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Improved convergence rate for the simulation of stochastic differential equations driven by subordinated Lévy processes

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Abstract

We consider the Euler approximation of stochastic differential equations (SDE's) driven by Lévy processes in the case where we cannot simulate the increments of the driving process exactly. In some cases where the driving process Y is a *subordinated* stable process, i.e. $Y = Z(V)$ with V a subordinator and Z a stable process, we propose an approximation of Y by $Z(V^n)$ where V^n is an approximation of V. We then compute the rate of convergence for the approximation of the solution X of an SDE driven by Y using results about the stability of SDE's.

Key words: stochastic differential equation, numerical approximation, convergence rate, Lévy process, shot noise representation, subordination.

2000 Maths Subject Classification: Primary 60H10, 65C30; Secondary 60G51, 60F17

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1 Introduction

Recently various Lévy processes have been used as the driving process in stock price modeling as an alternative to the Wiener process and simulation of Lévy processes has received new attention. Rydberg (1997) used a Normal inverse Gaussian Lévy process, which is of type G, as a model for financial data. A more extensive treatment of Normal inverse Gaussian processes can be found in Barndorff-Nielsen (1998) (see also Barndorff-Nielsen & Pérez-Abreu, 1999).

We consider the Euler approximation of a stochastic differential equation (SDE) driven by a Lévy process. This scheme is possible to perform only if the driving process can be simulated, this is the case for example for a Brownian motion or for a stable process in $\mathbb{R}$. The rate of convergence for the Euler scheme has been studied in Jacod & Protter (1998), Jacod (2002). If the driving process cannot be simulated exactly, one has to resort upon an approximation for it, usually using a large number of simulations for a good approximation. The error induced by this procedure adds up with the error due to the Euler scheme and the rate of convergence (expressed in term of the number of actually simulated random variables) becomes slower. This is indeed shown in Rubenthaler (2001), where the simulated approximation of the driving process is based upon a compound Poisson approximation neglecting the small jumps of the process. If we want a difference of order ϵ between the exact solution and the approximated solution, the amount of work needed in the scheme in Rubenthaler (2001) is of order $\epsilon^{-\sigma}$ for some σ which can be very large for unfavorable cases, but the method works for all Lévy processes.

We here try to improve the above result by finding a better approximation of the driving process under some extra assumptions on the probabilistic structure of the driving process. That is, we try to find a scheme whose complexity is of order $\epsilon^{-\sigma'}$ for an error of order ϵ with $\sigma' < \sigma$ as ϵ tends to zero.

2 Main results

2.1 Setting and notation

We consider the following stochastic differential equation (SDE) :

$$X_t = X_0 + \int_0^t f(X_{s-}) \, \mathrm{d}Y_s \ . \tag{2.1}$$

where X_0 is random variable in $\mathbb{R}^d$ ($d \geq 1$). X takes values in $\mathbb{R}^d$ and f denotes a C^1 function taking values in $\mathcal{M}_{d,q}$ ($q \geq 1$, where $\mathcal{M}_{d,q}$ is the set of real matrices with d rows and q columns) with f and f' bounded and where Y is given by :

$$Y_t = \sigma W_t + bt + Z(V_t) \text{ on } [0,1]$$

where σW_t is a q-dim Brownian motion with covariance $\sigma\sigma^T$ (σ^T being the transpose of the matrix σ), b is a $\mathbb{R}^q$-valued constant, V is a subordinator (taking values in $\mathbb{R}^+$)

which we cannot simulate exactly and Z is a stable process of index γ $(0 < \gamma \leq 2)$ taking values in $\mathbb{R}^q$. The fact that Y has this particular form will allow us to build an approximation of it in a special way. This special form has some connections with mathematical finance where changed time Brownian motion are used (see for example Geman, Madan & Yor (2002), Geman, Madan & Yor (2001)).

The process Y is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq 1})$ We can write $f = (f_1, ..., f_q)$ where $f_1, f_2, ...$ are the column components of f. We will by $|...|$ denote the norm $|(x_1, ..., x_N)| = |x_1| + ...|x_N|$ in any N-dimensional space on $\mathbb{R}$ (including the matrices). We recapitulate the general form of the characteristic function of a q-dimensional γ-stable random process Z, which can be found in Theorem 14.10 of Sato (1999). For any such non-trivial Z, there then exists a finite non-zero measure λ_1 on S^{q-1} the unit sphere of $\mathbb{R}^q$ and a vector τ in $\mathbb{R}^q$ and $B \in \mathcal{M}_{q,q}$ such that :

$$\mathrm{E}\, e^{i\langle\theta,Z_1\rangle} = \exp\left(-\int_{S^{q-1}} |\langle\theta,s\rangle|^\gamma \left(1 - i\tan\left(\frac{\pi\gamma}{2}\right)\mathrm{sign}(\langle\theta,s\rangle)\right)\lambda_1(\mathrm{d}s) + i\langle\theta,\tau\rangle\right) \quad \text{for } \gamma \notin \{1,2\} \qquad (2.2)$$

$$\mathrm{E}\, e^{i\langle\theta,Z_1\rangle} = \exp\left(-\int_{S^{q-1}} \left(|\langle\theta,s\rangle| + i\frac{2}{\pi}\langle\theta,s\rangle\log(|\langle\theta,s\rangle|)\right)\lambda_1(\mathrm{d}s) + i\langle\theta,\tau\rangle\right) \quad \text{for } \gamma = 1, \qquad (2.3)$$

$$\mathrm{E}\, e^{i\langle\theta,Z\rangle} = \exp(-\langle\theta,B\theta\rangle + i\langle\theta,\tau\rangle) \quad \text{for } \gamma = 2 \qquad (2.4)$$

The Euler approximation of X is the discretized process $(X^n_{\frac{[nt]}{n}})_{0\leq t\leq 1}$ where X^n is defined by :

$$X^n_t = X_0 + \int_0^t f\left(X^n_{\frac{[ns]}{n}-}\right) dY_s \ .$$

When we say that $X^n_{\frac{[n.]}{n}}$ is an *approximation* of X, we mean that $X^n_{\frac{[n.]}{n}} \xrightarrow[n\to+\infty]{\text{law}} X$ (this is here the convergence in law of a sequence of processes for the Skorohod topology and it will be proved in Lemma 4.1). We also say that $X^n_{\frac{[n.]}{n}}$ *approximates* X. As we have said, if the driving process Y cannot be simulated exactly we cannot use the traditional Euler scheme, so our aim is to define an Euler scheme based on some approximation of the increments of Y. This is comparable to the case where want to use higher order strong methods for SDE's driven by Brownian motion and do not have any exacts methods of simulating the iterated Itô integral we need for the scheme (See e.g (Kloeden & Platen, 1995, Ch. 10.6) and Wiktorsson (2001a)).

We define $\overline{X}^n$ to be the solution of :

$$\overline{X}^n_t = X_0 + \int_0^t f\left(\overline{X}^n_{\frac{[ns]}{n}-}\right) \mathrm{d}\overline{Y}^n_s \qquad (2.5)$$

where $\overline{Y}^n$ is an approximation of Y. This approximation is such that :

$$\overline{Y}_t^n = \sum_{k=0}^{[nt]} \delta \overline{Y}_k^n$$

where the $\delta\overline{Y}_k^n$'s are i.i.d. and approximate the increments $\delta Y_k^n = Y_{\frac{k+1}{n}} - Y_{\frac{k}{n}}$, $0 \leq k \leq n-1$. Since we know how to simulate Z but in general not how to simulate V, the approximation of δY_k will be obtained by replacing V by an approximation $\overline{V}^n$, that is : $\delta\overline{Y}_k^n \overset{\text{law}}{=} \sigma(W_{\frac{k+1}{n}} - W_{\frac{k}{n}}) + \frac{1}{n}b + Z(\overline{V}^n_{\frac{k+1}{n}}) - Z(\overline{V}^n_{\frac{k}{n}})$ will approximate $\delta Y_k = \sigma(W_{\frac{k+1}{n}} - W_{\frac{k}{n}}) + \frac{1}{n}b + Z(V_{\frac{k}{n}}) - Z(V_{\frac{k}{n}})$. The construction of $\overline{Y}^n$ will be given in the next subsection.

We notice that this setting implies that $\overline{X}_t^n = \overline{X}^n_{\frac{[nt]}{n}}$ for all n and for all t. As we will see, Y and the $\overline{Y}^n$'s are defined on a common probability space, so we can look at the error at time t, which is :

$$\widehat{e}_t^n = \overline{X}^n_{\frac{[nt]}{n}} - X_{\frac{[nt]}{n}} = e_t^n + \overline{e}_t^n$$

where

$$e_t^n = X^n_{\frac{[nt]}{n}} - X_{\frac{[nt]}{n}} \ ,$$

$$\overline{e}_t^n = \overline{X}^n_{\frac{[nt]}{n}} - X^n_{\frac{[nt]}{n}} \ .$$

We want to show that the sequences of processes (e^n) and $(\overline{e}^n)$ are going to zero (in law for the Skorohod topology) and we want to find their *rates of convergence*, that is sequences (u_n) and $(\overline{u}_n)$ such that $(u_n e^n)$ is tight (for the Skorohod topology) with some non null limit point and the same for $(\overline{u}_n\overline{e}^n)$. This will be sufficient to obtain that $((u_n \wedge \overline{u}_n)(\widehat{e}^n))_t^*$ is tight for all t (where for any process K, $K_t^* = \sup_{0\leq s\leq t}|K_t|$), but not quite enough for $u_n \wedge \overline{u}_n$ to be the rate for $(\widehat{e}^n)$. These are the main results and they are stated in Theorems 2.2 and 2.4.

2.2 The approximation of the increments : series representations and approximation of subordinators

We propose a new approximation of δY_k^n based on the series representations of subordinators.

Using the ideas of Bondesson (1982) (see also Rosiński, 2001, for a general discussion on series representations of Lévy processes) and adapting them to Lévy processes, we can represent the jump part of V as :

$$\sum_{0\leq s\leq t} \Delta V_s = \sum_{k=1}^{\infty} g(T_k)\mathbf{1}_{U_k \leq t}, \ 0 \leq t \leq 1$$

and so

$$V_t = D_V t + \sum_{k=1}^{\infty} g(T_k) 1_{U_k \le t},\ 0 \le t \le 1\ , \tag{2.6}$$

where D_V is the drift of V and $\{T_k, k \ge 1\}$ are the points of an homogeneous Poisson process index by $t \in \mathbb{R}^+$ and $\{U_k, k \ge 1\}$ is a sequence of i.i.d r.v. uniformly distributed on $(0,1)$ and g is the right-continuous inverse function of the tail-measure $F_V(x,\infty)$ defined as :

$$g(u) = \inf\{t > 0 : F_V(t,\infty) < u\} \tag{2.7}$$

where F_V is the Lévy measure of V_1

The simple approximation of the increments of the subordinator can now be expressed in term of the series representation for V. We have that $V_{\frac{k}{n}} - V_{\frac{k-1}{n}}$ is approximated as $V^n_{\frac{k}{n}} - V^n_{\frac{k-1}{n}}$ where

$$V^n_t = D_V t + \sum_{k:T_k \le a_n} g(T_k) 1_{U_k \le t},\ 0 \le t \le 1, \tag{2.8}$$

and a_n is a sequence of truncation levels going to $+\infty$ chosen later. One interpretation of a_n is that on average we have to sum a_n terms to obtain this approximation of V. V^n contains the jumps of V which are bigger than $g(a_n)$ plus the drift. The error $\widetilde{V}^n_t = V_t - V^n_t$ is just the remaining terms in the series i.e.

$$\widetilde{V}^n_t = \sum_{k:T_k > a_n} g(T_k) 1_{U_k \le t},\ 0 \le t \le 1 \tag{2.9}$$

and so contains the jumps of V smaller than $g(a_n)$. We set $g_n = g(a_n)$. Approximating V by V^n is natural and this is what is done in Rubenthaler (2001) but it seems that

$$\overline{V}^n_t = V^n_t + \mathrm{E}\,\widetilde{V}^n_t,\ 0 \le t \le 1$$

would in most cases be a far better approximation of V.

So we want to approximate V by something which has the law of $\overline{V}$ and so, we want to have some i.i.d. r.v. $(\delta\overline{Y}^n_k)_{1 \le k \le n}$ for each n, which have the law of $Z(V^n_{\frac{1}{n}} + \mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}})$. It is easy to simulate independent variables having the law of $Z(V^n_{\frac{1}{n}} + \mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}})$ and so, from a practical point of view, we know enough about the $\delta\overline{Y}^n_k$'s to simulate them. But as we want to study the difference $\overline{X}_{\frac{[n.]}{n}} - X_{\frac{[n.]}{n}}$, we have to define the $\delta\overline{Y}^n_k$'s on the same probability space as Y. That is, we have to find a coupling between the $\delta\overline{Y}^n_k$'s and Y. This is the subject of the following.

For the approximation of the increments

$$Z\left(V_{\frac{k}{n}}\right) - Z\left(V_{\frac{k-1}{n}}\right)$$

it is at a first glance tempting to use the approximation

$$Z\left(\overline{V}^n_{\frac{k}{n}}\right) - Z\left(\overline{V}^n_{\frac{k-1}{n}}\right) .$$

The problem is that the two processes obtained from these increments (the true and the approximated) do not have simultaneous independent increments with respect to any common filtration. This would cause considerably technical difficulties when analyzing the error-process for the approximative Euler scheme. In order to avoid this technical difficulties we instead propose another approximation which has the same law but which retains the simultaneous independent increment property. We do not believe that this new approximation is the best possible that retains the simultaneous independent increment property, but we have not been able to find any better approximation which covers the general case.

For the general case where the driving process also may have a Brownian part and a drift we propose the approximation :

$$\begin{aligned} \delta\overline{Y}^n_k &= Z\left(V_{\frac{k-1}{n}} + V^n_{\frac{k}{n}} - V^n_{\frac{k-1}{n}} + \left(\mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}}\right) \wedge \left(\widetilde{V}^n_{\frac{k}{n}} - \widetilde{V}^n_{\frac{k-1}{n}}\right)\right) - Z\left(V_{\frac{k-1}{n}}\right) \\ &\quad + Z^k\left(\left[\mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}} - \left(\widetilde{V}^n_{\frac{k}{n}} - \widetilde{V}^n_{\frac{k-1}{n}}\right)\right]^+\right) + \sigma\left(W_{\frac{k}{n}} - W_{\frac{k-1}{n}}\right) + \frac{1}{n}b \qquad (2.10) \end{aligned}$$

where $\{Z^k(\cdot)\}_{k=1}^n$ are independent processes, with the same distribution as the original process Z, constructed on an extension of the original probability space. In this setting, we have the following easy lemma (based on the fact that all the processes used here have stationary and independent increments).

Lemma 2.1 *The $(\delta\overline{Y}^n_k)_{1\le k\le n}$ are i.i.d. for each n and they have the law of $Z(V^n_{\frac{1}{n}} + \mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}}) + \sigma\left(W_{\frac{k}{n}} - W_{\frac{k-1}{n}}\right) + \frac{1}{n}b$.*

2.3 Main theorems

We use here the notations of Jacod (2002) to separate some cases. We set F_Y to be the Lévy measure of Y. We set for all $\beta > 0$:

$$\theta_V(\beta) = F_V([\beta, +\infty)) ,$$

$$\theta_Y(\beta) = F_Y(\{x : |x| \ge \beta\}) ,$$

and if $q = 1$:

$$\theta^+_Y(\beta) = F_Y([\beta, +\infty)) ,$$

$$\theta^-_Y(\beta) = F_Y((-\infty, -\beta]) .$$

We introduce the following assumptions :

Assumption (H0) : The process Y has a non-null Brownian part.

Assumption (H1-α') : The process Y has no Brownian part and we have $\theta_Y(\beta) \leq \frac{C}{\beta^{\alpha'}}$ for all $\beta \in (0,1]$ (for some constant C). (We notice that we always have $\theta_y(\beta) \leq \frac{C}{\beta^2}$ and so $\alpha' \leq 2$.)

Assumption (H2-α') : We have $q = 1$. The process Y has no Brownian part and we have $\beta^{\alpha'}\theta_Y^+(\beta) \to \theta_Y^+$ and $\beta^{\alpha'}\theta_Y^-(\beta) \to \theta_Y^-$ as $\beta \to 0$, and further $\theta_Y = \theta_Y^+ + \theta_Y^- > 0$. We also set $\theta'_Y = \theta_Y^+ - \theta_Y^-$, and we observe that $\theta_Y(\beta) \sim \frac{\theta_Y}{\beta^{\alpha'}}$ when $\beta \to 0$.

Assumption (H3) : The measure F_Y is symmetrical about 0.

Assumption (H4) : The process Y has no drift part.

For convenience, we restate the definition of the sequence (u_n) given in Jacod (2002) :

- **Case 0 :** We have **(H0)**, then $u_n = \sqrt{n}$.
- **Case 1 :** We have **(H1-α')** for some $\alpha' > 1$, then $u_n = \left(\frac{n}{\log n}\right)^{1/\alpha'}$.
- **Case 2-a :** We have **(H1-α')** for $\alpha' = 1$, then $u_n = \frac{n}{(\log n)^2}$.
- **Case 2-b :** We have **(H1-α')** for $\alpha' = 1$ and **(H3)**, then $u_n = \frac{n}{\log n}$.
- **Case 3-a :** We have **(H1-α')** for some $\alpha' < 1$, then $u_n = n$.
- **Case 3-b :** We have **(H1-α')** for some $\alpha' < 1$ and **(H3)** and **(H4)**, then $u_n = \left(\frac{n}{\log n}\right)^{1/\alpha'}$

We introduce the following assumption on Z and V :

Assumption (Z0) : One of the following conditions holds :

- $\gamma > 1$
- $\gamma = 1$ and the measure λ_1 is symmetrical about 0
- $\gamma < 1$ and Z has no drift (i.e. $\tau = 0$)

Assumption (V0) : The Lévy measure of V satisfies : $\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \int_0^\epsilon x^2 F_V(dx)$.

We set $\mu_n = \mathrm{E}\,|\widetilde{V}^n_{\frac{1}{n}} - \mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}}|$. We recall that Z is stable with index γ. We define the sequences (v_n), (w_n) by :

$$v_n = \frac{1}{(n\mu_n)^{\frac{1}{\gamma}}}\ ,$$

$$\begin{cases} w_n \log(w_n) = \frac{1}{\sqrt{n\,\mathrm{E}\left(\left(\widetilde{V}^n_{\frac{1}{n}} - \mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}}\right)^2\right)}} & \text{if } \gamma = 1 \\ w_n = \frac{1}{\sqrt{n\,\mathrm{E}\left(\left(\widetilde{V}^n_{\frac{1}{n}} - \mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}}\right)^2\right)}} & \text{if } \gamma < 1 \end{cases}$$

(we do not use w_n then $\gamma > 1$). And we define the sequence $(\overline{u}_n)$ by :

$$\overline{u}_n = \begin{cases} v_n & \text{if } \mathbf{(Z0)} \text{ holds} \\ v_n \wedge w_n & \text{if } \mathbf{(Z0)} \text{ does not hold} \end{cases}$$

We have

$$n\mu_n = n\,\mathrm{E}\,|\widetilde{V}^n_{\frac{1}{n}} - \mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}}| \leq 2n\,\mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}} = 2\int_0^{g_n} xF_V(dx) \underset{n\to+\infty}{\longrightarrow} 0$$

and

$$n\,\mathrm{E}((\widetilde{V}^n_{\frac{1}{n}} - \mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}})^2) = \int_0^{g_n} x^2 F_V(dx) \underset{n\to+\infty}{\longrightarrow} 0$$

and so $\overline{u}_n \underset{n\to+\infty}{\longrightarrow} +\infty$. We set $\widehat{u}_n = u_n \wedge \overline{u}_n$. In the following theorem, we speak of the UT property, the reader can consult Subsection 5.1 for a more thorough description of this property. We recall that for any process K : $K^*_t = \sup_{s\in[0,t]} |K_s|$.

Theorem 2.2 *If either* **(H0)** *or* **(H1-α')** *holds and if* **(V0)** *holds and if the sequence (a_n) is chosen such that :*

$$\frac{1}{g(a_n)^2}\int_0^{g(a_n)} x^2 F_V(dx) \geq n, \text{ for all } n \tag{2.11}$$

then the sequence $\widehat{u}_n\widehat{e}^n$ is UT with respect to the the filtrations $(\mathcal{F}^n_t) = (\mathcal{F}^n_{\frac{[nt]}{n}})$ and in particular, we have that the sequence of real random variables $((\widehat{u}_n\widehat{e}^n)^*_t)$ is tight for all t.

Remark 2.3 *For given Z and V, it is not straightforward to check whether Y satisfies assumption* **(H1-α')**. *That is why we have added some technical Lemmas in Section 5 to show one can move back and forth between the assumptions on V and on Y.*

We are not able to prove that $(\widehat{u}_n\widehat{e}^n)$ is tight as a process under the assumption of Theorem 2.2 and so we cannot show that $\widehat{u}_n$ is the rate of convergence of $(\widehat{e}^n)$.

However, we have $\widehat{e}^n = e^n + \overline{e}^n$ and we show in the following that $(u_n e^n)$ and $(\overline{u}_n \overline{e}^n)$ are tight.

If **(H0)** holds then $(u_n e^n)$ is tight because it has a limit (described in Jacod & Protter (1998)) and this limit is non-degenerate. If Y has no Brownian part, we can apply Jacod (2002) to say that : $(u_n e^n)$ is tight under **(H1-α')** and $(u_n e^n)$ has a non-degenerate limit under **(H2-α')** (described in Jacod (2002)). The sequence $(\overline{u}_n \overline{e}^n)$ is tight because of the following theorem.

We define $\widehat{Z}$ as a stable process of index γ such that $\mathrm{E}(e^{i\langle\theta,\widehat{Z}(1)\rangle}) = e^{-\int_{S^{q-1}} |\langle\theta,s\rangle|^\gamma \lambda_1(ds)}$. For $s \in \mathbb{R}^q$, we set s_i to be the i:th coordinate of s. We define $B^{(\gamma)}$ as a Brownian motion with covariance matrix M_γ defined by :

$$M_\gamma \begin{cases} = 2\tau\tau^T & \text{if } \gamma \neq 1 \\ = \left(\frac{8}{\pi^2}\int_{S^{q-1}} s_i\lambda_1(ds) \times \int_{S^{q-1}} s_i\lambda_1(ds)\right)_{1\leq i,j\leq q} & \text{if } \gamma = 1 \end{cases}$$

where τ^T is the transposed vector of the column vector τ.

Theorem 2.4 *If* **(V0)** *holds and if the sequence (a_n) is chosen such that Equation (2.11) holds then the sequence $(\overline{u}_n\overline{e}^n)$ is tight. If in addition,* **(Z0)** *holds or if* **(Z0)** *does not hold and $\frac{v_n}{w_n} \underset{n\to+\infty}{\longrightarrow} l \in [0,+\infty]$ then $\overline{u}_n\overline{e}^n \underset{n\to+\infty}{\overset{law}{\longrightarrow}} \overline{U}$ where $\overline{U}$ is defined on an extension of the probability space as the unique solution of :*

$$\overline{U}_t = \sum_{i=1}^q \int_0^t \nabla f_i(X_{s-})U_{s-}\ \mathrm{d}Y_s^{(i)} + \int_0^t f(X_{s-})\ \mathrm{d}\widetilde{Z}_s \tag{2.12}$$

where $Y_s^{(i)}$ denotes the i:th coordinate of Y_s, and

$$\widetilde{Z} = \begin{cases} \widehat{Z} & \text{if } \mathbf{(Z0)} \text{ holds} \\ \frac{1}{l\vee 1}\widehat{Z} + (l\wedge 1)B^{(\gamma)} & \text{if } \mathbf{(Z0)} \text{ does not hold and } \frac{v_n}{w_n} \underset{n\to+\infty}{\longrightarrow} l \in [0,+\infty]\ , \end{cases}$$

and where $\widehat{Z}$ and $B^{(\gamma)}$ are independent of each other and of Y.

Remark 2.5 *If* **(Z0)** *does not hold then we can describe the limit of $(\overline{u}_n\overline{e}^n)$ only if $\frac{v_n}{w_n}$ has a limit. We recall that the sequences (v_n) and (w_n) are determined by the choice of (a_n).*

Let us now explain how we will proceed to prove Theorem 2.2 and Theorem 2.4. We have for all (x, y):

$$f(x) - f(y) \quad = \quad \int_0^1 \nabla f(tx + (1-t)y)(x-y)\ \mathrm{d}t.$$

We set : $\phi(x,y) = \int_0^1 \nabla f(tx + (1-t)y)\ \mathrm{d}t$. We have that ϕ is a bounded continuous function. If we write f with its column functions as $f(x) = [f_1(x)\cdots f_q(x)]$ we can then identify the result of the linear operator $f'(u)$ applied on a column vector v as the $d \times q$-matrix where the i:th column is given by $\nabla f_i(u)v$. And so we have $\phi(u,u)v = f'(u)v = \sum_{i=1}^q \nabla f_i(u)v$.

We have :

$$\begin{aligned}
\overline{e}_t^n &= \int_0^t f(\overline{X}^n_{\frac{[ns]}{n}-})\,\mathrm{d}\overline{Y}^n_{\frac{[ns]}{n}} - \int_0^t f(X^n_{\frac{[ns]}{n}-})\,\mathrm{d}Y_{\frac{[ns]}{n}} \\
&= \int_0^t \phi(\overline{X}^n_{\frac{[ns]}{n}-}, X^n_{\frac{[ns]}{n}-})(\overline{X}^n_{\frac{[ns]}{n}-} - X^n_{\frac{[ns]}{n}-})\,\mathrm{d}\overline{Y}^n_{\frac{[ns]}{n}} + \int_0^t f(X^n_{\frac{[ns]}{n}-})\,\mathrm{d}\Big(\overline{Y}^n_{\frac{[ns]}{n}} - Y_{\frac{[ns]}{n}}\Big) \\
&= \int_0^t \phi(\overline{X}^n_{\frac{[ns]}{n}-}, X^n_{\frac{[ns]}{n}-})\overline{e}^n_{\frac{[ns]}{n}-}\,\mathrm{d}\overline{Y}^n_{\frac{[ns]}{n}} + \int_0^t f(X^n_{\frac{[ns]}{n}-})\,\mathrm{d}\Big(\overline{Y}^n_{\frac{[ns]}{n}} - Y_{\frac{[ns]}{n}}\Big)\ .
\end{aligned}$$

In view of this equation, some results on the stability of the solutions of SDE's will allow us to say that the rate of convergence of $(\overline{e}^n)$ is equal to the rate of convergence of $\overline{Y}^n_{\frac{[n.]}{n}} - Y_{\frac{[n.]}{n}}$ (the results about the stability of SDE's can be found in Subsection 5.1). The rate of convergence of (e^n) can be found in Jacod (2002) and Jacod & Protter (1998). Then we can use these results to prove Theorem 2.2 and Theorem 2.4. So we devote the next section to the study of $\widetilde{Y}^n = \overline{Y}^n_{\frac{[n.]}{n}} - Y_{\frac{[n.]}{n}}$.

3 Convergence rate for the process $\widetilde{Y}^n$

Recall that

$$\begin{aligned}
\widetilde{Y}^n_t &= \sum_{i=1}^{[nt]} \delta Y^n_k - \delta\overline{Y}^n_k \\
&= \sum_{i=1}^{[nt]} Z\left(V_{\frac{k}{n}}\right) - Z\left(V_{\frac{k-1}{n}}\right) - Z\left(V_{\frac{k-1}{n}} + V^n_{\frac{k}{n}} - V^n_{\frac{k-1}{n}} + \left(\mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}}\right) \wedge \left(\widetilde{V}^n_{\frac{k}{n}} - \widetilde{V}^n_{\frac{k-1}{n}}\right)\right) \\
&\quad + Z\left(V_{\frac{k-1}{n}}\right) - Z^k\left(\left[\mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}} - \left(\widetilde{V}^n_{\frac{k}{n}} - \widetilde{V}^n_{\frac{k-1}{n}}\right)\right]^+\right) \\
&= \sum_{k=1}^{[nt]} Z\left(V_{\frac{k}{n}}\right) - Z\left(V_{\frac{k}{n}} - \widetilde{\delta}^+_{n,k}\right) - Z^k\left(\widetilde{\delta}^-_{n,k}\right) \qquad (3.1)
\end{aligned}$$

where $\widetilde{\delta}^+_{n,k} = \left[\widetilde{V}^n_{\frac{k}{n}} - \widetilde{V}^n_{\frac{k-1}{n}} - \mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}}\right]^+$ and $\widetilde{\delta}^-_{n,k} = \left[\widetilde{V}^n_{\frac{k}{n}} - \widetilde{V}^n_{\frac{k-1}{n}} - \mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}}\right]^-$ and the $Z^{(k)}$'s are independant copies of Z. We set $\mu_n = \mathrm{E}\,|\widetilde{V}^n_{\frac{1}{n}} - \mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}}|$. We denote the variance by the symbol Var. We recall that if **(Z0)** does not hold then $\overline{u}_n = v_n \wedge w_n$.

Proposition 3.1 *If* **(V0)** *holds and if the sequence* (a_n) *is chosen such that Equation (2.11) holds then* $(\overline{u}_n\widetilde{Y}^n)$ *is tight.*

- *If, in addition,* **(Z0)** *holds, then* $\overline{u}_n\widetilde{Y}^n \xrightarrow[n\to+\infty]{law} \widehat{Z}$ *where* $\widehat{Z}$ *is a symmetric stable process with index* γ *such that*

$$\mathrm{E}(\exp(i\langle\theta, \widehat{Z}(1)\rangle)) = \exp\left(-\int_{S^{q-1}} |\langle\theta, s\rangle|^\gamma \lambda_1(ds)\right) .$$

- *If, in addition,* **(Z0)** *does not hold and* $\frac{v_n}{w_n} \xrightarrow[n\to+\infty]{} l \in [0, +\infty]$ *then*

$$\overline{u}_n\widetilde{Y}^n \xrightarrow[n\to+\infty]{law} \frac{1}{l\vee 1}\widehat{Z} + (l\wedge 1)B^{(\gamma)}$$

where $B^{(\gamma)}$ *and* $\widehat{Z}$ *are independent.*

In order to prove the proposition we first need two lemmas. The first one from Marcus & Rosiński (2001) we state without proof.

Lemma 3.2 **(L^1 inequality)**
If A *is an infinitely divisible random variable with no Gaussian component,* $\mathrm{E}\,A = 0$ *and* $\mathrm{E}\,|A| < \infty$ *such that*

$$\mathrm{E}\exp(iA\theta) = \exp\int_{-\infty}^{\infty}(e^{ix\theta} - 1 - ix\theta)\,N(\mathrm{d}x),$$

then $0.25\rho \le \mathrm{E}\,|A| \le 2.125\rho$ *where* ρ *is the solution to the equation* $\xi(z) = 1$ *with* $\xi(z) = \int_{-\infty}^{\infty}\min(x^2/z^2, x/z)\,N(\mathrm{d}x)$.

Using Lemma 3.2 on a Lévy measure with bounded support we obtain the following lemma.

Lemma 3.3 *If* M *is the Lévy measure of a subordinator which satisfies :*

$$\lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon^2}\int_0^\varepsilon x^2 M(\mathrm{d}x) = +\infty$$

and if we choose a non-increasing truncation sequence (q_n) *going to* 0 *such that*

$$\frac{1}{q_n^2}\int_0^{q_n} x^2 M(dx) > n \text{ for all large enough } n$$

and if we choose a family of subordinators $Q^n,\ n\ge 1$ *with*

$$\mathrm{E}\exp(iQ_1^n\theta) = \exp\left(\int_0^{q_n}(\exp(i\theta x) - 1)M(\mathrm{d}x)\right)$$

we then have that for all n *large enough :*

$$1 \le \frac{\sqrt{\mathrm{Var}\,Q^n_{\frac{1}{n}}}}{\mathrm{E}\,|Q^n_{\frac{1}{n}}\,\mathrm{E}\,Q^n(1/n)|} \le 4\ .$$

PROOF. The lower bound straightforward. For the upper bound we start by noting that by Lemma 3.2, we have that for all n :

$$\frac{1}{4}\xi_n^{-1}(1) \leq \mathrm{E}\,|Q^n_{\frac{1}{n}} - \mathrm{E}\,Q^n_{\frac{1}{n}}| \ , \tag{3.2}$$

where

$$\xi_n(z) = \frac{1}{n}\int_0^{q_n} \min\left(\frac{x^2}{z^2}, \frac{x}{z}\right) M(dx) \ .$$

Let $z = \sqrt{\mathrm{Var}\, Q^n_{\frac{1}{n}}}$. By construction, we have :

$$z^2 = \frac{1}{n}\int_0^{q_n} x^2 M(dx) \geq q_n^2 \ .$$

So :

$$\xi_n(z) = \frac{1}{n}\int_0^{q_n} \frac{x^2}{z^2} M(\mathrm{d}x) = 1 \ .$$

So, by Equation (3.2) :

$$\frac{\sqrt{\mathrm{Var}\, Q_{\frac{1}{n}}}}{\mathrm{E}\,|Q^n_{\frac{1}{n}} - \mathrm{E}\,Q^n_{\frac{1}{n}}|} \leq 4 \ .$$

□

PROOF OF PROPOSITION 3.1. We set $\Psi_Z(\theta) = -\log \mathrm{E}\exp(i\langle\theta, Z_1\rangle)$. We introduce the following convenient notations :

$$D_n = \widetilde{V}^n_{\frac{1}{n}} - \mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}}$$

and

$$A_n = -\Re(\Psi_Z(\overline{u}_n\theta))|D_n| - i\Im(\Psi_Z(\overline{u}_n\theta))D_n \ ,$$

where $\Re$ means the real part and $\Im$ means the imaginary part. Due to our Lemma 2.1 and to Lemma 2.1 of Jacod (2002), the convergence of the sequence of $\mathbb{R}^q$ valued random variables $(\widetilde{Y}^n_1)$ implies the convergence of the sequence of processes $(\widetilde{Y}^n)$, and the tightness of $(\widetilde{Y}^n_1)$ implies the tightness of $(\widetilde{Y}^n)$. So we look at :

$$\begin{aligned}
\mathrm{E}\exp(i\overline{u}_n\langle\theta,\widetilde{Y}_1^n\rangle) &= (\mathrm{E}\exp(i\overline{u}_n\langle\theta,\delta\widetilde{Y}_1^n\rangle)^n\rangle \\
&= \mathrm{E}\left(\mathrm{E}(\exp(i\overline{u}_n\langle\theta,Z(\widetilde{\delta}_{n,1}^+)\rangle - i\overline{u}_n\langle\theta,Z^{(1)}(\widetilde{\delta}_{n,1}^-)\rangle)\middle|\widetilde{\delta}_{n,1}^+,\widetilde{\delta}_{n,1}^-)\right) \\
&= (\mathrm{E}\exp(-\Psi_Z(\overline{u}_n\theta)\widetilde{\delta}_1^+ - \Psi_Z(-\overline{u}_n\theta)\widetilde{\delta}_1^-))^n \\
&= \left(\mathrm{E}\exp\left\{-\Re\Psi_Z\left(\overline{u}_n\theta\right)\left|\widetilde{V}_{\frac{1}{n}}^n - \mathrm{E}\,\widetilde{V}_{\frac{1}{n}}^n\right|\right.\right. \\
&\qquad \left.\left. -i\Im\Psi_Z\left(\overline{u}_n\theta\right)\left(\widetilde{V}_{\frac{1}{n}}^n - \mathrm{E}\,\widetilde{V}_{\frac{1}{n}}^n\right)\right\}\right)^n \\
&= (\mathrm{E}\exp\left\{-\Re\Psi_Z\left(\overline{u}_n\theta\right)|D_n| - i\Im\Psi_Z\left(\overline{u}_n\theta\right)D_n\right\})^n \\
&= (\mathrm{E}(e^{A_n}))^n\ .
\end{aligned}$$

We set : $\varphi_n(\overline{u}_n\theta) = (\mathrm{E}\exp\left\{-\Re\Psi_Z\left(\overline{u}_n\theta\right)|D_n| - i\Im\Psi_Z\left(\overline{u}_n\theta\right)D_n\right\})$. To simplify notation we set

$$f_\gamma(\theta) = -\int_{S^d}|\langle\theta,s\rangle|^\gamma\lambda_1(\mathrm{d}s)\ .$$

We first suppose that **(Z0)** holds.
To show that

$$\lim_{n\to\infty}\varphi_n(\theta\overline{u}_n)^n = \exp(-\Re\Psi_Z(\theta)) = \exp(f_\gamma(\theta))\ ,$$

we proceed in two steps. We note that this convergence holds if the following two statements are true

$$\lim_{n\to\infty}(1-\Re\Psi_Z(\overline{u}_n\theta)\mu_n)^n = \exp(f_\gamma(\theta)), \tag{3.3}$$

where $\mu_n = \mathrm{E}\left|\widetilde{V}_{\frac{1}{n}}^n - \mathrm{E}\,\widetilde{V}_{\frac{1}{n}}^n\right|$, and

$$\lim_{n\to\infty}|\varphi_n(\theta\overline{u}_n)^n - (1-\Re\Psi_Z(\overline{u}_n\theta)\mu_n)^n| = 0\ . \tag{3.4}$$

To prove the statements we need two inequalities: First

$$|z^n - w^n| \le n|z-w| \text{ if } w,z\in\mathbb{C} \text{ with } |z|\le 1,\ |w|\le 1 \tag{3.5}$$

(this is a simple consequence of (Lemma 1 Billingsley , 1986, p. 369)), and secondly

$$|\exp(z)-1-z| \le |z|^2/2,\ \text{if } z\in\mathbb{C} \text{ with } \Re z\le 0 \tag{3.6}$$

As $\lim_{n\to+\infty} -n\mu_n \Re\Psi_Z(\overline{u}_n\theta) = f_\gamma(\theta)$, we have Equation (3.3).

For Equation (3.4), we proceed by using Equation (3.5) with $z = \varphi_n(\overline{u}_n\theta)$ and $w = (1 - \Re\Psi_Z(\theta\overline{u}_n)\mu_n)$ to obtain

$$|\varphi_n(\theta\overline{u}_n)^n - (1 - \Re\Psi_Z(\overline{u}_n\theta)\mu_n)^n| \leq n|\varphi_n(\theta\overline{u}_n) - (1 - \Re\Psi_Z(\theta\overline{u}_n)\mu_n)|.$$

Noting that

$$\begin{aligned} |\varphi_n(\theta\overline{u}_n) - (1 - \Re\Psi_Z(\theta\overline{u}_n)\mu_n)| &= |\mathrm{E}\left[\exp\left\{-\Re\Psi_Z\left(\overline{u}_n\theta\right)|D_n| - i\Im\Psi_Z\left(\overline{u}_n\theta\right)D_n\right\}\right. \\ &\qquad \left. - \left(1 - \Re\Psi_Z\left(\overline{u}_n\theta\right)|D_n| - i\Im\Psi_Z\left(\overline{u}_n\theta\right)D_n\right)\right]| \\ &\leq \mathrm{E}\left|\exp\left\{-\Re\Psi_Z\left(\overline{u}_n\theta\right)|D_n| - i\Im\Psi_Z\left(\overline{u}_n\theta\right)D_n\right\}\right. \\ &\qquad \left. - \left(1 - \Re\Psi_Z\left(\overline{u}_n\theta\right)|D_n| - i\Im\Psi_Z\left(\overline{u}_n\theta\right)D_n\right)\right| \end{aligned}$$

By using Equations (2.2), (2.3), (2.4), (3.6) and because **(Z0)** holds, we obtain that for some constant C :

$$\begin{aligned} |\varphi_n(\theta\overline{u}_n)^n - (1 - \Re\Psi_Z(\theta\overline{u}_n)\mu_n)^n| &\leq \frac{n}{2}\,\mathrm{E}\left(|\Re\Psi_Z(\overline{u}_n\theta)|D_n| + i\Im\Psi_Z(\overline{u}_n\theta)D_n|^2\right) \\ &= Cn\overline{u}_n^{2\gamma}\,\mathrm{E}(|D_n|^2) \end{aligned}$$

We now use Lemma 3.3 to see that

$$n\overline{u}_n^{2\gamma}\,\mathrm{E}(|D_n|^2) = \frac{1}{n\mu_n^2}\,\mathrm{Var}\,\widetilde{V}^n_{\frac{1}{n}} \leq \frac{4}{n}$$

which shows that

$$\lim_{n\to\infty} |\varphi_n(\theta\overline{u}_n)^n - (1 - \Re\Psi_Z(\theta\overline{u}_n)\mu_n)^n| = 0\ .$$

So we have shown that $\overline{u}_n\widetilde{Y}^n \xrightarrow[n\to+\infty]{\text{law}} \widehat{Z}$ if **(Z0)** holds.

We now suppose that **(Z0)** does not hold.
In the following, C is a constant which may change from line to line. We recall the following inequality :

$$\left|e^z - 1 - z - \frac{z^2}{2}\right| \leq \frac{|z|^3}{6}\ ,\ \text{if } z \in \mathbb{C} \text{ with } \Re z \leq 0\ .$$

Using this inequality, we have :

$$\begin{aligned}
\left|\varphi_n\left(\theta\overline{u}_n\right)^n-\left(\mathrm{E}\left(1-A_n-\frac{A_n^2}{2}\right)\right)^n\right| &\leq n\left|\mathrm{E}\left(e^{A_n}\right)-\mathrm{E}\left(1-A_n-\frac{A_n^2}{2}\right)\right| \\
&\leq n\,\mathrm{E}\left(\frac{|A_n|^3}{6}\right) \\
&= \frac{n}{6}|\Psi_Z(\overline{u}_n\theta)|^3\,\mathrm{E}(|D_n|^3) \\
&\leq Cn|\Psi_Z(\overline{u}_n\theta)|^3\sqrt{\mathrm{E}(D_n^2)}\sqrt{\mathrm{E}(D_n^4)}
\end{aligned}$$

In the same way as above, if we show that :

- $n\,\mathrm{E}(A_n) \underset{n\to+\infty}{\sim} -\left(\frac{\overline{u}_n}{v_n}\right)^\gamma \Re(\Psi_Z(\theta))$
- $n\,\mathrm{E}(A_n^2) \begin{cases} \underset{n\to+\infty}{\sim} -\left(\frac{\overline{u}_n}{w_n}\right)^2 (\langle\theta,\tau\rangle)^2 & \text{if } \gamma<1 \\ \underset{n\to+\infty}{\sim} -\left(\frac{\overline{u}_n\log\overline{u}_n}{w_n\log w_n}\right)^2 \left(\frac{2}{\pi}\right)^2 \int_{S^{q-1}}|\langle\theta,s\rangle|^2\lambda_1(ds) & \text{if } \gamma=1 \end{cases}$
- $n|\Psi_Z(\overline{u}_n\theta)|^3\sqrt{\mathrm{E}(D_n^2)}\sqrt{\mathrm{E}(D_n^4)} \underset{n\to+\infty}{\longrightarrow} 0,$

this will finish the proof because : $\frac{\overline{u}_n}{v_n} \underset{n\to+\infty}{\longrightarrow} \frac{1}{l\vee 1}$, $\frac{\overline{u}_n}{w_n} \underset{n\to+\infty}{\longrightarrow} l\wedge 1$ and $\frac{\overline{u}_n\log\overline{u}_n}{w_n\log w_n} \underset{n\to+\infty}{\longrightarrow} l\wedge 1$.

We have for all n :

$$\begin{aligned}
n\,\mathrm{E}(A_n) &= -n\Re(\Psi_Z(\overline{u}_n\theta))\,\mathrm{E}(|D_n|) \\
&= -\overline{u}_n^\gamma n\Re(\Psi_Z(\theta))\,\mathrm{E}(|D_n|) \\
&= -\left(\frac{\overline{u}_n}{v_n}\right)^\gamma \Re(\Psi_Z(\theta))
\end{aligned}$$

and if $\gamma<1$:

$$\begin{aligned}
n\,\mathrm{E}(A_n^2) &\underset{n\to+\infty}{\sim} n(\Im\Psi_Z(\overline{u}_n\theta))^2\,\mathrm{E}(D_n^2) \\
&\underset{n\to+\infty}{\sim} -n\overline{u}_n^2(\langle\theta,\tau\rangle)^2\,\mathrm{E}(D_n^2) \\
&= -\left(\frac{\overline{u}_n}{w_n}\right)^2(\langle\theta,\tau\rangle)^2
\end{aligned}$$

and if $\gamma=1$:

$$\begin{aligned}
n\,\mathrm{E}(A_n^2) &\underset{n\to+\infty}{\sim} n(\Im\Psi_Z(\overline{u}_n\theta))^2\,\mathrm{E}(D_n^2) \\
&\underset{n\to+\infty}{\sim} -n\left(\frac{2}{\pi}\right)^2\left(\int_{S^{q-1}}(\langle\overline{u}_n\theta,s\rangle\log\langle\overline{u}_n\theta,s\rangle)\lambda_1(ds)\right)^2\mathrm{E}(D_n^2) \\
&\underset{n\to+\infty}{\sim} -\left(\frac{\overline{u}_n\log\overline{u}_n}{w_n\log w_n}\right)^2\left(\frac{2}{\pi}\right)^2\left(\int_{S^{q-1}}(\langle\theta,s\rangle)\lambda_1(ds)\right)^2.
\end{aligned}$$

By using that for $g(a_n) < 1$:

$$\mathrm{E}\left(\widetilde{V}^n_{\frac{1}{n}} - \mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}}\right)^4 = \frac{1}{n}\int_0^{g(a_n)} x^4 F_V(\mathrm{d}x) + 3\left(\mathrm{Var}\,\widetilde{V}^n_{\frac{1}{n}}\right)^2 \leq g(a_n)^2\,\mathrm{Var}\,\widetilde{V}^n_{\frac{1}{n}} + 3\left(\mathrm{Var}\,\widetilde{V}^n_{\frac{1}{n}}\right)^2,$$

and by Equation (2.11) and because $\overline{u}_n \leq w_n$, we have :

$$\begin{aligned}
n|\Psi_Z(\overline{u}_n\theta)|^3\sqrt{\mathrm{E}(D_n^2)}\sqrt{\mathrm{E}(D_n^4)} &\leq n|\Psi_Z(\overline{u}_n\theta)|^3\sqrt{\mathrm{E}(D_n^2)}\sqrt{g(a_n)^2\,\mathrm{E}(D_n^2) + 3(\mathrm{E}(D_n^2))^2} \\
&\leq n|\Psi_Z(\overline{u}_n\theta)|^3\sqrt{\mathrm{E}(D_n^2)}\sqrt{(\mathrm{E}(D_n^2))^2 + 3(\mathrm{E}(D_n^2))^2} \\
&= 2n|\Psi_Z(\overline{u}_n\theta)|^3(\mathrm{E}(D_n^2))^{\frac{3}{2}} \\
&\leq \begin{cases} Cn\overline{u}_n^3(\mathrm{E}(D_n^2))^{\frac{3}{2}} & \text{if } \gamma < 1 \\ Cn(\overline{u}_n\log\overline{u}_n)^3(\mathrm{E}(D_n^2))^{\frac{3}{2}} & \text{if } \gamma = 1 \end{cases} \\
&\leq \frac{C}{n^{\frac{1}{2}}}\,.
\end{aligned}$$

So, in all cases we have : $n|\Psi_Z(\overline{u}_n\theta)|^3\sqrt{\mathrm{E}(D_n^2)}\sqrt{\mathrm{E}(D_n^4)} \leq \frac{C}{n^{\frac{1}{2}}} \underset{n\to+\infty}{\longrightarrow} 0$ which finishes the proof.

□

4 Convergence of the approximative Euler scheme

4.1 Proofs of Theorems 2.2 and 2.4

We will first prove Theorem 2.4 but before going into the proof of this theorem, we need to prove an easy lemma.

Lemma 4.1 *Under the assumptions of Proposition 3.1, we have :*

$$(X^n_{\frac{[n.]}{n}}, \overline{X}^n_{\frac{[n.]}{n}}) \underset{n\to+\infty}{\overset{law}{\longrightarrow}} (X, X)\,. \tag{4.1}$$

PROOF. The convergence in Equation (4.1) can be found in many papers (see for example Jacod (2002) or Jacod & Protter (1998)) but we recall the proof in a few words, as a part of it will soon be useful. We have that $X^n_{\frac{[n.]}{n}}$ is solution of the equation :

$$X^n_{\frac{[nt]}{n}} = X_0 + \int_0^t f\left(X^n_{\frac{[ns]}{n}-}\right)\,\mathrm{d}Y_{\frac{[ns]}{n}}\,. \tag{4.2}$$

We have that $\overline{X}^n_{\frac{[n.]}{n}}$ is the solution of :

$$\overline{X}^n_{\frac{[nt]}{n}} = X_0 + \int_0^t f\left(\overline{X}^n_{\frac{[ns]}{n}-}\right) \mathrm{d}\overline{Y}^n_s \ . \tag{4.3}$$

By Lemma 2.1 of Jacod (2002), we have that :

$$Y_{\frac{[n.]}{n}} \xrightarrow[n\to+\infty]{\text{law}} Y \tag{4.4}$$

and that the sequence $(Y_{\frac{[n.]}{n}})$ is UT w.r.t the filtrations $(\mathcal{F}^n)$. By Proposition 3.1, we have that $\overline{Y}^n_{\frac{[n.]}{n}} - Y_{\frac{[n.]}{n}} \xrightarrow[n\to+\infty]{\text{law}} 0$. So, by Equation (4.4), $(\overline{Y}^n_{\frac{[n.]}{n}}, Y_{\frac{[n.]}{n}}) \xrightarrow[n\to+\infty]{\text{law}} (Y,Y)$. The sequence $(\overline{Y}^n_{\frac{[n.]}{n}})$ is UT w.r.t the filtrations $(\mathcal{F}^n)$ (by Lemma 2.1 of Jacod (2002)) and so the sequence $(\overline{Y}^n_{\frac{[n.]}{n}}, Y_{\frac{[n.]}{n}})$ is UT w.r.t. the filtrations $(\mathcal{F}^n)$. Thus, by Equations (4.2) and (4.3) and Theorem 5.3, we have :
$(X^n_{\frac{[n.]}{n}}, \overline{X}^n_{\frac{[n.]}{n}}) \xrightarrow[n\to+\infty]{\text{law}} (X,X)$. □

PROOF OF THEOREM 2.4. We make the proof under the assumption **(Z0)**. If **(Z0)** does not hold, the proof is almost the same. For technical reasons, we introduce :

$$\widehat{Y}^n_t = \sum_{k=0}^{[nt]} Z(V_{\frac{k-1}{n}} + V^n_{\frac{k}{n}} - V^n_{\frac{k-1}{n}}) - Z(V_{\frac{k-1}{n}}) \ .$$

We have :

$$\widehat{Y}^n_{\frac{[nt]}{n}} - Y_{\frac{[nt]}{n}} = \sum_{k=1}^{[nt]} Z\left(V_{\frac{k}{n}} - \left(\widetilde{V}^n_{\frac{k}{n}} - \widetilde{V}^n_{\frac{k-1}{n}}\right)\right) - Z\left(V_{\frac{k}{n}}\right) \ .$$

We first want to show that $\widehat{Y}^n_{\frac{[n.]}{n}} - Y_{\frac{[n.]}{n}} \xrightarrow[n\to+\infty]{\text{law}} 0$. Since the variables $\left(Z\left(V_{\frac{k}{n}} - \left(\widetilde{V}^n_{\frac{k}{n}} - \widetilde{V}^n_{\frac{k-1}{n}}\right)\right) - Z\left(V_{\frac{k}{n}}\right)\right)_{1\le k\le n}$ are i.i.d. then by Lemma 2.1 Jacod (2002), it is sufficient to look at the value at time 1.

$$\begin{aligned}
\widehat{Y}^n_1 - Y_1 &= \sum_{k=1}^{n} Z\left(V_{\frac{k}{n}} - \left(\widetilde{V}^n_{\frac{k}{n}} - \widetilde{V}^n_{\frac{k-1}{n}}\right)\right) - Z\left(V_{\frac{k}{n}}\right) \\
&\overset{\text{law}}{=} -\sum_{k=1}^{n} Z\left(\widetilde{V}^n_{\frac{k}{n}} - \widetilde{V}^n_{\frac{k-1}{n}}\right) \\
&\overset{\text{law}}{=} -Z\left(\widetilde{V}^n_1\right).
\end{aligned}$$

We have $\widetilde{V}_1^n \xrightarrow[n\to+\infty]{\text{law}} 0$ and since Z has right continuous sample paths it follows that $Z\left(\widetilde{V}_1^n\right) \xrightarrow[n\to+\infty]{\text{law}} 0$ and thus we have that $\widehat{Y}^n_{\frac{[n.]}{n}} - Y_{\frac{[n.]}{n}} \xrightarrow[n\to+\infty]{\text{law}} 0$.

We now want to show the independence of $\widehat{Y}^n_{\frac{[n.]}{n}}$ and $Y_{\frac{[n.]}{n}} - \overline{Y}^n_{\frac{[n.]}{n}}$. We note that $\widehat{Y}^n_{\frac{[n.]}{n}}$ is the sum of all increments of Z between $V_{\frac{k-1}{n}}$ and $\left[V_{\frac{k-1}{n}} + \left(V^n_{\frac{k}{n}} - V^n_{\frac{k-1}{n}}\right)\right]$, $k=1,...,n$. Moreover by Equation (3.1), $Y_{\frac{[n.]}{n}} - \overline{Y}^n_{\frac{[n.]}{n}}$ is the sum of all increments of Z between $\left[V_{\frac{k}{n}} - \left(\widetilde{V}^n_{\frac{k}{n}} - \widetilde{V}^n_{\frac{k-1}{n}} - \mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}}\right)^+\right]$ and $V_{\frac{k}{n}}$, $k=1,...,n$, plus all the increments of $Z^{(k)}$ between 0 and $\left(\widetilde{V}^n_{\frac{k}{n}} - \widetilde{V}^n_{\frac{k-1}{n}} - \mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}}\right)^-$,$k=1,...,n$.

Now since

$$V_{\frac{k}{n}} - \left(\widetilde{V}^n_{\frac{k}{n}} - \widetilde{V}^n_{\frac{k-1}{n}} - \mathrm{E}\,\widetilde{V}^n_{\frac{1}{n}}\right)^+ \geq V_{\frac{k-1}{n}} + \left(V^n_{\frac{k}{n}} - V^n_{\frac{k-1}{n}}\right), k=1,\cdots,n$$

and by the stationary independent increments property of Z and the independence of $\left[V^n_{\frac{k}{n}} - V^n_{\frac{k-1}{n}}\right]$ and $\left[\widetilde{V}^n_{\frac{k}{n}} - \widetilde{V}^n_{\frac{k-1}{n}}\right]$ $(k=1,...,n)$ it follows that $\widehat{Y}^n$ is independent of $Y_{\frac{[n.]}{n}} - \overline{Y}^n_{\frac{[n.]}{n}}$.

From the independence of $\widehat{Y}^n$ and $Y_{\frac{[n.]}{n}} - \overline{Y}^n_{\frac{[n.]}{n}}$, Proposition 3.1 and the convergence $\widehat{Y}^n_{\frac{[n.]}{n}} - Y_{\frac{[n.]}{n}} \xrightarrow[n\to+\infty]{\text{law}} 0$, it then follows that :

$$\left(\overline{u}_n\left(Y_{\frac{[n.]}{n}} - \overline{Y}^n_{\frac{[n.]}{n}}\right), Y_{\frac{[n.]}{n}}\right) = \left(\overline{u}_n\left(Y_{\frac{[n.]}{n}} - \overline{Y}^n_{\frac{[n.]}{n}}\right), \widehat{Y}^n_{\frac{[n.]}{n}} + Y_{\frac{[n.]}{n}} - \widehat{Y}^n_{\frac{[n.]}{n}}\right) \xrightarrow[n\to+\infty]{\text{law}} (\widehat{Z}, Y)$$

with $\widehat{Z}$ and Y independent and $\widehat{Z}$ is a stable process of index γ such that

$$\mathrm{E}(\exp(i\langle\theta, \widehat{Z}(1)\rangle)) = \exp\left(-\int_{S^{q-1}} |\langle\theta, s\rangle|^\gamma \lambda_1(ds)\right) .$$

By Equations (4.2) and (4.3), we have for all t :

$$\begin{aligned}\overline{u}_n\overline{e}^n_t = \overline{u}_n\left(X^n_{\frac{[nt]}{n}} - \overline{X}^n_{\frac{[nt]}{n}}\right) &= \int_0^t \phi\left(X^n_{\frac{[ns]}{n}-}, \overline{X}^n_{\frac{[ns]}{n}-}\right)\overline{u}_n\overline{e}^n \,\mathrm{d}Y_{\frac{[ns]}{n}} \\ &\quad + \int_0^t f\left(\overline{X}^n_{\frac{[ns]}{n}-}\right) \mathrm{d}\left(\overline{u}_n\left(Y^n_{\frac{[ns]}{n}} - \overline{Y}^n_{\frac{[ns]}{n}}\right)\right) \quad (4.5)\end{aligned}$$

By Lemma 2.1 p. 7 of Jacod (2002), the sequences $\left(\overline{u}_n\left(Y_{\frac{[n.]}{n}} - \overline{Y}^n_{\frac{[n.]}{n}}\right)\right)$ and $\left(Y_{\frac{[n.]}{n}}\right)$ are UT w.r.t the filtrations $(\mathcal{F}^n)$ and so, by Lemma 4.1 and Theorem 5.3 :

$$\overline{u}_n\overline{e}^n \xrightarrow[n\to+\infty]{\text{law}} \overline{U} .$$

where $\overline{U}$ is the solution of Equation (2.12) with $\widehat{Z}$ and Y independent.

□

We set $W_t^n = \int_0^{\frac{[nt]}{n}} \left(f(X_{s-}) - f(X_{\frac{[ns]-}{n}}) \right) \mathrm{d}Y_s$. By Jacod (2002), we have :

$$u_n e_t^n = u_n W_t^n + \int_0^t \phi\left(X_{\frac{[ns]}{n}}, X^n_{\frac{[ns]}{n}} \right) u_n e^n_{s-} \, \mathrm{d}Y_{\frac{[ns]}{n}} . \tag{4.6}$$

So, in order to prove Theorem 2.2 using Lemma 5.4 (and using the fact that the sum of two UT sequences is also UT), we first need to prove the following lemma.

Lemma 4.2 *If* **(H1-**α'**)** *holds then the sequence* $(u_n W^n)$ *has the UT property with respect to the filtrations* $(\mathcal{F}^n)$.

PROOF. As shown in Jacod (2002), $u_n W^n$ can be written $u_n W_t^n = \sum_{i=1}^{[nt]} \zeta_i^n$ with some variables ζ_i^n which are $\mathcal{F}_{\frac{i}{n}}$-measurable and such that :

$$\begin{cases} |\,\mathrm{E}(\zeta_i^n \mathbf{1}_{|\zeta_i^n| \leq 1} | \mathcal{F}_{\frac{i-1}{n}})| \leq \frac{\xi_n}{n} , \\ \mathrm{E}(|\zeta_i^n|^2 \mathbf{1}_{|\zeta_i^n| \leq 1} | \mathcal{F}_{\frac{i-1}{n}}) \leq \frac{\xi'_n}{n} , \\ \mathrm{P}(|\zeta_i^n| > y | \mathcal{F}_{\frac{i-1}{n}}) \leq \frac{\xi''_{n,y}}{n} , \ \forall y > 1 \end{cases} \tag{4.7}$$

with

$$\limsup_{n\to+\infty} \xi_n < \infty, \ \limsup_{n\to+\infty} \xi'_n < \infty, \ \limsup_{y\to+\infty} \lim_{n\to+\infty} \xi''_{n,y} = 0 . \tag{4.8}$$

A careful reading of Jacod (2002) shows that Equation (4.7) holds for all $y > 0$ and that $\sup_n \xi''_{n,y} < \infty$ for all $y > 0$. And so the result is a consequence of Theorem 1-4 (ii) of Mémin & Slominski (1991).

□

PROOF OF THEOREM 2.2. If **(H0)** holds then $(u_n e^n)$ is UT with respect to the filtration $(\mathcal{F}_t)$ by Theorem 6.1 of Jacod & Protter (1998) and so $(u_n e^n)$ is UT with respect to the filtrations $(\mathcal{F}^n)$.

If **(H1-**α'**)** holds then by Theorem 1.1 of Jacod (2002), the sequence $(u_n e^n)$ is tight. By Lemmas 4.2 and 5.4, the sequence $(u_n e^n)$ is then UT with respect to the filtrations $(\mathcal{F}^n)$.

We now look at Equation (4.5). By Lemma 3.1 p. 7 of Jacod (2002), the sequences $(\overline{u}_n(Y^n_{\frac{[n.]}{n}} - \overline{Y}^n_{\frac{[n.]}{n}}))$ and $(Y_{\frac{[n.]}{n}})$ are UT w.r.t $(\mathcal{F}^n)$. So, by Lemma 5.2, the sequences $\left(\int_0^{\cdot} \phi(X^n_{\frac{[ns]}{n}-}, \overline{X}^n_{\frac{[ns]}{n}-}) dY_{\frac{[ns]}{n}} \right)$ and $\left(\int_0^{\cdot} f(\overline{x}^n_{\frac{[ns]}{n}-}) d(\overline{u}_n(Y^n_{\frac{[ns]}{n}} - \overline{Y}^n_{\frac{[ns]}{n}})) \right)$ are UT w.r.t $(\mathcal{F}^n)$. As $(\overline{u}_n e^n)$ is tight by Theorem 2.4 then, by Lemma 5.4, $(\overline{u}_n \overline{e}^n)$ is UT w.r.t $(\mathcal{F}^n)$. So the sequence $(\widehat{u}_n \widehat{e}^n)$ is UT with respect to the filtrations $(\mathcal{F}^n)$. Lemma 1.1 of Jakubowski, Mémin & Pagès (1989) now allows us to say that for all t, the sequence $((\widehat{u}_n \widehat{e}^n)^*_t)$ is tight.

□

4.2 The choice of the number of terms (a_n) in the approximation of the increments and the amount of work needed to generate the solution

In order to get non-trivial limit processes for both sequences $(\overline{e}^n)$ and (e^n) we need to balance their rates $\overline{u}_n$ and u_n so that $u_n \asymp \overline{u}_n$ (for two sequences of real (b_n) and (c_n) going to $+\infty$, we write $b_n \asymp c_n$ if there exists constants $C, C' > 0$ such that $Cb_n \leq c_n \leq C'b_n$ for n large enough). This is done by choosing the sequence a_n or equivalently by choosing the truncation sequence $g_n = g(a_n)$ in an appropriate way. We have that a_n is the average number of terms in the approximation of the subordinator and therefore proportional to the amount of work needed to approximate the increments of the driving process Y. The total amount of work is thus of the order of $a_n + n$ where n is the number of steps in the Euler scheme.

By now, we consider the following special case : we suppose that **(Z0)** holds and that we are in case **1**, **2b** or **3b** (with $0 < \alpha' < 2$) and

$$\beta^\alpha \theta_V(\beta) \underset{\beta \to 0}{\longrightarrow} \theta_V \ .$$

This implies that $\lim_{u \to +\infty} u^{1/\alpha} g(u) = \theta_V^{1/\alpha}$. We suppose that $\alpha = \alpha'/\gamma$ and that $0 < \alpha < 1$; this can be the case under the assumptions of Lemma 5.6. We decide to take :

$$a_n = \frac{n^{\frac{2+\alpha}{2-\alpha}}}{(\log n)^{\frac{2}{2-\alpha}}} \ . \tag{4.9}$$

We will show that for such a_n, we have :

$$\frac{1}{g_n^2} \int_0^{g_n} x^2 F_V(dx) \geq n \tag{4.10}$$

and $u_n \asymp \overline{u}_n$. Running the computations in the opposite way would easily show that if $u_n \asymp \overline{u}_n$ and Equation (4.10) holds then $a_n \asymp \frac{n^{\frac{2+\alpha}{2-\alpha}}}{(\log n)^{\frac{2}{2-\alpha}}}$. However, we are not able to see whether there exists a_n's such that $u_n \asymp \overline{u}_n$ and Equation (4.10) does not hold.

Using an integration by parts formula, we obtain that :

$$\frac{1}{g_n^2}\int_0^{g_n} x^2 F_V(dx) \underset{n\to+\infty}{\sim} \frac{\theta_V}{g_n^\alpha} \underset{n\to+\infty}{\sim} a_n ,$$

and, since the exponent of n in Equation (4.9) is bigger than 1, we have Equation (4.10) for n large enough. So we can apply Lemma 3.3 to get that $\mu_n \asymp \sqrt{\operatorname{Var} \widetilde{V}^n_{\frac{1}{n}}}$ and by a change of variable, we get that :

$$\operatorname{Var} \widetilde{V}^n_{\frac{1}{n}} = \frac{1}{n}\int_0^{g_n} x^2 F(dx) = \frac{1}{n}\int_{a_n}^{\infty} g(u)^2 \text{ d}u \underset{n\to+\infty}{\sim} \frac{a_n^{1-2/\alpha}}{n}\frac{\theta_V^{1/\alpha}}{1-2/\alpha} .$$

Thus :

$$\overline{u}_n = (n\mu_n)^{-1/\gamma} \asymp n^{-\frac{1}{\gamma}}(a_n^{\frac{1}{2}-\frac{1}{\alpha}} n^{-\frac{1}{2}})^{-\frac{1}{\gamma}} = \left(\frac{n}{\log n}\right)^{\frac{1}{\alpha'}} .$$

From above and Jacod (2002), we recall that :

$$u_n = \left(\frac{n}{\log n}\right)^{1/\alpha'} ,$$

and so $u_n \asymp \overline{u}_n$.

We see that the amount of work is dominated by a_n this since the exponent of n in Equation (4.9) is bigger than 1 for all $\alpha > 0$ and thus the asymptotic amount of work to obtain the solution will be governed by a_n. We now want to express u_n in term of the amount of work a_n. Using that a function of the form

$$f(x) = \frac{x^a}{\log(x)^b}$$

has an asymptotic inverse of the form

$$g(x) = x^{\frac{1}{a}}\left(\frac{1}{a}\log(x)\right)^{\frac{b}{a}} ,$$

i.e.

$$\frac{f(g(x))}{x} \to 1 \text{ and } \frac{g(f(x))}{x} \to 1 \text{ as } x \to \infty,$$

we get that :

$$\begin{aligned} u_n = a_n^{\frac{2-\alpha}{2\alpha'}} n^{-\frac{\alpha}{2\alpha'}} &\underset{n\to+\infty}{\sim} a_n^{\frac{2-\alpha}{2\alpha'}} a_n^{\frac{2-\alpha}{2+\alpha}\frac{(-\alpha)}{2\alpha'}} \left(\frac{2-\alpha}{2+\alpha}\log a_n\right)^{\frac{-\alpha}{2\alpha'}\frac{2}{2+\alpha}} \\ &\asymp a_n^{\frac{2-\alpha}{\alpha\gamma(2+\alpha)}} (\log a_n)^{-\frac{1}{\gamma(2+\alpha)}} . \end{aligned}$$

The worst case, however not attainable since then V is not a subordinator in that case (but we can come arbitrarily close), is when $\alpha = 1$. We see then that the rate u_n drops to $a_n^{\frac{1}{3\gamma}} \log(a_n)^{-\frac{1}{3\gamma}}$. So for $\gamma = 2$, the rate drops to $a_n^{\frac{1}{6}} \log(a_n)^{-\frac{1}{6}}$ and this is the lowest rate our scheme will have. The rate $a_n^{1/6} \log(a_n)^{-1/6}$ may seem rather low but we should have in mind that for the comparable case the rate in Rubenthaler (2001) dropped to a_n^0. Although that last rate was obtained with considerably less assumptions on the probabilistic structure of the increments of Y.

5 Technical facts

5.1 The UT Property and stability of solutions of SDE's.

We use several results about the UT (for Uniform Tightness) condition and stability of solutions of SDE's in the sequel and we write here the results we need. The complete definitions and proofs can be found in Jakubowski, Mémin & Pagès (1989), Kurtz & Protter (1990), Kurtz & Protter (1991a), Kurtz & Protter (1991b), Mémin & Slominski (1991). Lemma 5.2 is (almost) Lemma 1-6 of Mémin & Slominski (1991). Theorem 5.3 comes from Proposition 5.1 of Kurtz & Protter (1990). Lemma 5.4 comes from Corollary 6.20 p. 381 of Jacod & Shiryaev (2003). We recall here the definition of the UT property found in Kurtz & Protter (1991b) (other equivalent properties are given in this paper). For $n = 1, 2, ...$, let $\Theta_n = (\Omega_n, \mathcal{G}^n, (\mathcal{G}^n)_{t\geq 0}, \mathrm{P}^n)$ a filtered probability space. We set :

$$\begin{aligned}\mathcal{H}^n &= \{H^n : H_t^n = Y_0^n + \sum_{i=1}^{p-1} Y_i^n \mathbf{1}_{[t_i,t_{i+1})}(t), \\ &\qquad \text{with } Y_i^n \in \mathcal{G}_{t_i}^n,\ p \in \mathbb{N},\ 0 = t_0 < t_1 < ... < t_p < \infty, |Y_i^n| \leq 1\} .\end{aligned}$$

Definition 5.1 *A sequence of semi-martingale (K^n) defined on Θ_n satisfies the UT condition if for each $t > 0$ the set $\{\int_0^t H_s^n dK_s^n,\ H^n \in \mathcal{H}^n,\ n \in \mathbb{N}\}$ is stochastically bounded.*

We see that the definition of the UT property is made with respect to some filtrations $(\mathcal{G}^n)$. In the following lemmas and theorems, the sequences of processes are supposed to be defined on the same filtered probability spaces Θ_n for each n. In this paper, we use the UT property with respect to $\Theta_n = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq 1}, \mathrm{P})$ for all n or with respect to $\Theta_n = (\Omega, \mathcal{F}^n, (\mathcal{F}_t^n)_{0\leq t\leq 1}, \mathrm{P}) = (\Omega, \mathcal{F}, (\mathcal{F}_{\frac{[nt]}{n}})_{0\leq t\leq 1}, \mathrm{P})$ and we say that the processes are UT with respect to the filtration $(\mathcal{F}_t)$ or to the filtrations $(\mathcal{F}_t^n)$ depending on the case.

Lemma 5.2 *If the sequence of semi-martingales $(Z^n)_{n\geq 0}$ (taking values in $\mathbb{R}^r$) satisfies UT and if $(H^n)_{n\geq 0}$ is a sequence of predictable (relatively to the filtration in which the Z^n's are given) and locally uniformly bounded processes (taking values in $\mathcal{M}_{s,r}(\mathbb{R})$), then the sequence $(\int_0^\cdot H_s^n\, \mathrm{d}Z_s^n)_{n\geq 0}$ satisfies UT.*

Theorem 5.3 *Let (Z^n) and (H^n) be two sequences of semi-martingales, Z^n taking values in $\mathbb{R}^s$ and H^n taking values in $\mathbb{R}^r$. Let $\psi : \mathbb{R}^r \to \mathcal{M}_{r,s}$ be a Lipschitz function, and U^n be the solution of :*

$$U_t^n = H_t^n + \int_0^t \psi(U_{s-}^n)\,\mathrm{d}Z_s^n.$$

If $(Z^n, H^n) \xrightarrow[n\to+\infty]{law} (Z^\infty, H^\infty)$ and if the sequence (Z^n) is UT, then $(U^n, Z^n, H^n) \xrightarrow[n\to+\infty]{law} (U^\infty, Z^\infty, H^\infty)$ where U^∞ is the solution of :

$$U_t^\infty = H_t^\infty + \int_0^t \psi(U_{s-}^\infty)\,\mathrm{d}Z_s^\infty.$$

Lemma 5.4 *If we have two sequences of semi-martingales (U^n) and (A^n) (defined in some filtration $\mathcal{G}^n$ for each n) where A^n takes values in $\mathbb{R}^s$ and U^n take values in $\mathcal{M}_{r,s}$ and such that (A^n) is UT and (U^n) is tight.*
Then $K^n = \int_0^\cdot U_{s-}^n dA_s^n$ is UT.

5.2 Relations between $\theta_{Z(V)}$ and θ_V

We recall here that if $0 < \gamma < 2$ then $\theta_Z(\beta) \le C\beta^{-\gamma}$ and that if $\gamma = 2$ then $\theta_Z(\beta) = 0$ for all β. We recall that D_V is the drift of V and that τ is the drift of Z. The reader can consult Theorem 30.1 of Sato (1999) for the relation between the characteristics of Y, Z, V.

Lemma 5.5 *We have for some constant C'' :*

$$\mathrm{P}(|Z_t| \ge \beta) \le C'' \left(\frac{t}{\beta^\gamma} + \mathbf{1}_{|\tau t| \ge \frac{\beta}{2}} \right) . \tag{5.1}$$

PROOF. We set $Z'_t = Z_t - \tau t$. By Proposition 4 p. 221 of Bertoin (1996), we have that for some constant C and for all $\beta > 0$:

$$\mathrm{P}(|Z'_t| \ge \beta) = \mathrm{P}(|Z'_1| \ge \beta t^{-\frac{1}{\gamma}}) \le C \frac{t}{\beta^\gamma}$$

(this is stated in Bertoin (1996) for $0 < \gamma < 2$ but it is also true for $\gamma = 2$). And this finishes the proof. □

Lemma 5.6 *If we have for some constant C and some $0 < \alpha < 1$, for all $0 < \beta \le 1$:*

$$\theta_V(\beta) \le \frac{C}{\beta^\alpha} ,$$

then we have for some constant C' :

$$\theta_Y(\beta) \leq C' \left(\frac{D_V}{\beta^\gamma} + \frac{1}{\beta^{\alpha\gamma}} + \mathbf{1}_{\tau\neq 0} \frac{1}{\beta^\alpha} \right) .$$

PROOF. By Theorem 30.1 of Sato (1999), we have that for all $\beta > 0$:

$$\theta_Y(\beta) = D_V \theta_Z(\beta) + \int_0^{+\infty} \mathrm{P}(|Z_t| \geq \beta) F_V(\mathrm{d}t) . \tag{5.2}$$

By using Lemma 5.5, we have that, for some constant C'' :

$$\theta_Y(\beta) \leq D_V\theta_Z(\beta) + \int_0^{\frac{\beta^\gamma}{C''}} \frac{C''t}{\beta^\gamma} F_V(\mathrm{d}t) + \theta_V\left(\frac{\beta^\gamma}{C''}\right) + \mathbf{1}_{\tau\neq 0} \int_{\frac{\beta}{2\tau}}^{+\infty} F_V(\mathrm{d}t).$$

We have that : $\mathbf{1}_{\tau\neq 0} \int_{\frac{\beta}{2\tau}}^{+\infty} F_V(\mathrm{d}t) \leq \mathbf{1}_{\tau\neq 0} \frac{C}{\beta^\alpha}$ for some constant C. Using the integration by parts formula for functions of bounded variation, we obtain :

$$\begin{aligned} \theta_V\left(\frac{\beta^\gamma}{C''}\right) + \int_0^{\frac{\beta^\gamma}{C''}} \frac{C''t}{\beta^\gamma} F_V(\mathrm{d}t) &= \frac{C''}{\beta^\gamma} \int_0^{\frac{\beta^\gamma}{C''}} \theta_V(t)\ \mathrm{d}t \\ &\leq \frac{C''}{\beta^\gamma} \int_0^{\frac{\beta^\gamma}{C''}} \frac{C}{t^\alpha}\ \mathrm{d}t \\ &\leq \frac{C'}{\beta^{\alpha\gamma}} \end{aligned}$$

for some constant C', and as $\beta^\gamma\theta_Z(\beta)$ is bounded, this finishes the proof.

□

Lemma 5.7 *If $\tau = 0$ and if we have for some constant C', for some $0 < \alpha' \leq 2$, for all $0 < \beta \leq 1$:*

$$\theta_Y(\beta) \leq \frac{C'}{\beta^{\alpha'}} ,$$

then we have for some constant C :

$$\theta_V(\beta) \leq \frac{C}{\beta^{\frac{\alpha'}{\gamma}}} .$$

PROOF. There exists $C'' > 0$ such that $\mathrm{P}(|Z_1| > C'') > 0$. So we have for some constant C changing from line to line :

$$\begin{aligned}\theta_V(\beta) &\leq \int_\beta^{+\infty} \frac{\mathrm{P}(|Z_1| \geq C''\beta^{\frac{1}{\gamma}}t^{-\frac{1}{\gamma}})}{\mathrm{P}(|Z_1| > C'')} F_V(\mathrm{d}t) \\ &\leq C\int_\beta^{+\infty} \mathrm{P}(|Z_t| \geq C''\beta^{\frac{1}{\gamma}})F_V(\mathrm{d}t)\end{aligned}$$

So, by Equation (5.2), we have for some constant C changing from line to line :

$$\begin{aligned}\theta_V(\beta) &\leq C\theta_Y(C''\beta^{\frac{1}{\gamma}}) \\ &\leq \frac{C}{\beta^{\frac{\alpha'}{\gamma}}} .\end{aligned}$$

□

Lemma 5.8 *We suppose that :*

- Z *is 1-dimensional,*
- $\tau = 0$
- $D_V = 0$.

If we have for some $0 < \alpha < 1$ *and some* $\theta_V > 0$ *:*

$$\beta^\alpha \theta_V(\beta) \underset{\beta\to 0}{\longrightarrow} \theta_V ,$$

then we have for some $\theta_Y^+, \theta_Y^- > 0$ *:*

$$\beta^{\alpha\gamma}\theta_Y^+(\beta) \underset{\beta\to 0}{\longrightarrow} \theta_Y^+ , \beta^{\alpha\gamma}\theta_Y^-(\beta) \underset{\beta\to 0}{\longrightarrow} \theta_Y^- .$$

PROOF. We make the proof for the θ_Y^+ part, the proof for θ_Y^- is the same. We denote by f_{Z_1} the density of Z_1. Using the same properties as in the proofs of the two preceding Lemmas, we get for all $\beta > 0$:

$$\begin{aligned}\beta^{\alpha\gamma}\theta_Y^+(\beta) &= \beta^{\alpha\gamma}\int_0^{+\infty} \mathrm{P}(Z_t \geq \beta)F_V(\mathrm{d}t) \\ &= \beta^{\alpha\gamma}\int_0^{+\infty} \mathrm{P}(Z_1 \geq \beta t^{-\frac{1}{\gamma}})F_V(\mathrm{d}t) \\ &= \beta^{\alpha\gamma}\int_0^{+\infty}\int_{\beta t^{-\frac{1}{\gamma}}}^{+\infty} f_{Z_1}(u)\ \mathrm{d}u F_V(\mathrm{d}t) \\ &= \beta^{\alpha\gamma}\int_0^{+\infty} f_{Z_1}(u)\theta_V(\beta^\gamma u^{-\gamma})\ \mathrm{d}u\end{aligned}$$

By assumption, we have for some constant C'' :

$$\beta^{\alpha\gamma}\int_0^{\beta} f_{Z_1}(u)\theta_V(\beta^\gamma u^{-\gamma})\ \mathrm{d}u \ \leq\ \int_0^{\beta} f_{Z_1}(u)C''u^{\alpha\gamma}\ \mathrm{d}u \underset{\beta\to 0}{\longrightarrow} 0\ .$$

We notice that :

$$\beta^{\alpha\gamma}\int_\beta^{+\infty} f_{Z_1}(u)\theta_V(\beta^\gamma u^{-\gamma})\ \mathrm{d}u \ =\ \int_0^{+\infty} u^{\alpha\gamma} f_{Z_1}(u)\mathbf{1}_{u\geq\beta}\left(\frac{\beta}{u}\right)^{\alpha\gamma}\theta_V(\beta^\gamma u^{-\gamma})\ \mathrm{d}u\ .$$

By integration by parts, we get for some constant C :

$$\begin{aligned}\int_1^{+\infty} u^{\alpha\gamma} f_{Z_1}(u)\ \mathrm{d}u &= \mathrm{P}(Z_1\geq 1)+\int_1^{+\infty}\alpha\gamma u^{\alpha\gamma-1}\,\mathrm{P}(Z_1\geq u)\ \mathrm{d}u\\ &\leq \mathrm{P}(Z_1\geq 1)+C\int_1^{+\infty} u^{\alpha\gamma-1-\gamma}\ \mathrm{d}u\\ &< \infty\ ,\end{aligned}$$

and so $\int_0^{+\infty} u^{\alpha\gamma} f_{Z_1}(u)\ \mathrm{d}u<\infty$. Since $\mathbf{1}_{u\geq\beta}\left(\frac{\beta}{u}\right)^{\alpha\gamma}\theta_V(\beta^\gamma u^{-\gamma})\to\theta_V$ as $\beta\to 0$ for all $u>0$ and $t^\alpha\theta_V(t)$ is uniformly bounded for $t\leq 1$, then by bounded convergence :

$$\beta^{\alpha\gamma}\int_\beta^{+\infty} f_{Z_1}(u)\theta_V(\beta^\gamma u^{-\gamma})\ \mathrm{d}u \underset{\beta\to 0}{\longrightarrow}\int_0^{+\infty} u^{\alpha\gamma} f_{Z_1}(u)\theta_V\ \mathrm{d}u,$$

which finishes the proof. □

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