EXERCISES 8

SDES

In all the exercises, $(\Omega, \mathcal{A}, \mathbb{P})$ denotes the current probability space and $(B_t)_{t\geq 0}$ a (real) Brownian motion.

Exercise 1. For some point $x \in \mathbb{R}$, we consider the SDE:

$$\forall t \geq 0,\ X_t = x + \int_0^t bX_s ds + \int_0^t \sigma X_s dB_s.$$

(1) Show that the SDE admits a unique solution.
(2) Show that

$$X_t = x \exp\big((b - \frac{\sigma^2}{2})t + \sigma B_t\big).$$

Exercise 2. For some point $x \in \mathbb{R}$, we consider the SDE:

$$\forall t \geq 0,\ X_t = x + \int_0^t bX_s ds + B_t.$$

(1) Show that the SDE admits a unique solution.
(2) Show that

$$X_t = \exp(bt)\big(x + \int_0^t \exp(-bs) dB_s\big).$$

Exercise 3. For some point $x \in \mathbb{R}$, we consider the SDE:

$$\forall t \geq 0,\ X_t = x + \int_0^t \sin(2X_s) ds + \int_0^t \big(1 + \cos^2(X_s)\big)^{1/2} dB_s.$$

(1) Show that the SDE admits a unique solution.
(2) Show that $(\phi(X_t))_{t\geq 0}$ is a martingale, where

$$\phi(x) = \int_0^x \big(1 + \cos^2(u)\big)^2 du.$$

Exercise 4. We consider the SDE

$$\forall t \geq 0,\ dX_t = -\frac{\sin(t)}{2 + \cos(t)} X_t dt + (2 + \cos(t)) dB_t.$$

(1) Show that the SDE admits a unique solution.
(2) Show that it is given by

$$\forall t \geq 0,\ X_t = (2 + \cos(t)) B_t.$$

Exercise 5. Let $x \in \mathbb{R}$.

(1) Show that the SDE

$$\forall t \geq 0,\ X_t = x + \int_0^t sX_s ds + \int_0^t \exp(s) dB_s,$$

admits a unique solution.

(2) Let y be the solution of the ordinary differential equation:

$$y_t = 1 + \int_0^t s y_s ds.$$

Give the form of y.

(3) Show that $Z = y^{-1} X$ is an Itô process and compute its infinitesimal variation.

(4) Show that the solution of the SDE is given by

$$\forall t \geq 0,\ X_t = \exp\bigl(t^2/2\bigr)\left[x + \int_0^t \exp(s - s^2/2) dB_s\right].$$

(5) Deduce that X is a Gaussian process. Give its law.

Exercise 6. Let $x \in \mathbb{R}$. We consider the SDE

$$(\star) \qquad \forall t \geq 0,\ X_t = x + \int_0^t \cos(s) ds + \int_0^t s^{1/2} X_s dB_s.$$

(1) Show that the process M given, for any $t \geq 0$, by $M_t = \exp(-\int_0^t s^{1/2} dB_s - t^2/4)$ is a solution of the SDE

$$\forall t \geq 0,\ M_t = 1 - \int_0^t s^{1/2} M_s dB_s.$$

(2) Show that $(\star)$ admits a unique solution, denoted by $(X_t)_{t \geq 0}$.

(3) Show that

$$\forall t \geq 0,\ \exp\bigl(t^2/2\bigr) X_t M_t = x + \int_0^t \left[\exp(s^2/2) \cos(s) M_s\right] ds$$

(4) Deduce that

$$\forall t \geq 0,\ X_t = \exp\left(\int_0^t s^{1/2} dB_s - t^2/4\right)\left[x + \int_0^t \left(\exp\left(-\int_0^s r^{1/2} dB_r + s^2/4\right) \cos(s)\right) ds\right].$$

Exercise 7. We here consider the two-dimensional SDE

$$\forall t \geq 0, \quad \begin{cases} X_t = 1 - \dfrac{1}{2}\displaystyle\int_0^t X_s ds - \int_0^t Y_s dB_s, \\ Y_t = -\dfrac{1}{2}\displaystyle\int_0^t Y_s ds + \int_0^t X_s dB_s. \end{cases}$$

(1) Show that there is a unique solution. (We admit that the existence and uniqueness theorem is still true.)

(2) By Itô's formula, show, that for every $t \geq 0$, $X_t^2 + Y_t^2 = 1$.

(3) It is thus natural to seek for a solution as $(X_t, Y_t) = (\cos(\theta_t), \sin(\theta_t))$, where θ is an Itô process.

Give the necessary form of $d\theta_t$.

(4) Deduce the solution of the equation.