EXERCISES 2

INDEPENDENCE OF GAUSSIAN VECTORS – GAUSSIAN PROCESSES – BROWNIAN MOTION

In all the exercises, $(\Omega, \mathcal{A}, \mathbb{P})$ denotes the current probability space.

1. Independence

Exercise 1. Let $(X_i)_{1\le i\le n}$, $n \ge 2$, be n independent and identically distributed r.v. of Gaussian law $\mathcal{N}(0,1)$. Prove that the r.v. $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$ and $\max_{1\le i\le n} X_i - \min_{1\le i\le n} X_i$ are independent.

Hint: Consider the vector $(\bar{X}_n, X_1 - \bar{X}_n, \dots, X_n - \bar{X}_n)^t$.

Exercise 2. Let $(X_n)_{n\ge 1}$ be a sequence of I.I.D. r.v. of Gaussian law $\mathcal{N}(0,1)$. We set:

$$B_0 = 0,\ \forall n \ge 1,\ B_n = \sum_{k=1}^n X_k.$$

(1) Give the covariance matrix of $(B_1, \dots, B_n)$ as well as its probability density (if exists).
(2) For $1 \le m \le n$, set $Z_m = B_m - (m/n)B_n$. Prove that Z_m and B_n are independent.

2. Law of a Process

Exercise 3. Let $(X_t)_{0\le t\le 1}$ be a real-valued continuous process.

(1) Show that the following mapping is a random variable:

$$\omega \in \Omega \mapsto \int_0^1 X_s(\omega) ds.$$

(Hint: think of Riemann sums.)

(2) Let $(Y_t)_{0\le t\le 1}$ be another real-valued continuous process.
  (a) Assume that X and Y have the same law, prove that $\int_0^1 X_s ds$ and $\int_0^1 Y_s ds$ have the same law.
  (b) Assume that X and Y are independent, prove that $\int_0^1 X_s ds$ and $\int_0^1 Y_s ds$ are independent.

3. Gaussian Processes

Exercise 4. Let $(X_t)_{t\ge 0}$ be a Gaussian process. For a function ψ from $\mathbb{R}_+$ into itself, show that $(X_{\psi(t)})_{t\ge 0}$ is also Gaussian.

Exercise 5. Let $(X_t)_{0\le t\le 1}$ be a real-valued continuous Gaussian process. We suppose that the functions $t \mapsto \mathbb{E}(X_t)$ and $(t,s) \mapsto \mathbb{E}(X_s X_t)$ are continuous. Show that $\int_0^1 X_s ds$ has a Gaussian law. Compute its mean and its covariance.

Exercise 6. Let $(B_t)_{t\ge 0}$ be a (real) Brownian motion and $(Z_t)_{0\le t\le 1}$ be the process:

$$\forall t \in [0,1],\ Z_t = B_t - tB_1.$$

(1) Show that $(Z_t)_{0\le t\le 1}$ is a Gaussian process and is independent of B_1. Compute the mean and the covariance functions of Z.

(2) We define the time reversal of Z by:

$$\forall t \in [0,1],\ Y_t = Z_{1-t}.$$

Show that both processes have the same law.

4. Brownian Motion

Exercise 7. Let $(B_t)_{t\geq 0}$ be a (real) Brownian motion. Show that $(-B_t)_{t\geq 0}$ is a Brownian motion.

Exercise 8. Let $(B_t)_{t\geq 0}$ be a (real) Brownian motion. For a real $a > 0$, show that $(B_{a+t} - B_a)_{t\geq 0}$ is a Brownian motion and is independent of $(B_t)_{0\leq t\leq a}$.

Exercise 9. Let $(B_t)_{t\geq 0}$ be a (real) Brownian motion et and $(\widetilde{B}_t)_{t\geq 0}$ be the family of random variables given by:

$$\widetilde{B}_0 = 0,\ \forall t > 0,\ \widetilde{B}_t = tB_{t^{-1}}.$$

(1) Show that $(\widetilde{B}_t)_{t\geq 0}$ is a centered Gaussian process with $(s,t) \in \mathbb{R}_+^2 \mapsto s \wedge t$ as covariance function.
(2) Deduce that $(\widetilde{B}_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ have the same law.

Exercise 10. A d-dimensional Brownian motion is a process of the form $(B_t = (B_t^1, \ldots, B_t^d))_{t\geq 0}$, where $(B_t^i)_{t\geq 0}$, $1 \leq i \leq d$, are independent (real) Brownian motions. Show that for such a B and for a matrix U of size $d \times d$ with UU^* equal to the identity matrix, the process $(UB_t)_{t\geq 0}$ is also a d-dimensional Brownian motion.
(To simplify, you may choose $d = 2$.)

Exercise 11. Show that the probability that a Brownian motion is non-decreasing on a given interval $[a,b]$, $0 \leq a < b$, is zero.

Exercise 12. Let $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$ be two independent Brownian motions. Show that $(B_t = 2^{-1/2}(B_t^1 + B_t^2))_{t\geq 0}$ is a Brownian motion.