Université Nice Sophia Antipolis **Exercises 2**

Stochastic calculus

Exercice 1 Let $(Y_n)_{n\geq 1}$ be an i.i.d sequence of uniform r.v. on $[0,1]$. We set $\mathcal{F}_n = \sigma(Y_1,...,Y_n)$, and we define the process $(X_n)_{n\in\mathbb{N}}$ by $X_0 = 1$ and

$$X_{n+1} = \begin{cases} X_n + 1 & \text{if } 0 \leq Y_{n+1} < \frac{X_n}{X_n+1} \\ \\ 0 & \text{if} \frac{X_n}{X_n+1} \leq Y_{n+1} \leq 1 \end{cases}$$

1. Show X_n converges almost surely towards an r.v. X_∞.
2. Show that the process $(X_n)_{n\in\mathbb{N}}$ is a $(\mathcal{F}_n)-$martingale.
3. Do we have $X_n = \mathbb{E}[X_\infty|\mathcal{F}_n]$? Is the martingale $(X_n)_{n\in\mathbb{N}}$ uniformly integrable ?
4. Let $T := \inf\{n \geq 0, X_n = 0\}$. Show that T is a stopping and that T is almost surely finite. Do we have $\mathbb{E}[X_T] = \mathbb{E}[X_0]$?

Exercice 2 Let (X_n) be a submartingale and $a > 0$.

1. Show the *maximal inequality*

$$a\,\mathbb{P}(\sup_{0\leq k\leq n} X_k \geq a) \leq \mathbb{E}[X_n \mathbb{1}_{\{\sup_{0\leq k\leq n} X_k \geq a\}}] \leq \mathbb{E}[X_n^+]\,.$$

   Hint : use the decomposition $\{\sup_{0\leq k\leq n} X_k \geq a\} = F_0 \cup F_1 \cup \cdots \cup F_n$ with $F_k = \{X_0 < a\} \cap \cdots \cap \{X_{k-1} < a\} \cap \{X_k \geq a\}$.

2. Show that :

$$a\,\mathbb{P}(\sup_{0\leq k\leq n} |X_k| \geq a) \leq 2\mathbb{E}[|X_n|] - \mathbb{E}[X_0] \leq 2\mathbb{E}[|X_n|] + \mathbb{E}[|X_0|]\,.$$

   Hint : use the stopping time $T = \inf\{k, X_k \leq -a\}$.

Exercice 3 Let (X_n) be a martingale.

1. For Z nonnegative r.v. and $p \geq 1$, show that

$$\mathbb{E}[Z^p] = \int_0^\infty p a^{p-1} \mathbb{P}(Z \geq a) \mathrm{d}a.$$

2. We set $S_n := \sup_{0\leq k\leq n} |X_k|$. Let $p > 1$ and $q = \frac{p}{p-1}$. Show that $(|X_n|)$ is a submartingale, then that

$$\mathbb{E}\left[S_n^p\right] \leq q\,\mathbb{E}\left[|X_n|\,S_n^{p-1}\right]\,.$$

3. Show *Doob's inequality* :

$$\|S_n\|_p \leq q\|X_n\|_p$$