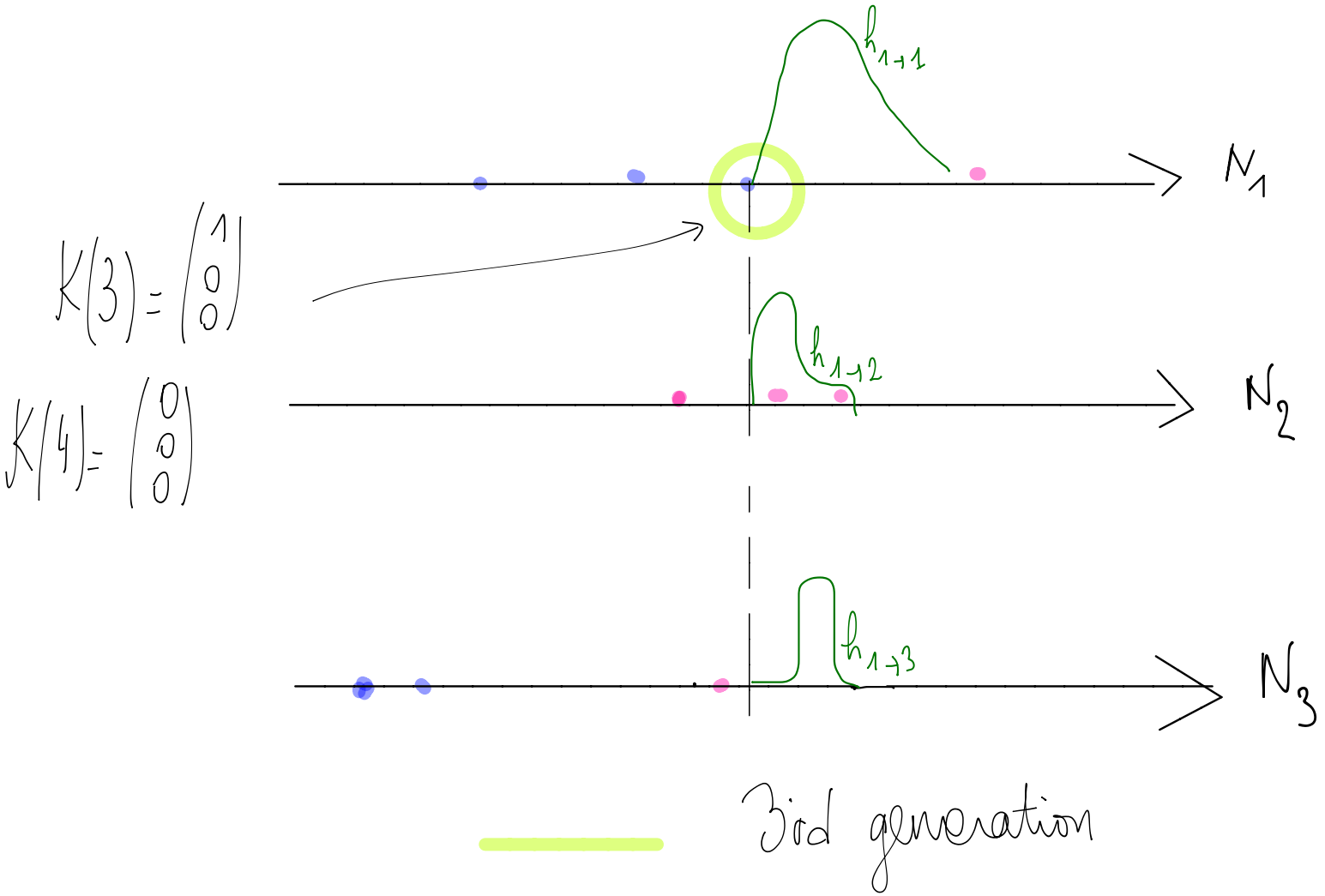

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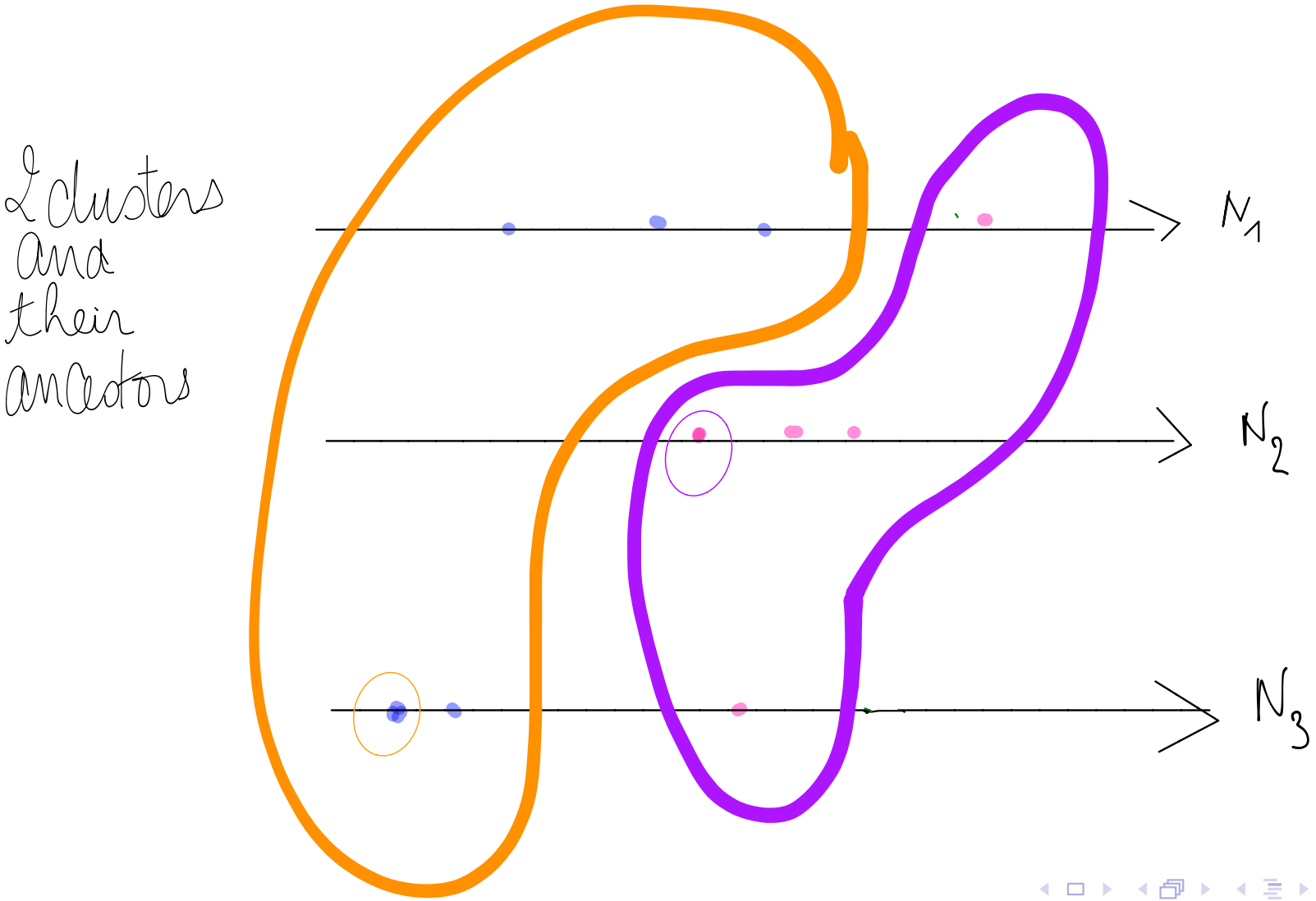
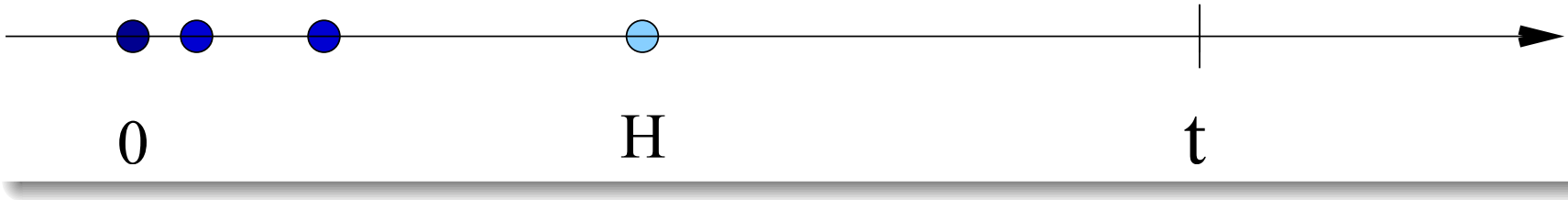


Univariate and multivariate clusters

Cluster process

ancestor in 0 + all its family



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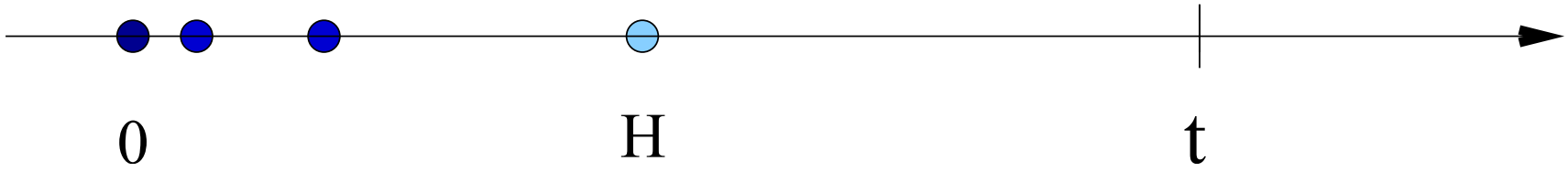

If multivariate, the distribution of the cluster depends on the type of the ancestor.

Number of points of one cluster

If ancestor of type $\ell \to \mathbb{E}_\ell$.

- $K(n)$ vector of the number of points per type in the nth generation
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- If yes,

$$\mathbb{P}_\ell(N_{tot} \geq x) \leq \mathbb{E}_\ell \left(e^{\min_i(\theta_i) N_{tot}} \right) e^{-\min_i(\theta_i)x} \leq \mathbb{E}_\ell \left(e^{\theta' W(\infty)} \right) e^{-\min_i(\theta_i)x}$$

exponentially small

Laplace transform for the cluster

$$\phi(\theta)' = (\phi_1(\theta), ..., \phi_M(\theta)),$$
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&= \mathbb{E}_\ell\left(e^{\theta' W(n-1)} e^{\phi(\theta)' K(n-1)}\right) \\
&= \exp(u_n(\theta)_\ell)
\end{aligned}$$

with $u_n(\theta) = \theta + \phi(u_{n-1}(\theta))$, $u_0(\theta) = \theta$

Laplace transform for the cluster(2)

If ϕ local contraction for a certain norm $\|.\|$:
there exists $r > 0$ and $C < 1$ st if $\|s\| < r$ then

$$\|\phi(s)\| \leq C\|s\|.$$

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If $\|\theta\| \leq r(1 - C)$,

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If $\|\theta\| \leq r(1-C)$,

$$\begin{aligned}
\|u_n(\theta)\| &\leq \|\theta\| + \|\phi(u_{n-1}(\theta))\| \\
&\leq \|\theta\| + C\|u_{n-1}(\theta)\| \\
&\leq \|\theta\|(1 + C + ... + C^n) \\
&\leq \frac{\|\theta\|}{1-C} \leq r
\end{aligned}$$

Hence each coordinate of $u_n(\theta)$ increases (since $W(n)$ increases) but remains in a compact set. Therefore it converges and for θ small enough, $W(\infty)$ has a Laplace transform.

Contraction ?

$K(1)$ has a Laplace transform (only thing needed $\Longleftarrow$ it's Poisson).

$$\phi_\ell(\theta) = \log \mathbb{E}_\ell(e^{\theta' K(1)}).$$

Hence $\partial\phi(0) = \Gamma = (\int h_\ell^{(m)})_{\ell,m}$ with spectral radius < 1.

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Since $\phi(0) = 0$, by continuity

$$\exists C \in (0,1), r > 0, \quad \forall \|s\| < r, \quad \|\phi(s)\| \leq C\|s\|.$$

Number of points for Hawkes processes per interval

Assume

- stationary (spectral radius of $\Gamma < 1$)
- h has bounded support.

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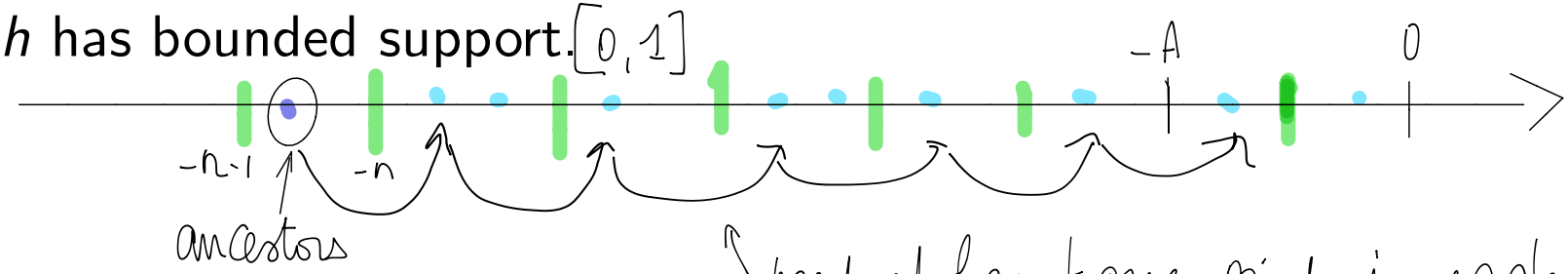

$\tilde{N}_{A,\ell}$ number of points in $[-A, 0)$ whose ancestor is of type ℓ

number of ancestors of type ℓ in $[-n-1, n]$

$$\tilde{N}_{A,\ell} \leq \sum_{n} \sum_{k=1}^{A_n} \max\{W_{n,k} - n + \lceil A \rceil, 0\}$$

$W_{n,k}$: number of points in the cluster of ancestor k

Number of points for Hawkes processes per interval(2)

Let $H_n(\theta_\ell) = \mathbb{E}_\ell \left(e^{\theta_\ell \max\{W-n+\lceil A \rceil, 0\}} \right)$, then

$$\begin{aligned} \mathbb{E}\left(e^{\theta_\ell \tilde{N}_{A,\ell}} \right) &\leq \prod_n \mathbb{E}(H_n(\theta_\ell)^{A_n}) \\ &\leq \prod_n \exp\left(\nu_\ell (H_n(\theta_\ell) - 1)\right) \end{aligned}$$

Since W has a Laplace transform, H_n exponentially decreasing with n and it converges.

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Since W has a Laplace transform, H_n exponentially decreasing with n and it converges.
$\rightarrow$ Laplace transform of $N_{[-A,0)}$ exists
$\rightarrow \mathbb{P}(N_{[-A,0)} > y)$ exponentially small...
NB : true as soon as the branching distribution has a Laplace,
constant unknown... depends on M ! ! !

The expectation of G

Recall that

$$\eta_{\mathbf{a}}(t) = \mathbf{R}\mathbf{c}_t'\mathbf{a} = \nu + \sum_k a_k \int_{-\infty}^{t} \delta^{-1/2} \mathbf{1}_{[t-(k+1)\delta, t-k\delta]} dN_{t-u}$$

$$\mathbf{a}'\mathbf{G}\mathbf{a} = \int_0^T \eta_{\mathbf{a}}(t)^2 dt$$

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Hence

$$\mathbb{E}(\mathbf{G}) \geq cI \iff \forall \mathbf{a}, \mathbb{E}\left(\int_0^T \eta_{\mathbf{a}}(t)^2 dt\right) \geq c\|\mathbf{a}\|^2.$$

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Hence

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If the process is stationnary $\mathbb{E}(\eta_{\mathbf{a}}(t)^2)$ does not depend on t.
Hence we want to show that $c = T\alpha$ with $\mathbb{E}(\eta_{\mathbf{a}}(0)^2) \geq \alpha\|\mathbf{a}\|^2$.
Then we need to concentrate $\frac{1}{T}\mathbf{G}$ around its expectation.

Minoration of the expectation

- If

$$\eta_{\mathbf{a}}(t) = \nu + \sum_k a_k \int_{-\infty}^{t} \delta^{-1/2} \mathbf{1}_{[t-(k+1)\delta, t-k\delta]} dN_{t-u}$$

and N homogeneous Poisson process, then we can find α s.t. $\mathbb{E}(\eta_{\mathbf{a}}(0)^2) \geq \alpha \|\mathbf{a}\|^2$.

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  and N homogeneous Poisson process, then we can find α s.t. $\mathbb{E}(\eta_{\mathbf{a}}(0)^2) \geq \alpha \|\mathbf{a}\|^2$.
- Use the likelihood and Girsanov theorem, to transfer the minoration for Poisson to another minoration for Hawkes....
- because of the likelihood, need again to have a Laplace transform of the number of points...

Clusters and their size

(Univariate)

- Cluster = ancestor in 0 + all its family

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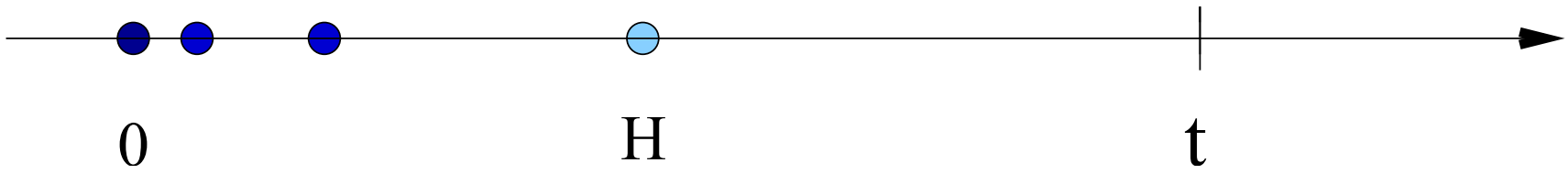

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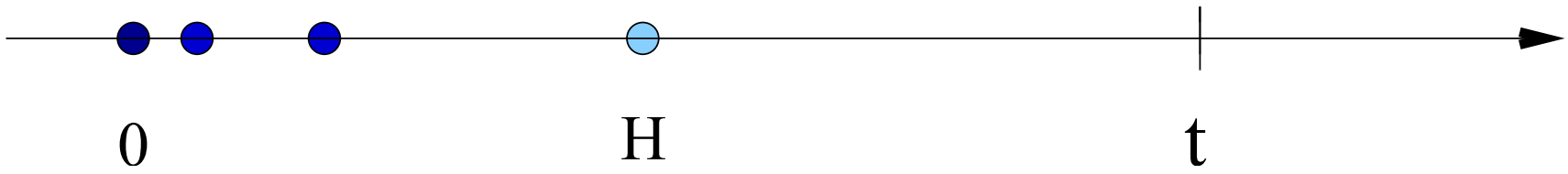

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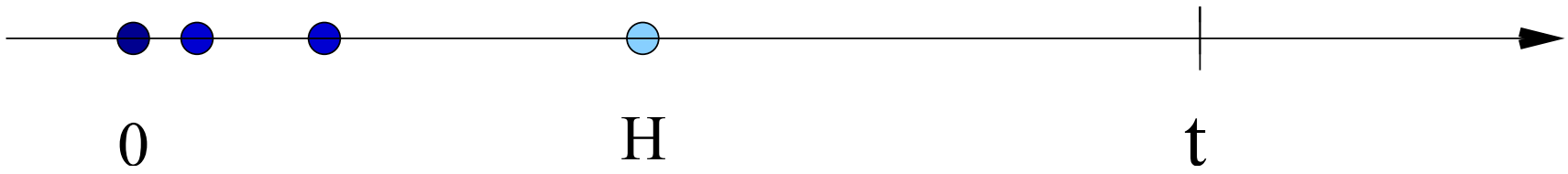
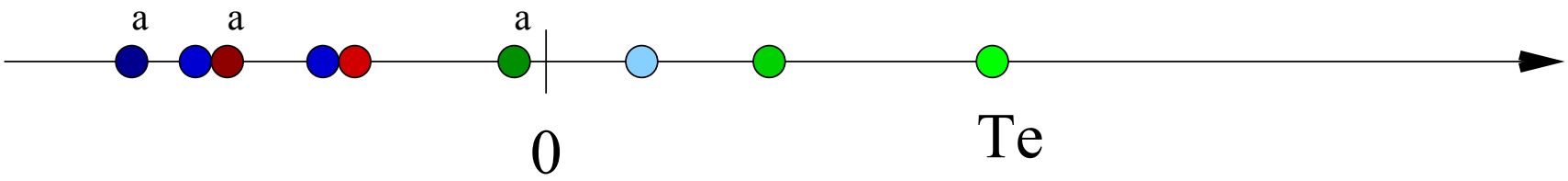


- Size of cluster H $\leq GW * A$ where A maximal support size for h
- and GW total number of births in a Galton-Watson process (Poisson($\int h$)).
- Hence $\mathbb{P}(\mathrm{H} > t)$ exponentially small

Extinction time

- Extinction time Te = time from 0 until the last birth, if no ancestor after 0



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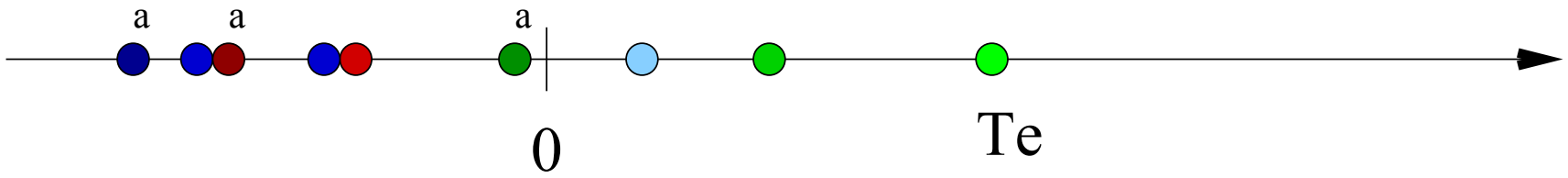

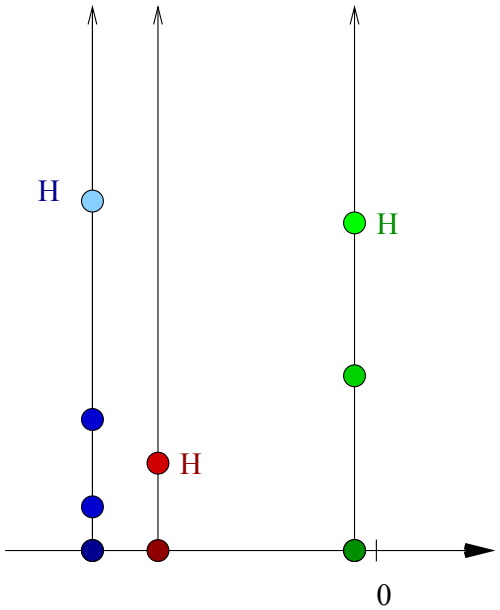

Conditionnally to the ancestor at t,
$P(H > t-a)$ = pty that it reaches a.

$$\Rightarrow P(T_e \le a \mid \text{ancestors}) = \prod_{anc} P(H > anc - a).$$

$$= \exp \int_{-\infty}^{0} \log P(H > t-a)\, dN_{anc}(t)$$

- Poisson ancestor process before 0 marked by the size of the associated cluster, H

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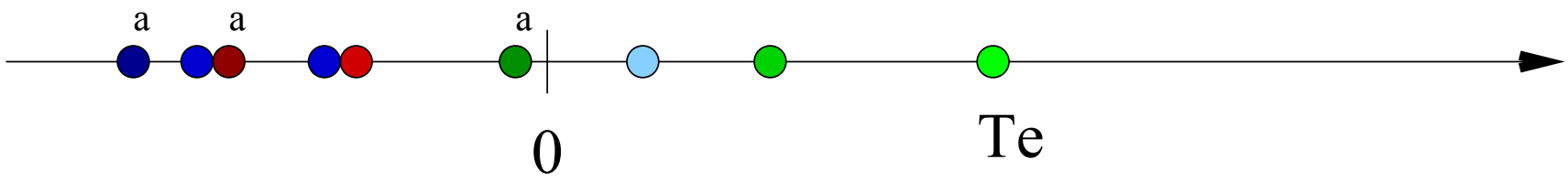

- $\mathbb{P}(\mathsf{Te} \leq a) = \exp\left(-\nu \int_a^{+\infty} \mathbb{P}(\mathsf{H} > t)dt\right)$

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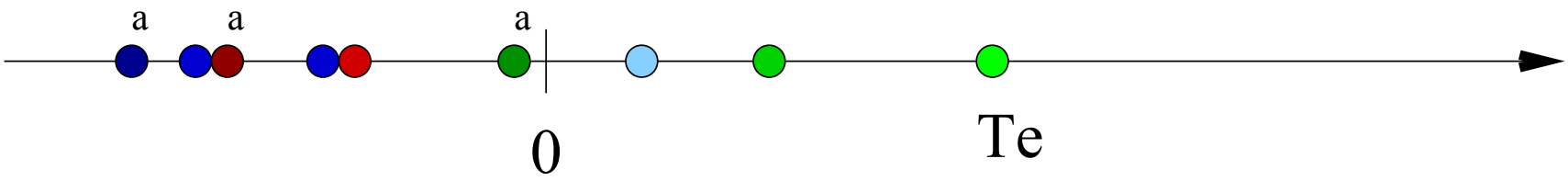

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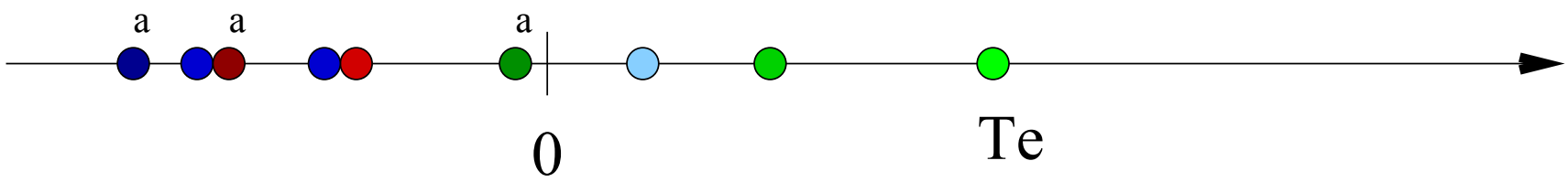

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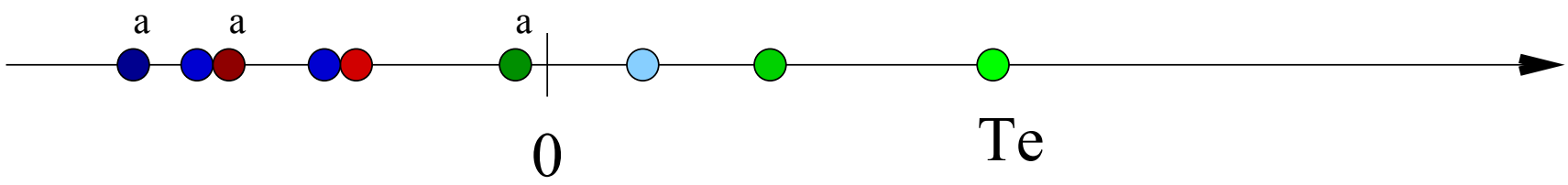
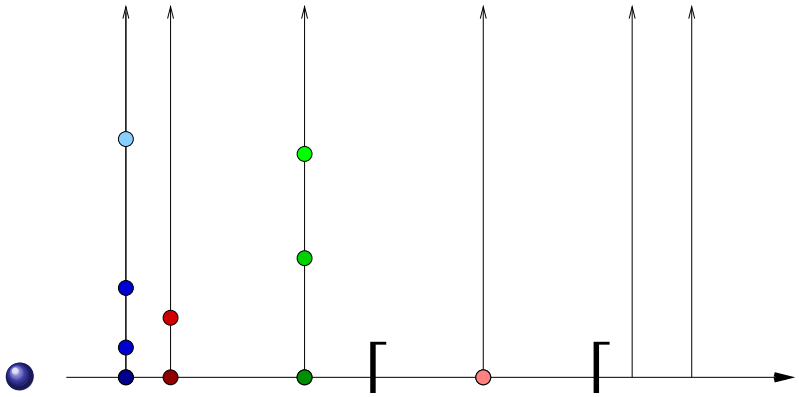


- Hence $\mathbb{P}(\mathrm{Te} > a)$ exponentially small
- Approximated simulation of a stationnary Hawkes process.
- For exact simulation see work of Möller etc

Towards independence



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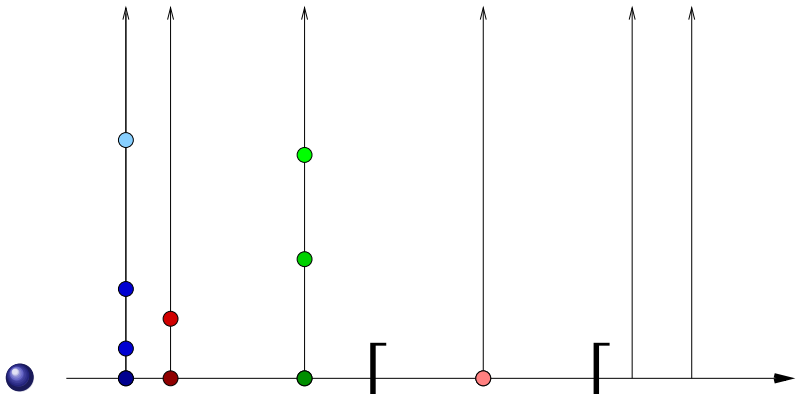

Cut $[0, T]$ in almost independent pieces (cf Berbee's lemma, discrete case (Baraud,Comte et Viennet))

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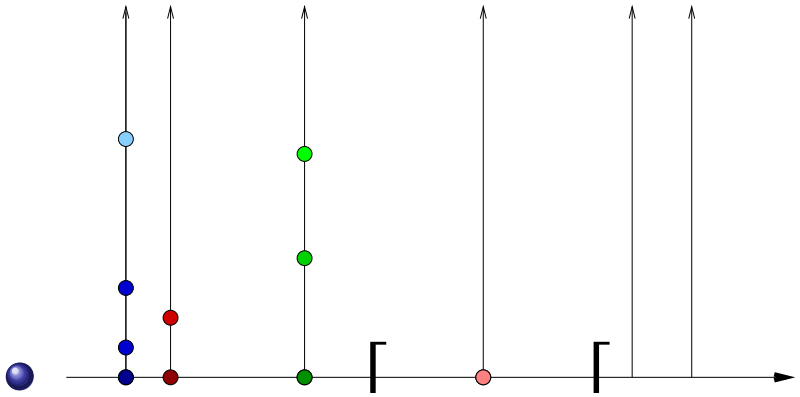
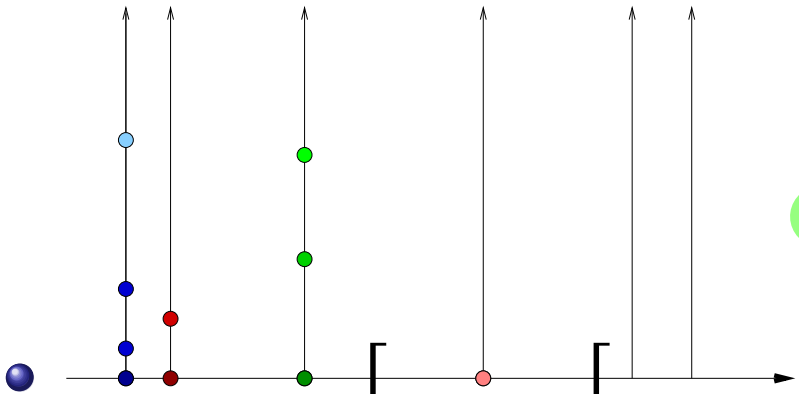
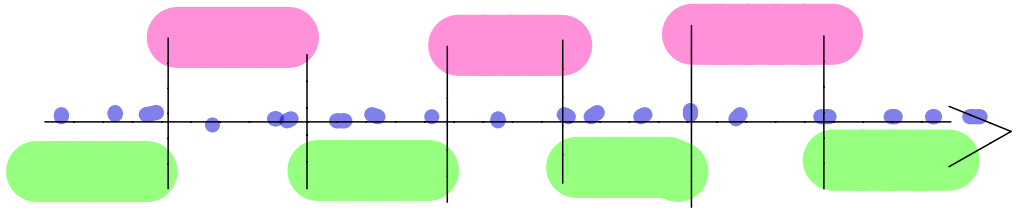
- 

  Cut $[0, T]$ in almost independent pieces (cf Berbee's lemma, discrete case (Baraud,Comte et Viennet))
- Control of the total variation distance.

Towards independence





- Cut $[0, T]$ in almost independent pieces (cf Berbee's lemma, discrete case (Baraud,Comte et Viennet))
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- Up to this error = sum of two groups of independent variables.

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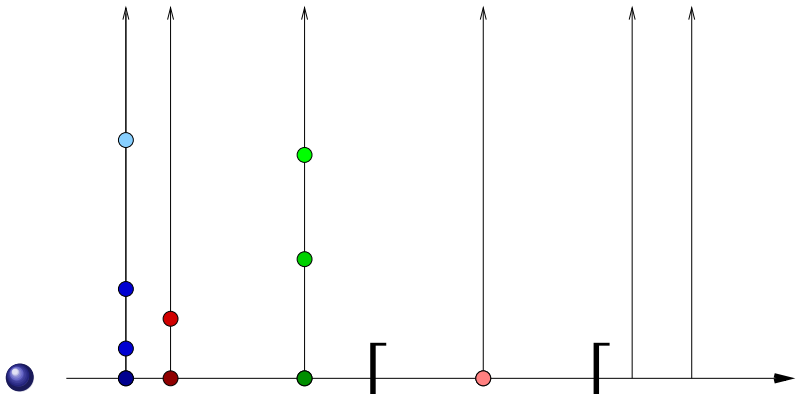

Cut $[0, T]$ in almost independent pieces (cf Berbee's lemma, discrete case (Baraud,Comte et Viennet))

- Control of the total variation distance.
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- All concentration inequalities for i.i.d. variables apply (Bernstein ...)

Table of Contents

Lasso criterion

$$\begin{aligned}\hat{\mathbf{a}}^{(r)} &= \operatorname{argmin}_{\mathbf{a}}\{\gamma_T(\lambda_{\mathbf{a}}^{(r)}) + \operatorname{pen}(\mathbf{a})\} \\ &= \operatorname{argmin}_{\mathbf{a}}\{-2\mathbf{a}'\mathbf{b}_r + \mathbf{a}'\mathbf{G}\mathbf{a} + 2(\mathbf{d}^{(r)})'|\mathbf{a}|\}\end{aligned}$$

M number of interacting processes, K number of bins, A maximal interaction delay. If linear stationnary Hawkes

Oracle inequality in probability

If $\mathbf{d} = \sqrt{2(1+\varepsilon)\hat{V}_T^{\mu} x} + bx/3$ then with probability larger than

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Lasso criterion...

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- good renormalization by T $\rightsquigarrow$, if piecewise constant true,

$$\text{loss on interaction function} \leq 0 + \square \frac{|\mathbf{a}^*|_0 \log(T)^3}{T}$$

with probability larger than $1 - \square M^2 K \log(T)^2 T^{-\gamma} - \square T^{-1}$

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- If distance bounded (reasonable),

$$\mathbb{E}(\text{ loss }) \leq \square \frac{|\mathbf{a}^*|_0 \log(T)^3}{T} + \square M^2 K \log(T)^2 T^{-\gamma} + \square T^{-1}$$

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- If distance bounded (reasonable),

$$\mathbb{E}(\text{ loss }) \leq \mathbb{I} \frac{|\mathbf{a}^*|_0 \log(T)^3}{T} + \square M^2 K \log(T)^2 T^{-\gamma} + \square T^{-1}$$

- Hence $\gamma > 1$

Choice of x

If $x = \gamma \log(T)$, with probability larger than

$$1 - M^2 K \log(T)^2 T^{-\gamma} - \square T^{-1}$$

$$\sum_r \|\lambda^{(r)} - \mathbf{R}\mathbf{c}_t \hat{\mathbf{a}}^{(r)}\|^2 \leq \square \inf_{\mathbf{a}} \left\{ \sum_r \|\lambda^{(r)} - \mathbf{R}\mathbf{c}_t \mathbf{a}\|^2 + \frac{1}{T} \sum_{i \in supp(\mathbf{a})} (d_i^{(r)})^2 \right\}.$$

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- Hence $\gamma > 1$
- Good reason to believe (proof in Poisson case) that $\gamma \in [1, \square]$, γ smaller changes the rate, γ larger changes the constant ...

Bernstein-type inequality for general counting process (Hansen, RB, Rivoirard)

Let $(H_s)_{s\geq 0}$ be a predictable process and $M_t = \int_0^t H_s(dN_s - \lambda(s)ds)$.
Let $b > 0$ and $v > w > 0$. For all $x, \mu > 0$ such that $\mu > \phi(\mu)$, let

$$\hat{V}_\tau^\mu = \frac{\mu}{\mu - \phi(\mu)} \int_0^\tau H_s^2 dN_s + \frac{b^2 x}{\mu - \phi(\mu)},$$

where $\phi(u) = \exp(u) - u - 1$.
Then for every stopping time τ and every $\varepsilon > 0$

$$\mathbb{P}\left(M_\tau \geq \sqrt{2(1+\varepsilon)\hat{V}_\tau^\mu x} + bx/3, \quad w \leq \hat{V}_\tau^\mu \leq v \text{ and } \sup_{s\in[0,\tau]} |H_s| \leq b \right)$$
$$\leq 2\frac{\log(v/w)}{\log(1+\varepsilon)} e^{-x}.$$

$\rightsquigarrow$ a generic choice of $\mathbf{d}$ for the Lasso whatever the underlying process....

Practical choice of **d**

Recall that the oracle inequality needs

$$\left|\mathbf{b}^{(r)} - \bar{\mathbf{b}}^{(r)}\right| \leq \mathbf{d}^{(r)}, \quad \forall r$$

with

$$\mathbf{b} = \int_0^T \mathbf{Rc}_t dN_t^{(r)} \text{ and } \bar{\mathbf{b}} = \int_0^T \mathbf{Rc}_t \lambda^{(r)}(t) dt.$$

- 'Bernstein Lasso' and 'Bernstein Lasso + OLS'

$$d_i = \sqrt{2\gamma \log(T) \hat{V}_i} + \frac{B_i \gamma \log(T)}{3},$$

$$\hat{V}_i = \int_0^T (\mathbf{Rc}_{t,i})^2 \mathrm{d}N_{t,r_i}, \quad B_i = \sup_{t \in [0,T], m} |\mathbf{Rc}_{t,i}|.$$

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- 'Adaptive Lasso' (Zou)

$$d_i = \frac{\gamma}{2|\hat{a}_i^{ols}|}.$$

Why +OLS ?

Lasso : Least-absolute Shrinkage and Selection Operator

If $\mathbf{G} = I$, minimizing

$$-2\mathbf{a}'b + \mathbf{a}'G\mathbf{a} + 2\mathbf{d}'|\mathbf{a}|$$

$\rightsquigarrow$ a soft-thresholding estimator, i.e.

$$\hat{\mathbf{a}}_{Lasso} = (\mathbf{b} - sign(\mathbf{b})\mathbf{d})_+.$$

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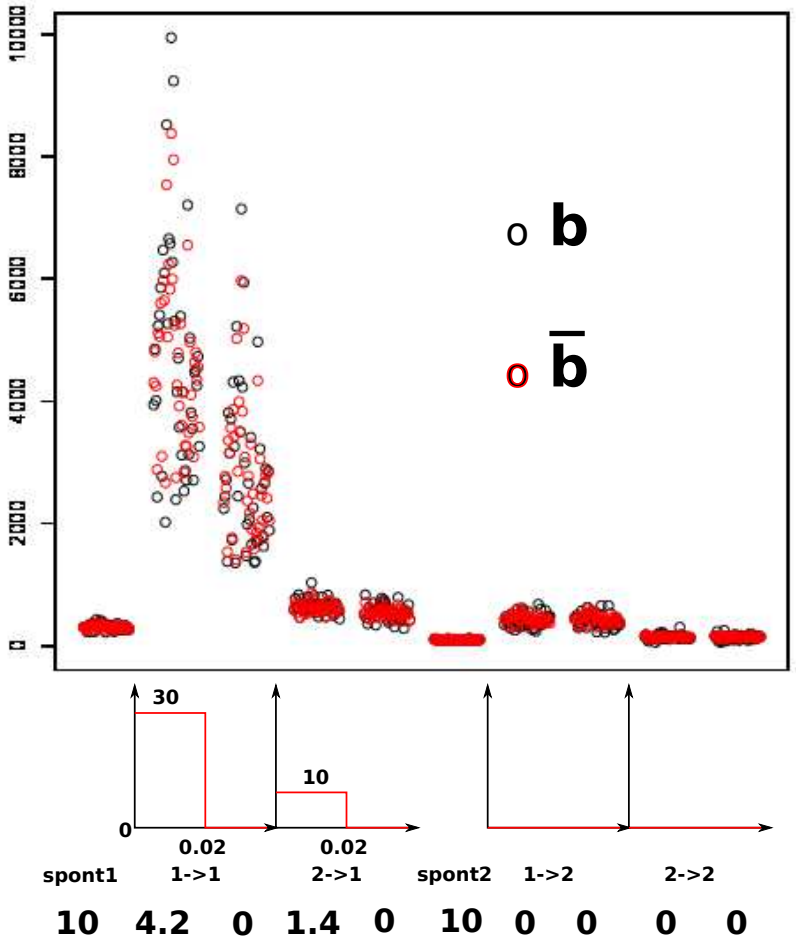
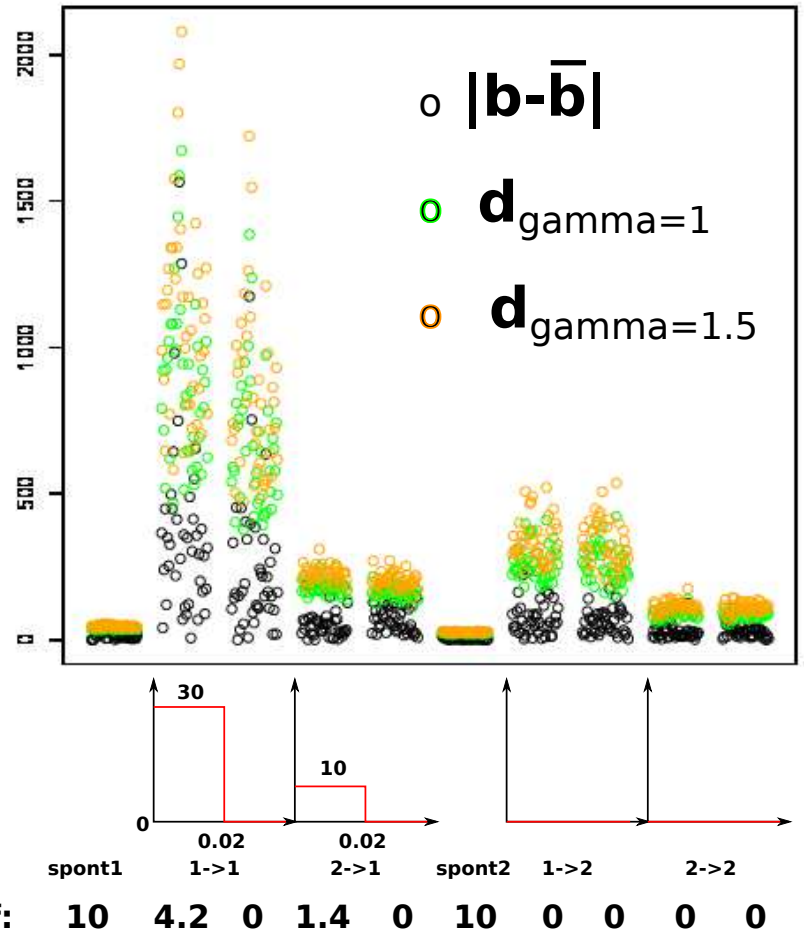
Hence small bias anytime $\rightsquigarrow$
once the support is estimated by Lasso, compute the ordinary least-square estimate (OLS) on the resulting support.

Choices of the weights d_λ





NB : Adaptive Lasso (Zou) $d_\lambda = \gamma/(2|\hat{a}_\lambda|)$

Simulation study - Support recovery

We perform 100 runs with $T = 2$, $M = 8$, $K = 4$ (264 coefficients to be estimated by using 636 observations)

	Bernstein Lasso			Adaptive Lasso		
Tuning parameter γ	0.5	1	2	2	200	1000
Correct clusters identif. $\in [0, 100]$	0	**32**	24	0	0	**32**
False non-zero interactions $\in [0, 55]$	17	6	**1**	55	13	**1**
False zero interactions $\in [0, 9]$	**0**	**0**	2	**0**	**0**	2
False zero spontaneous rates $\in [0, 8]$	**0**	**0**	**0**	**0**	1	3
False non-zero coeff. $\in [0, 238]$	22	7	**1**	199	17	**1**
False zero coeff. $\in [0, 18]$	1	2	7	**0**	2	7

Simulation study - Support recovery

We perform 100 runs with $T = 20$, $M = 8$, $K = 4$

Tuning parameter γ	Bernstein Lasso 0.5	Bernstein Lasso 1	Bernstein Lasso 2	Adaptive Lasso 2	Adaptive Lasso 200	Adaptive Lasso 1000
Correct clusters identif. $\in [0, 100]$	63	99	**100**	0	0	90
False non-zero interactions $\in [0, 55]$	3	1	**0**	55	10	**0**
False zero interactions $\in [0, 9]$	**0**	**0**	**0**	**0**	**0**	**0**
False zero spontaneous rates $\in [0, 8]$	**0**	**0**	**0**	**0**	**0**	**0**
False non-zero coeff. $\in [0, 238]$	4	1	**0**	197	13	**0**
False zero coeff. $\in [0, 18]$	**0**	**0**	**0**	**0**	**0**	**0**

For $T = 20$,

- $\gamma = 1$ or $\gamma = 2$ is convenient for Bernstein Lasso. It was also the case for $T = 2$.
- $\gamma = 1000$ is convenient for Adaptive Lasso. It was not the case for $T = 2$.

Simulation study - Influence of the OLS step

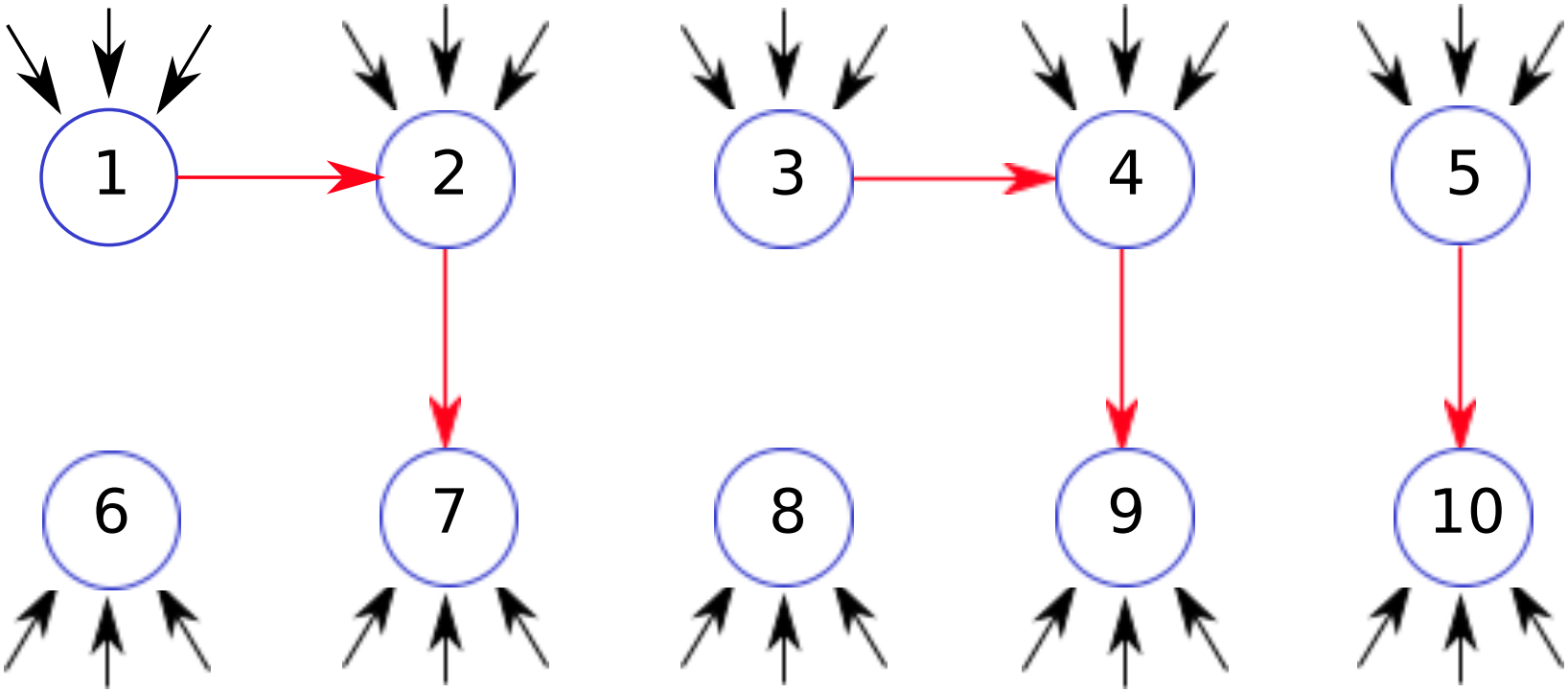
We perform 100 runs with $T = 20$, $M = 8$, $K = 4$.

Tuning parameter γ	Bernstein Lasso $\gamma = 1$	Bernstein Lasso $\gamma = 2$	Adaptive Lasso $\gamma = 1000$
MSE for spontaneous rates	37	69	27
MSE for spontaneous rates after OLS	10	9	10
MSE for interaction functions	3	6	0.5
MSE for interaction functions after OLS	0.5	0.4	0.4

$$\text{MSE for spontaneous rates } = \sum_{m=1}^{M} (\hat{\nu}^{(m)} - \nu^{(m)})^2$$

$$\text{MSE for interaction functions } = \sum_{m=1}^{M} \sum_{\ell=1}^{M} \int (\hat{h}_\ell^{(m)}(t) - h_\ell^{(m)}(t))^2 \mathrm{d}t$$

Another (more realistic ?) neuronal network : Integrate and Fire



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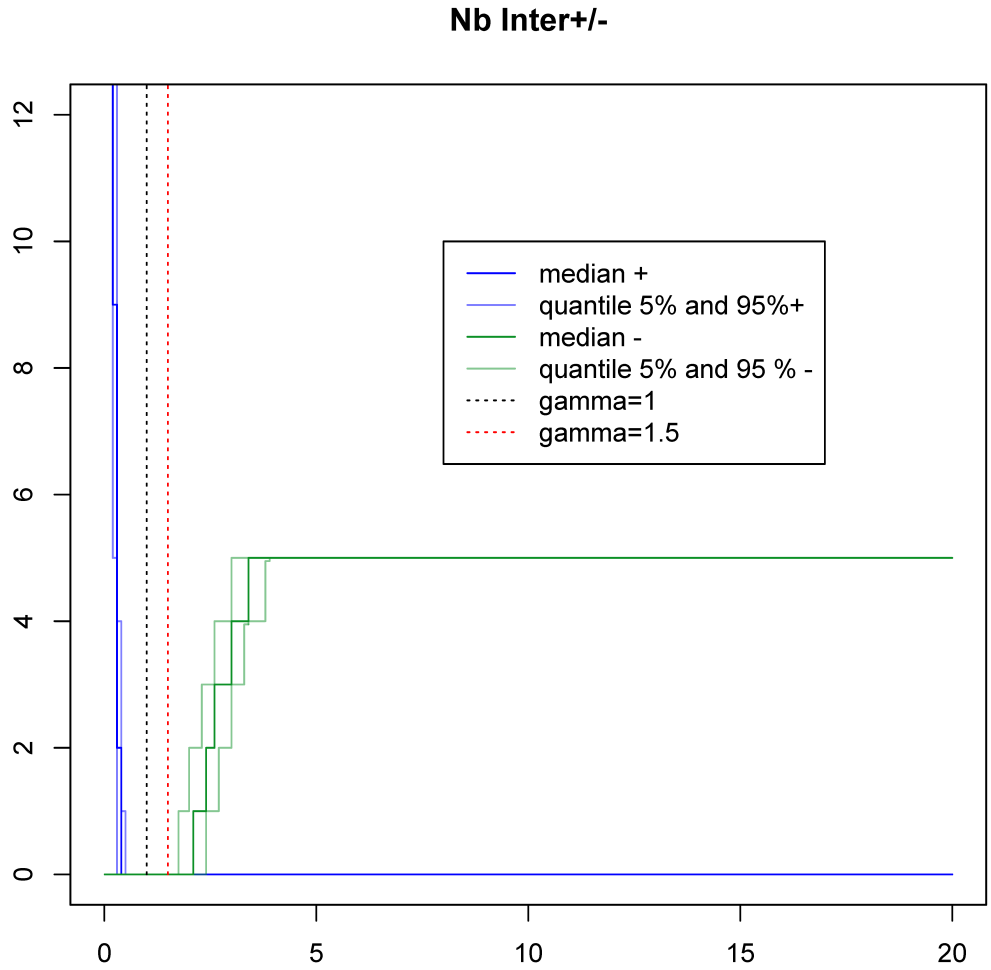

$T = 60s$

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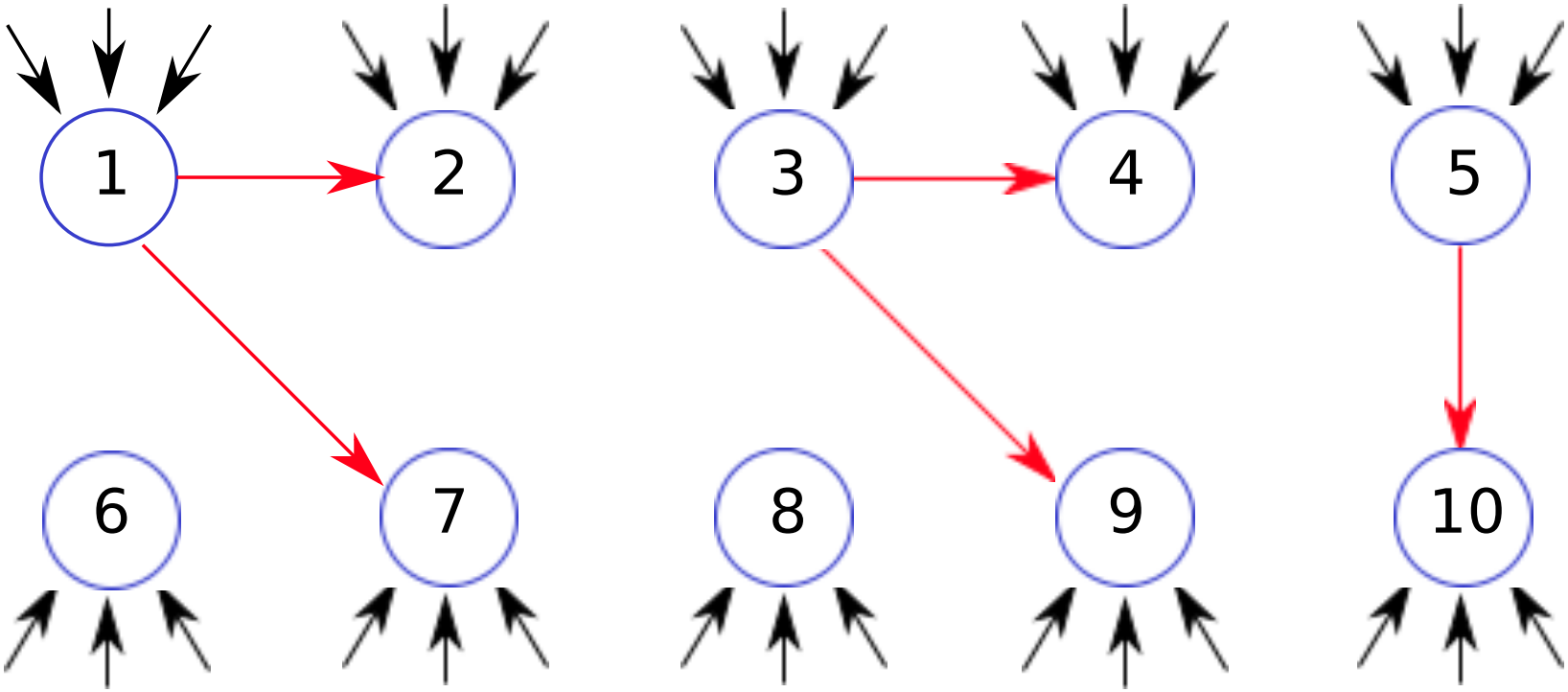
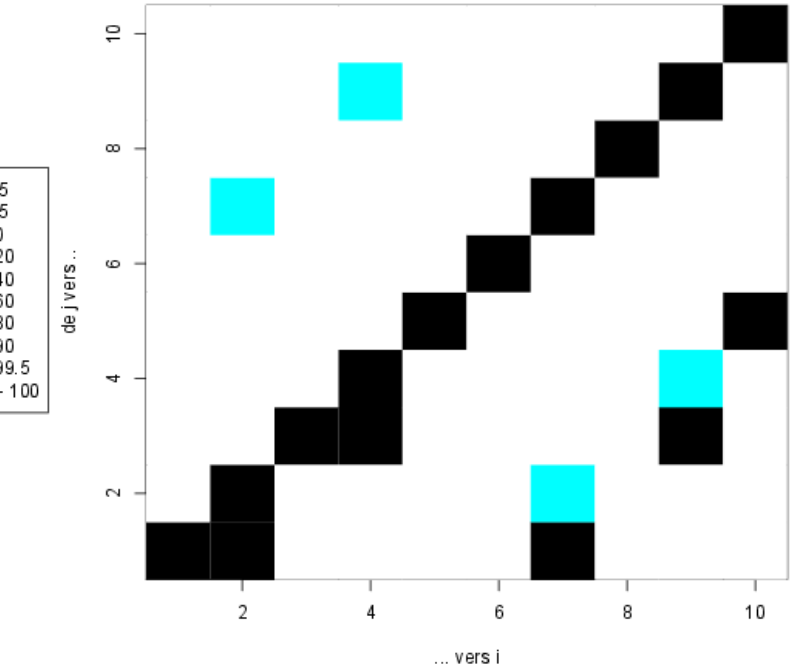
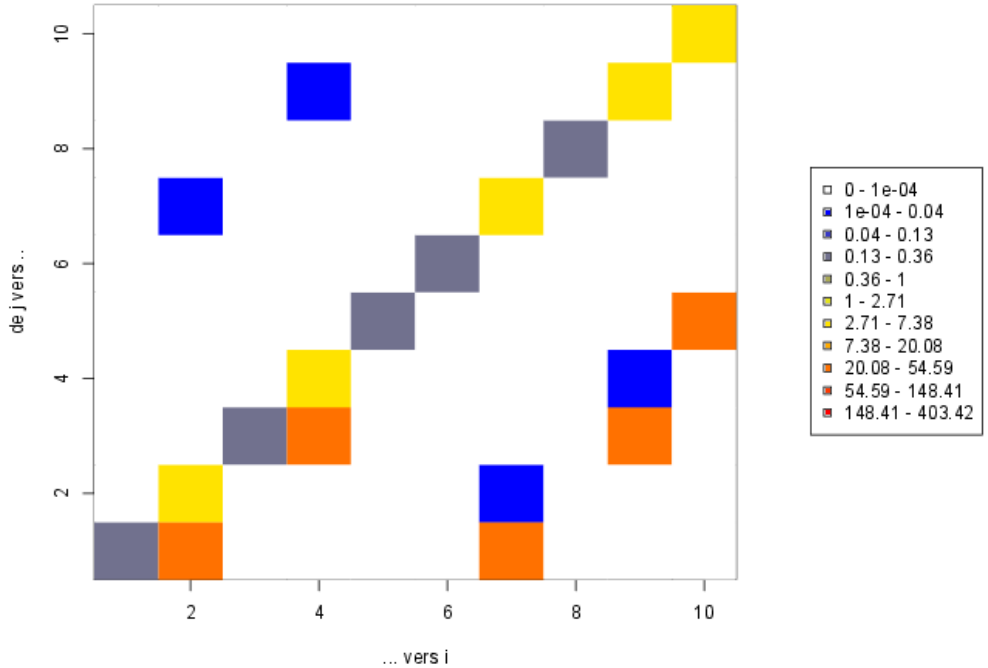


Another (more realistic ?) neuronal network : Integrate and Fire

Number of times that a given interaction is detected



Mean energy ($\int h^2$) when detected



Open questions for Lasso

- Branching arguments ok for linear Hawkes
- Hence number of points controlled if intensity smaller than stationnary Hawkes (thinning) and in particular for $(.)_+$ and inhibition.

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- The "Berbee's lemma" : no idea how to get it without linear Hawkes. Hence what about $(.)_+$? ? ?
- Still on simulation works even if not Hawkes and $\gamma = 1.5$ works !
- What about group Lasso ? (no sharp enough concentration inequality yet)
- What about non stationnarity ? Segmentation, clustering (F. Picard, C. Tuleau-Malot)

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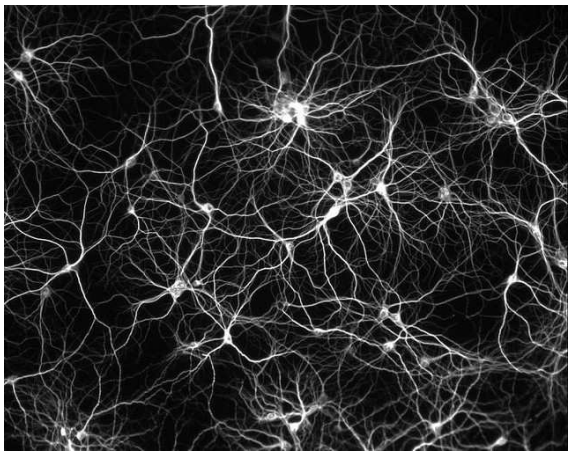
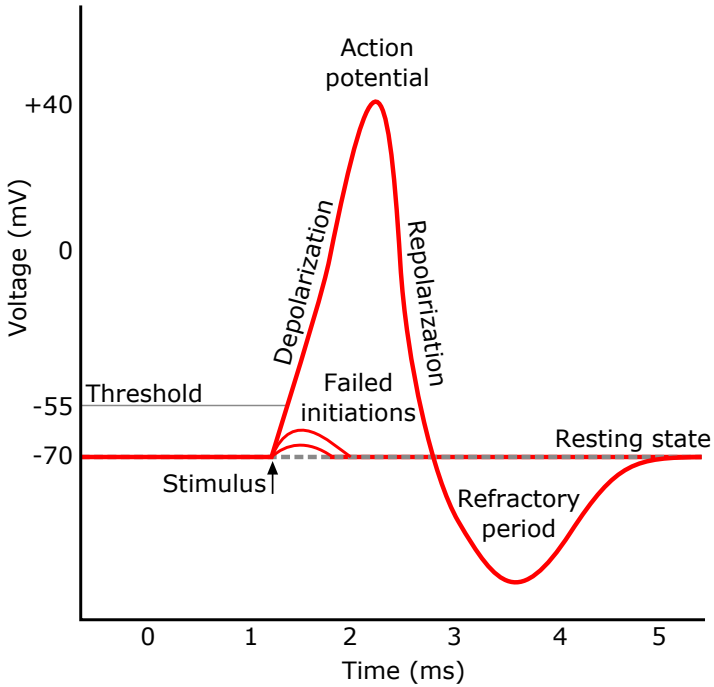
Biological context





Physiological constraint : refractory period.

Age structured equations (Pakdaman, Perthame, Salort, 2010)

Age = delay since last spike.

$$n(t,s) = \begin{cases} \text{probability density of finding a neuron with age } s \text{ at time } t. \\ \text{ratio of the population with age } s \text{ at time } t. \end{cases}$$

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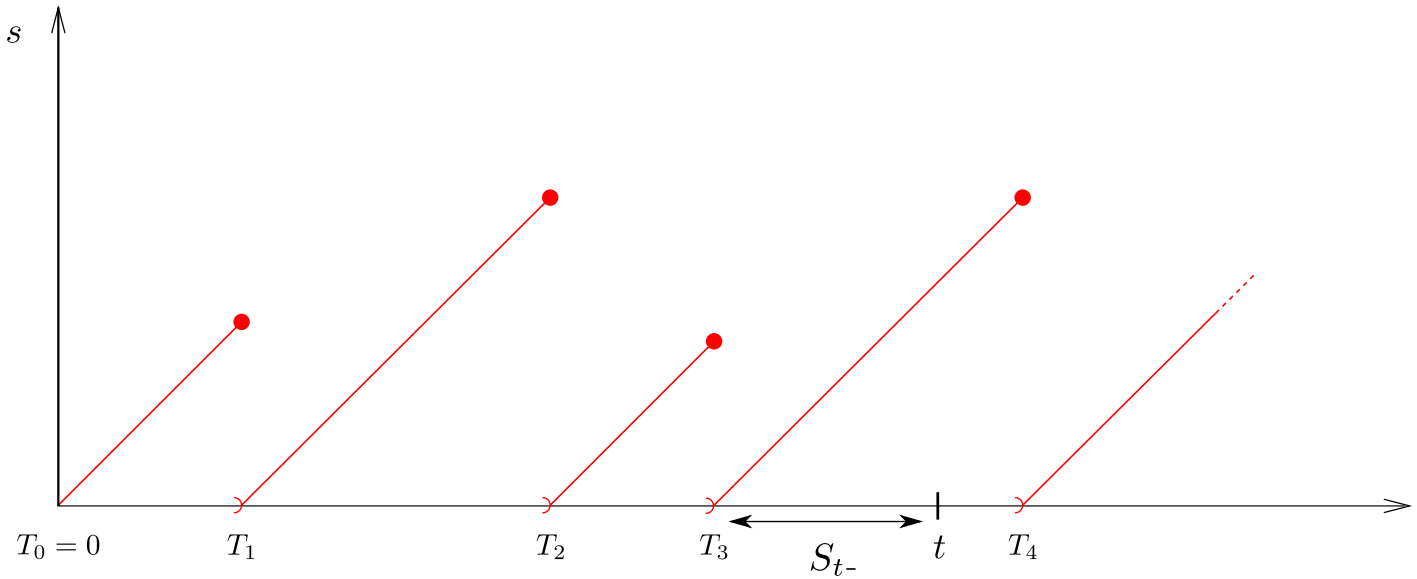
p represents the firing rate. For example, $p(s,X) = \mathbb{1}_{\{s>\sigma(X)\}}$.

$$X(t) = \int_0^t d(x) m(t-v) dv \quad \text{(global neural activity)}$$

d = delay function. For example, $d(v) = e^{-\tau v}$.

One neuron = point process $\rightsquigarrow$ age ?

- Age = delay since last spike.



Microscopic age

- We consider the continuous to the left (hence predictable) version of the age.
- The age at time 0 depends on the spiking times before time 0.
- The dynamic is characterized by the spiking times after time 0.

Framework

$\cdots < T_{-1} < T_0 \leq 0 < T_1 < \ldots$ the ordered sequence of points of N.
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- $N_- = N \cap (-\infty, 0]$ is a point process with distribution $\mathbb{P}_0$ (initial condition).
  The age at time 0 is finite $\Leftrightarrow N_- \neq \emptyset$.
- $N_+ = N \cap (0, +\infty)$ is a point process admitting some intensity $\lambda(t, \mathcal{F}_{t-}^N) \rightsquigarrow$ "probability to find a new point at time t given the past"

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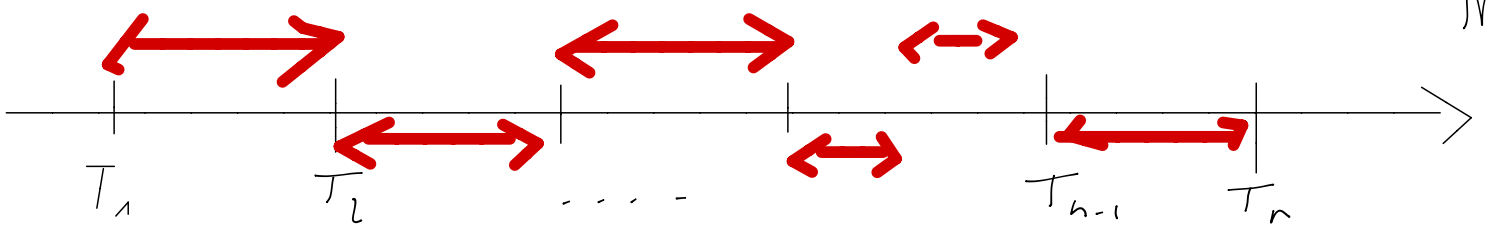
- $p(s, X(t))$ and $\lambda(t, \mathcal{F}_{t-}^N)$ are analogous.

Some classical point processes in neuroscience

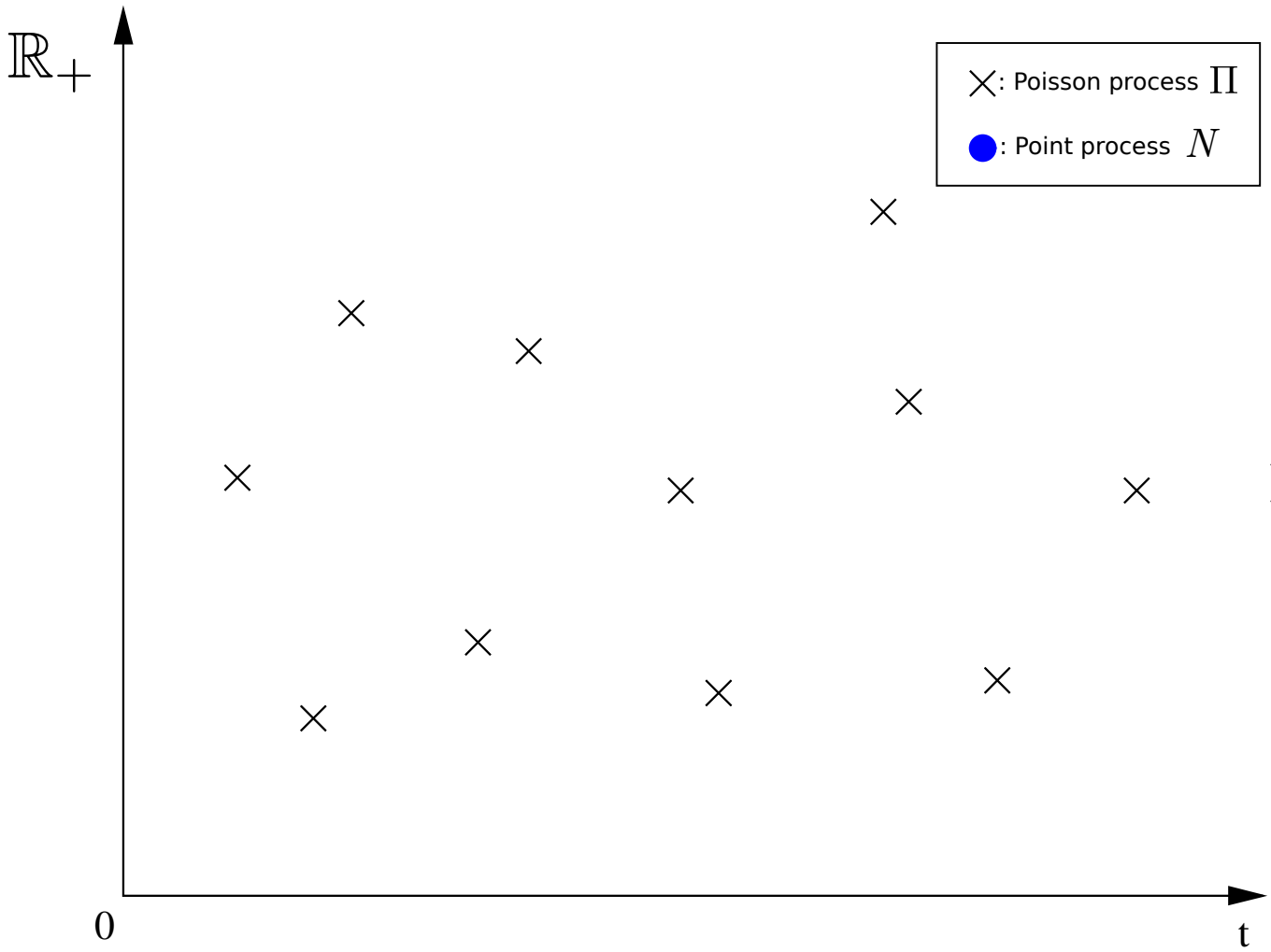
- Poisson process : $\lambda(t, \mathcal{F}_{t-}^N) = \lambda(t) =$ deterministic function.
- Renewal process : $\lambda(t, \mathcal{F}_{t-}^N) = f(S_{t-}) \Leftrightarrow$ i.i.d. ISIs.



- Hawkes process : $\lambda(t, \mathcal{F}_{t-}^N) = \mu + \int_{-\infty}^{t-} h(t-v)N(dv)$

$$\int_{-\infty}^{t-} h(t-v)N(dv) \quad \longleftrightarrow \quad \int_0^t d(v)m(t-v)dv = X(t).$$

Dynamic = Ogata's thinning



Π is a Poisson process with rate 1.

$\Pi(dt, dx) = \sum \delta_{\mathsf{X}}$.

$\mathbb{E}[\Pi(dt, dx)] = dtdx$.

λ is random.

N admits λ as an intensity.

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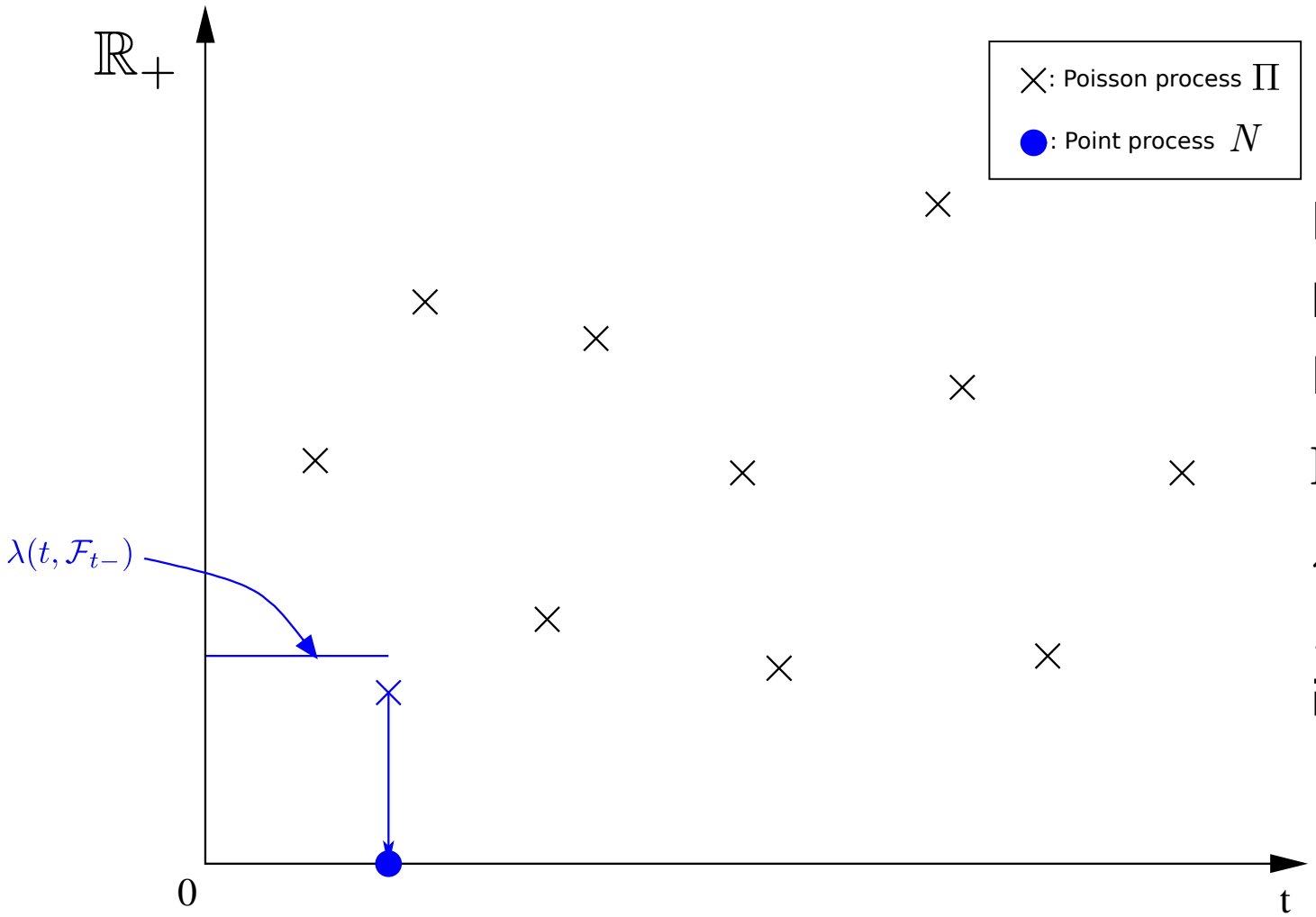

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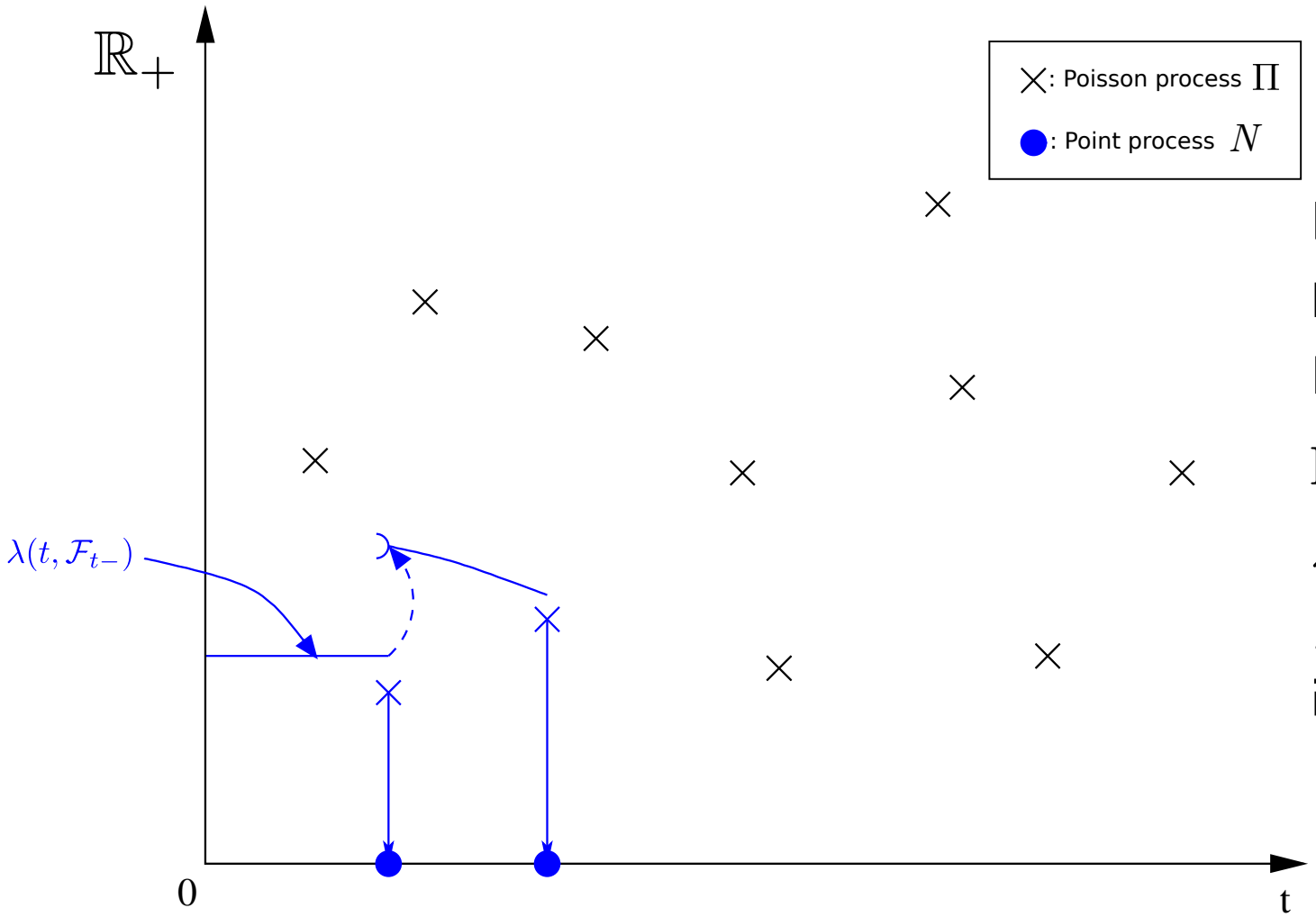

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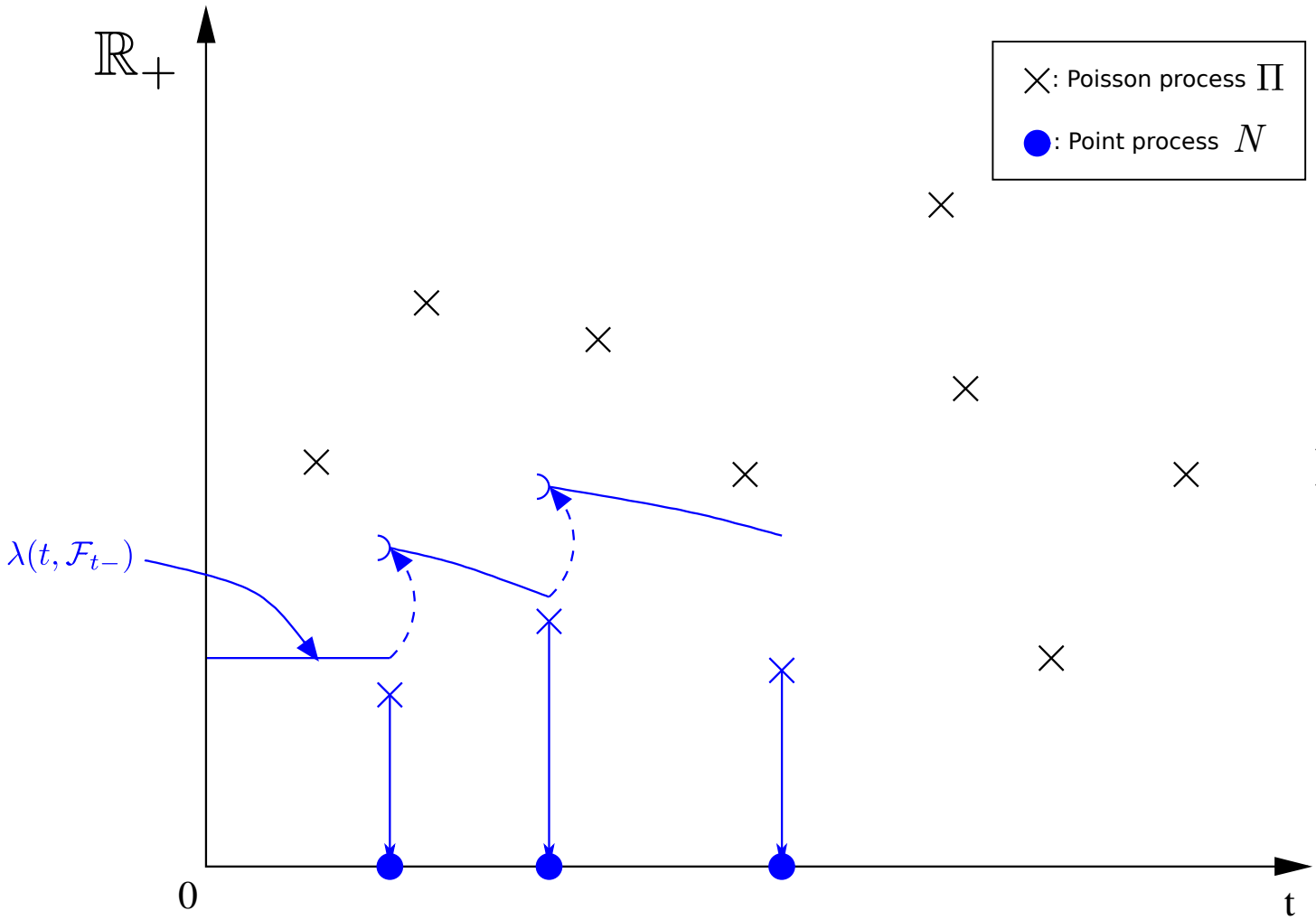

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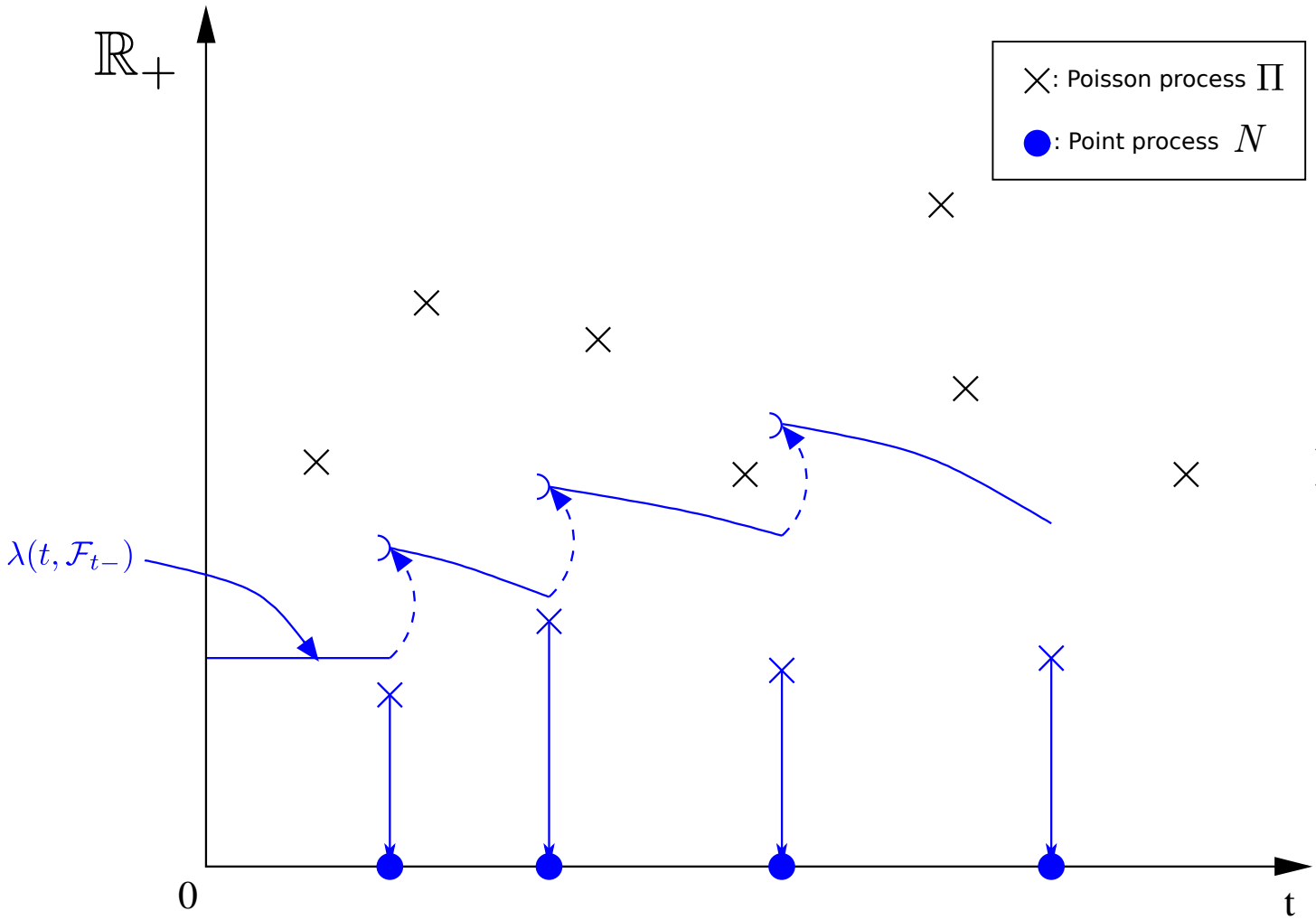

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- At fixed time t, we are looking at a Dirac mass at S_{t-}.

What we need

- Random measure U on $\mathbb{R}^2$.
- Action over test functions : $\forall \varphi \in C^{\infty}_{c,b}(\mathbb{R}^2_+)$,

$$\int \varphi(t,s)U(dt,ds) = \int \varphi(t,S_{t-})dt.$$

What we define

We construct an ad hoc random measure U which satisfies a system of stochastic differential equations similar to (PPS).

Microscopic equation (Chevallier, Cáceres, Doumic, RB)

Let Π be a Poisson measure. Let $\left(\lambda(t, \mathcal{F}_{t-}^N)\right)_{t>0}$ be some non negative predictable process which is L^1_{loc} a.s.
The measure U satisfies the following system a.s.

$$\begin{cases} (\partial_t + \partial_s)\{U(dt, ds)\} + \left(\int_{x=0}^{\lambda\left(t, \mathcal{F}_{t-}^N\right)} \Pi(dt, dx) \right) U(t, ds) = 0, \\ U(dt, 0) = \int_{s \in \mathbb{R}} \left(\int_{x=0}^{\lambda\left(t, \mathcal{F}_{t-}^N\right)} \Pi(dt, dx) \right) U(t, ds), \end{cases}$$

in the weak sense with initial condition $\lim_{t \to 0^+} U(t, \cdot) = \delta_{-T_0}$. ($-T_0$ is the age at time 0)

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$\rightsquigarrow$ $p(s, X(t))$ is replaced by $\int_{x=0}^{\lambda\left(t,\mathcal{F}_{t-}^N\right)} \Pi(dt,dx)$.

$$\mathbb{E}\left[\int_{x=0}^{\lambda\left(t,\mathcal{F}_{t-}^N\right)} \Pi(dt,dx) \middle| \mathcal{F}_{t-}^N\right] = \lambda\left(t,\mathcal{F}_{t-}^N\right) dt.$$

Taking the expectation

[...] defining $u = "\mathbb{E}(U)"$, $u(t,.)$ distribution of S_{t-}.
Let $\left(\lambda(t, \mathcal{F}_{t-}^N)\right)_{t>0}$ be some non negative predictable process which is L^1_{loc} a.s.
The measure U satisfies the following system,

$$\begin{cases} (\partial_t + \partial_s)\{U(dt,ds)\} + \left(\int_{x=0}^{\lambda\left(t,\mathcal{F}_{t-}^N\right)} \Pi(dt,dx)\right) U(t,ds) = 0, \\ U(dt,0) = \int_{s\in\mathbb{R}} \left(\int_{x=0}^{\lambda\left(t,\mathcal{F}_{t-}^N\right)} \Pi(dt,dx)\right) U(t,ds), \end{cases}$$

in the weak sense with initial condition $\lim_{t\to 0^+} U(t,\cdot) = \delta_{-T_0}$.

Taking the expectation

[...] defining $u = "\mathbb{E}(U)"$, $u(t,.)$ distribution of S_{t-}.
Let $\left(\lambda(t, \mathcal{F}_{t-}^N)\right)_{t>0}$ be some non negative predictable process which is L^1_{loc} in expectation, and which admits a finite mean.
The measure $u = "\mathbb{E}(U)"$ satisfies the following system,

$$\begin{cases} (\partial_t + \partial_s) u(dt, ds) + \rho_{\lambda, \mathbb{P}_0}(t,s) u(dt, ds) = 0, \\ u(dt, 0) = \displaystyle\int_{s \in \mathbb{R}} \rho_{\lambda, \mathbb{P}_0}(t,s) u(t, ds)\, dt, \end{cases}$$

in the weak sense where $\rho_{\lambda, \mathbb{P}_0}(t,s) = \mathbb{E}\left[\lambda\left(t, \mathcal{F}_{t-}^N\right) \middle| S_{t-} = s\right]$ for almost every t. The initial condition $\lim_{t \to 0^+} u(t, \cdot)$ is given by the distribution of $-T_0$.

Law of large numbers

- Law of large numbers.
- Population-based approach.

Theorem

Let $(N^i)_{i\geq 1}$ be some i.i.d. point processes on $\mathbb{R}$ with L^1_{loc} intensity in expectation. For each i, let $\left(S^i_{t-}\right)_{t>0}$ denote the age process associated to N^i. Then, for every test function φ,

$$\int \varphi(t,s)\left(\frac{1}{n}\sum_{i=1}^{n}\delta_{S^i_{t-}}(ds)\right)dt \xrightarrow[n\to\infty]{a.s.} \int \varphi(t,s)u(dt,ds),$$

with u satisfying the deterministic system.

Review of the examples

The system in expectation

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where $\rho_{\lambda, \mathbb{P}_0}(t, s) = \mathbb{E}\left[\lambda\left(t, \mathcal{F}_{t-}^N\right) \middle| S_{t-} = s\right]$.

- This result may seem OK to a probabilist,
- But analysts need some explicit expression for ρ.
- In particular, this system may seem linear, but it is non-linear in general.

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- Renewal process. $\rightarrow \rho_{\lambda, \mathbb{P}_0}(t, s) = f(s)$.

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- Renewal process. $\rightarrow \rho_{\lambda, \mathbb{P}_0}(t, s) = f(s)$.
- Hawkes process. $\rightarrow \rho_{\lambda, \mathbb{P}_0}$ is much more complex.

For Hawkes

Recall that

$$\int_{-\infty}^{t-} h(t-x)N(dx) \quad \longleftrightarrow \quad \int_0^t d(x)m(t-x)dx = X(t).$$

For Hawkes

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$$\int_{-\infty}^{t-} h(t-x)N(dx) \quad \longleftrightarrow \quad \int_0^t d(x)m(t-x)dx = X(t).$$

What we expected

Replacement of $p(s, X(t))$ by

$$\mathbb{E}\left[\lambda(t, \mathcal{F}_{t-}^N)\right] = \mu + \int_0^t h(t-x)\, u(dx, 0) \longleftrightarrow X(t)$$

What we find

$p(s, X(t))$ is replaced by $\rho_{\lambda, \mathbb{P}_0}(t, s)$ which is the conditional expectation, not the full expectation.

Conclusions : work in progress of J. Chevallier

- Univariate Hawkes processes do NOT lead to a PPS of the form $p(s, X_t) = \nu + X_t$, but to another completely explicit closed PDE system (NB : Non Markovian process)

Conclusions : work in progress of J. Chevallier

- Univariate Hawkes processes do NOT lead to a PPS of the form $p(s, X_t) = \nu + X_t$, but to another completely explicit closed PDE system (NB : Non Markovian process)
- Interacting Hawkes processes without refractory periods DO lead to this intuition (mean field)
- Even true for more realistic models with refractory periods and special $p(s, X_t)$.
- Propagation of chaos, CLT ...

Open questions

- But then when we pick some neurons and measure their activity via electrodes, why do we measure interactions ?
- What kind of models would allow both : independence with most of the neurons, sparse dependence with some (functional connectivity) ?
- What can we exactly infer in the reconstructed graphs ?
- Can it be couple to a more global measure of activity (LFP) to infer more information ?

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