Université de Nice - M2 Mathmods - Stochastic Calculus

EXERCISES 5

CONDITIONAL EXPECTATIONS

In all the exercises, $(\Omega, \mathcal{A}, \mathbb{P})$ *denotes the current probability space and* $(B_t)_{t\geq 0}$ *a (real) Brownian motion.*

Exercise 1. Let $\mathcal{G}$ and $\mathcal{F}$ be two σ-subalgebras of $\mathcal{A}$, with $\mathcal{G} \subset \mathcal{F}$. Let X be an integrable random variable. Prove that

$$\mathbb{P}\text{−a.s.}, \ \mathbb{E}\big[\mathbb{E}[X|\mathcal{F}]|\mathcal{G}\big] = \mathbb{E}[X|\mathcal{G}].$$

Exercise 2. Let $\mathcal{B}$ a σ-field of $\mathcal{A}$, X and Y be two r.v., and $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded Borel mapping. We assume that X is $\mathcal{B}$-measurable and that Y is independent of $\mathcal{B}$.

Prove that:

$$\mathbb{P}\text{−a.s.}, \ \mathbb{E}\big[f(X, Y)|\mathcal{B}\big] = \phi(X),$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ is given by:

$$\forall x \in \mathbb{R}^n, \ \phi(x) = \mathbb{E}[f(x, Y)].$$

Exercise 3. Let X and Y be two independent r.v. of uniform law on $[0, 1]$. We set $U = \inf(X, Y)$ and $V = \sup(X, Y)$. What is $\mathbb{E}(U|V)$?

Exercise 4. Show that $(B_t^4 - 6tB_t^2 + 3t^2)_{t\geq 0}$ is a martingale w.r.t. to the filtration $(\sigma(B_s, s \leq t))_{t\geq 0}$.