Université de Nice - M2 Mathmods - Stochastic Calculus

EXERCISES 4

WIENER'S INTEGRAL – ITO'S INTEGRAL – MARTINGALES

In all the exercises, $(\Omega, \mathcal{A}, \mathbb{P})$ denotes the current probability space and $(B_t)_{t\geq 0}$ a (real) Brownian motion.

Exercise 1. *A player begin a play with M_0 €. At each step, he wins 1€ with probability p and loses 1€ with probability $1-p$ ($p \in [0,1]$, $p \neq 1/2$). He stops the play when his wealth reaches a value M (M fixed $> M_0$) or reaches 0.*

(1) *We call S_n the wealth of the player after n steps of the game (we have $S_0 = M_0$). Write S_n as a sum of i.i.d. variables.*
(2) *Let $T = \inf\{n \geq 1 : S_n \in \{M, 0\}\}$. Show that T is a stopping time (relatively to the filtration $(\mathcal{F}_n = \sigma(S_0, \dots, S_n))_{n\geq 0}$).*
(3) *Find $\alpha \neq 0$ such that $(e^{\alpha S_n})_{n\geq 0}$ is a martingale. Show that $\forall n$, $\mathbb{E}(e^{\alpha S_{T\wedge n}}) = \mathbb{E}(e^{\alpha S_0})$.*
(4) *We set $q = \mathbb{P}(S_T = M)$. Find β such that $(S_n - \beta n)_{n\geq 0}$ is a martingale. Show that $\forall n$, $\mathbb{E}(S_{T\wedge n} - \beta(n \wedge T)) = \mathbb{E}(S_0)$.*
(5) *Show that $\mathbb{E}(T) < \infty$.*
(6) *Show that $\mathbb{E}(e^{\alpha S_T}) = \mathbb{E}(e^{\alpha S_0})$. Find $q = \mathbb{P}(S_T = M)$.*

Exercise 2. *We suppose we are in the same situation as in the Exercise above but with $p = 1/2$. Find $q = \mathbb{P}(S_T = M)$ (hint: show that $S_n^2 - n$ is a martingale).*

Exercise 3. *A player plays a game where, when you bet k€ , you win keuro with probability p (meaning you get back your k€ and you win an additional k€) or you loose k€ with probability $1-p$ ($p \in (0,1]$). The player is interested in his total gain. He starts with a gain $= 0$. He wants to reach the gain $+1$ using the following strategy.*

- *He bets 1 and if he wins, he stops; if not, he carries on to the next step.*
- *He bets 2 and if he wins, he stops; if not, he carries on to the next step.*
- *He bets 4 and if he wins, he stops; if not, he carries on to the next step.*
- *... etc ...*

(1) *Let $T = \inf\{n : G_n = 1\}$. Show that T is a stopping time relatively to the filtration $(\sigma(G_0, \dots, G_n))_{n\geq 0}$.*
(2) *Express the gain G_n after the n-th step using i.i.d. variables.*
(3) *Show that $T < +\infty$ a.s.*
(4) *The more the player looses, the more he needs on his bank account to keep on playing. The sum he will need during one game is $-\min_{n\in\{0,\dots,T-1\}}(G_n) + 2^n$. Compute $\mathbb{E}(\min_{n\in\{0,\dots,T\}} G_n)$.*
(5) *Suppose your start the game with $2^N - 1$€. You stop when your gain reaches -2^N or 1. What is the expectation of your final gain?*

Exercise 4. Let f be a deterministic locally admissible function.

(1) Show that

$$\forall t \geq 0,\ \mathbb{E}\left[\exp\left(\int_0^t f_s dB_s\right)\right] = \exp\left(\frac{1}{2}\int_0^t f_s^2 ds\right).$$

(2) Show that the process

$$\left(\exp\left(\int_0^t f_s dB_s - \frac{1}{2}\int_0^t f_s^2 ds\right)\right)_{t\geq 0}$$

is a martingale with respect to the natural filtration of B.

Exercise 5. Show that $(B_t^2 - t)_{t\geq 0}$ is a martingale. (With respect to the natural filtration of B.)

Exercise 6. Let $T > 0$. Show that

$$\lim_{n\to+\infty} \mathbb{E}\left[\left(\sum_{i=1}^n (B_{Ti/n} - B_{T(i-1)/n})^2 - T\right)^2\right] = 0.$$

Exercise 7. Let $T > 0$. Show that

$$\int_0^T (1 + \frac{B_t}{n})^n dB_t \underset{L^2}{\to} \int_0^T \exp(B_t) dB_t,$$

as $n \to +\infty$. Check first that the integrals are well-defined.

Exercise 8. Let $T > 0$. For a given $n \geq 1$, we define the process

$$\forall n \geq 0, \forall t \geq 0,\ B_t^n = \sum_{i=0}^{n-1} B_{Ti/n} \mathbf{1}_{(Ti/n, T(i+1)/n]}(t).$$

(1) Prove that $(B_t^n)_{t\geq 0}$ is a simple process w.r.t. the filtration generated by B.
(2) Show that

$$\lim_{n\to+\infty} \mathbb{E}\int_0^T |B_t^n - B_t|^2 dt = 0.$$

(3) What is the limit, in $L^2(\Omega)$, of

$$\left(\int_0^T B_t^n dB_t\right)_{n\geq 1}?$$

(4) Prove that

$$B_T^2 = 2\int_0^T B_t^n dB_t + \sum_{i=1}^n \left(B_{Ti/n} - B_{T(i-1)/n}\right)^2$$

(5) By the previous exercise, deduce

$$B_T^2 = 2\int_0^T B_t dB_t + T.$$