Université de Nice - M2 Mathmods - Stochastic Calculus

EXERCISES 1

RANDOM VARIABLES, CONDITIONAL EXPECTATION

In all the exercises, $(\Omega, \mathcal{A}, \mathbb{P})$ denotes the current probability space.

1. Random Variables

Exercise 1. Compute $\mathbb{V}(X)$ (if exists) in the following cases

(1) X is a r.v. of uniform law on $(0,1)$.
(2) X is a r.v. of Bernoulli law of parameter $p \in (0,1)$.
(3) X is a r.v. of Gaussian law $\mathcal{N}(m, \sigma^2)$ (i.e. of parameters $m \in \mathbb{R}$ and $\sigma > 0$.

Exercise 2. Let X be a r.v. of Gaussian law $\mathcal{N}(0,1)$ (i.e. of parameters 0 and 1). For $m \in \mathbb{R}$ and $\sigma > 0$, give the law of $m + \sigma X$.

Exercise 3. Let X be a r.v. of uniform law on $(0,1)$. Give the law of $1 - U$.

Exercise 4. Let X be a r.v. of Gaussian law $\mathcal{N}(0,1)$ (i.e. of parameters 0 and 1). Give the law of X^2.

Exercise 5. Let X be a r.v. and $t \in \mathbb{R}$. Compute $\mathbb{E}[\exp(tX)]$ in the following cases:

(1) X is a Bernoulli r.v. of parameter $0 < p < 1$.
(2) X is a Gaussian r.v. $\mathcal{N}(m, \sigma^2)$, $m \in \mathbb{R}$ and $\sigma > 0$.

Exercise 6. Show that the moments of a random variable X of Gaussian law $\mathcal{N}(0,1)$ are given by

$$\forall n \geq 0, \ \mathbb{E}(X^{2n}) = \frac{(2n)!}{2^n n!}, \ \mathbb{E}(X^{2n+1}) = 0.$$

Hint: Use the previous exercise.

2. Independent Random Variables

Exercise 7. Lett X, Y be two independent and identically distributed r.v. of law $\mathcal{N}(0,1)$. Prove that $X - Y$ and $X + Y$ are independent.

Exercise 8. Let U and V be two independent and identically distributed r.v. of uniform law on $(0,1)$. What is the law of $\max(U, V)$? What is the law of the pair $(\min(U, V), \max(U, V))$?

Exercise 9. Let X and Y be two independent and identically distributed r.v. of Gaussian law $\mathcal{N}(0,1)$. What is the law of X/Y? Is it possible to define $\mathbb{E}[X/Y]$?

Exercise 10. Let X and Y be two independent variables such that $\mathbb{E}(X^2 + Y^2) < +\infty$.

(1) Show that $\mathbb{E}(X)$ and $\mathbb{E}(Y)$ exist.
(2) Show that $\mathbb{V}(X + Y) = \mathbb{V}(X) + \mathbb{V}(Y)$.
(3) Find a counter-example to the above equality when X and Y aren't independent.

Exercise 11. We say that a r.v. X is an exponential r.v. of parameter $\lambda > 0$ if the law of X has the density

$$x \in \mathbb{R} \mapsto \lambda \mathbf{1}_{(0,+\infty)}(x) \exp(-\lambda x).$$

Let U and V be two independent exponential r.v. of parameter $\lambda > 0$. What is the law of $\min(U, V)$?

Exercise 12. Let X and Y be two independent Gaussian random variables $\mathcal{N}(m_1, \sigma_1^2)$ and $\mathcal{N}(m_2, \sigma_2^2)$, $m_1, m_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 > 0$. Using characteristic functions, give the law of $X_1 + X_2$.