EXERCISES 8

ITO'S FORMULA – GIRSANOV THEOREM

In all the exercises, $(\Omega, \mathcal{A}, \mathbb{P})$ denotes the current probability space and $(B_t)_{t\geq 0}$ a (real) Brownian motion.

Exercise 1. Let $(B_t^1, B_t^2, B_t^3)_{t\geq 0}$ be a three dimensional Brownian motion w.r.t. some filtration $(\mathcal{F}_t)_{t\geq 0}$. For a given vector $(b_1, b_2, b_3) \in \mathbb{R}^3$, we consider the process

$$\forall t \geq 0, \quad X_t = \exp\Big(\sum_{i=1}^{3} b_i B_t^i - \frac{1}{2}\sum_{i=1}^{3} b_i^2 t\Big).$$

(1) Prove that $(X_t)_{t\geq 0}$ is a square integrable martingale.
(2) Prove that the process $((B_t^1 + B_t^2 - (b_1 + b_2)t)X_t)_{t\geq 0}$ is also a martingale.

Exercise 2. Let $(B_t^1, B_t^2, B_t^3)_{t\geq 0}$ be a three dimensional Brownian motion w.r.t. some filtration $(\mathcal{F}_t)_{t\geq 0}$. For a given matrix σ of size 3×3, we consider the process

$$\forall t \geq 0, \quad X_t = \sigma \begin{pmatrix} B_t^1 \\ B_t^2 B_t^3 \end{pmatrix}.$$

Show that the process $(M_t = \sum_{i=1}^{3} (X_t^i)^2 - \text{Trace}(\sigma\sigma^*)t)_{t\geq 0}$ is a martingale.

Exercise 3. Let $(B_t)_{t\geq 0}$ be a Brownian and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration. For $\mu \in \mathbb{R}$ et $\sigma > 0$, we set

$$\forall t \geq 0, \ Y_t = \exp\big(\mu t + \sigma B_t\big),$$

referred as Geometric Brownian motion.

(1) We set $r = \mu + \sigma^2/2$ and we define

$$\forall t \geq 0, \ \widetilde{B}_t = B_t + \sigma^{-1} r t.$$

What can you say of $(\widetilde{B}_t)_{0\leq t\leq 1}$ under the probability:

$$\forall A \in \mathcal{A}, \ \mathbb{Q}(A) = \mathbb{E}\big[\exp\big(-\sigma^{-1} r B_1 - \frac{1}{2}\sigma^{-2} r^2\big)\mathbf{1}_A\big].$$

(2) Show that $(Y_t)_{0\leq t\leq 1}$ is, under the probability $\mathbb{Q}$, a martingale w.r.t. $(\mathcal{F}_t)_{0\leq t\leq 1}$.

Exercise 4. Let $(B_t)_{t\geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration, show that we can define a probabilty $\mathbb{Q}_1$ on $(\Omega, \mathcal{F}_1)$, equivalent to $\mathbb{P}$, such that $(B_t + B_t^3)_{0\leq t\leq 1}$ be a martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ under $\mathbb{Q}_1$. Hint: apply first Itô's formula to $(B_t + B_t^3)_{t\geq 0}$.

Exercise 5. Let $(B_t)_{t\geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration, show that we can define a probabilty $\mathbb{Q}_1$ on $(\Omega, \mathcal{F}_1)$, equivalent to $\mathbb{P}$, such that $((2+B_t^2)\exp(B_t))_{0\leq t\leq 1}$ be a martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ under $\mathbb{Q}_1$. Hint: apply first Itô's formula to $((2 + B_t^2)\exp(B_t))_{t\geq 0}$.