Université de Nice - M2 Mathmods - Stochastic Calculus

EXERCISES 3

WIENER'S INTEGRAL

In all the exercises, $(\Omega, \mathcal{A}, \mathbb{P})$ *denotes the current probability space and* $(B_t)_{t\geq 0}$ *a (real) Brownian motion.*

Exercise 1. Give the law of the random variable

$$Y = \int_0^{+\infty} \exp(-s) dB_s.$$

Check first that Y is well-defined.

Exercise 2. Find two admissible functions f and g such that $f \leq g$ and

$$\mathbb{P}\left\{ \int_0^1 f(s) dB_s > \int_0^1 g(s) dB_s \right\} > 0.$$

Exercise 3. Let f be an admissible function. Show that the process $(\int_0^t f_s dB_s)_{t\geq 0}$ is a Gaussian process. Compute its mean and its covariance.

Exercise 4. Let $(X_t)_{t\geq 0}$ be given by:

$$\forall t \geq 0,\ X_t = \int_0^{t^{1/2}} (2s)^{1/2} dB_s.$$

Show that it is Gaussian. Compute its mean and its covariance. Deduce that X is a Brownian motion.

Exercise 5. Let V_0 be a random variable independent of B and of Gaussian law $\mathcal{N}(0, 1/2)$. We define the process $(V_t)_{t\geq 0}$ (so-called Ornstein-Uhlenbeck stationary process) by:

$$\forall t \geq 0,\ V_t = \exp(-t) V_0 + \int_0^t \exp[-(t-s)] dB_s.$$

Show that it is a Gaussian process. For any $a > 0$, show that $(V_{a+t})_{t\geq 0}$ and $(V_t)_{t\geq 0}$ have the same distribution.