Université de Nice - M2 Mathmods - Stochastic Calculus

# EXERCISES 7

## ITO'S FORMULA AND GIRSANOV FORMULA

**Exercise 1.** Let $(B_t)_{t\geq 0}$ be a Brownian motion w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ and $(u_t)_{t\geq 0}$ be a continuous and $(\mathcal{F}_t)_{t\geq 0}$-adapted process such that

$$\forall t \geq 0,\ \forall \omega \in \Omega,\ |u_t(\omega)| \leq K,$$

for some constant $K > 0$. We admit the following inequality $\mathbb{E}\Big[\exp\Big(\int_0^t u_s dB_s\Big)\Big] \leq \exp(K^2 t/2)$.

(1) Show that the process $\forall t \geq 0,\ M_t = \exp\Big(\int_0^t u_s dB_s - \frac{1}{2}\int_0^t u_s^2 ds\Big)$, is a martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$. (Use Itô's formula.)

(2) We set $\forall t \geq 0,\ Y_t = -\int_0^t u_s ds + B_t$. Show that the process $(Y_t M_t)_{t\geq 0}$ is a martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$.

**Exercise 2.** Expand as an Itô process the process

$$\forall t \geq 0,\ X_t = (B_t^1)^2 + (B_t^2)^2 + (B_t^3)^2,$$

where $(B_t^1, B_t^2, B_t^3)_{t\geq 0}$ stands for a Brownian motion of dimension 3.

**Exercise 3.** Let $(B_t = (B_t^1, B_t^2))_{t\geq 0}$ be two independent Brownian motion w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ and $(u_t)_{t\geq 0}$ and $(v_t)_{t\geq 0}$ be two continuous and $(\mathcal{F}_t)_{t\geq 0}$-adapted processes, bounded by some constant $K$. Show that

$$\forall t \geq 0, \quad M_t = \exp\Big(\int_0^t u_s dB_s^1 + \int_0^t v_s dB_s^2 - \frac{1}{2}\int_0^t (u_s^2 + v_s^2) ds\Big),$$

is a martingale.

**Exercise 4.** Let $(B_t)_{t\geq 0}$ be a Brownian motion w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ and $(u_t)_{t\geq 0}$ and $(v_t)_{t\geq 0}$ ne two continuous and adapted processes such that

$$\forall t \geq 0, \quad \mathbb{E}\int_0^t (u_s^4 + v_s^4) ds < +\infty,$$

show that

$$\Big(\Big(\int_0^t u_s dB_s\Big)\Big(\int_0^t v_s dB_s\Big) - \int_0^t u_s v_s ds\Big)_{t\geq 0}$$

is a martingale.

**Exercise 5.** Let $(B_t^1, B_t^2)_{t\geq 0}$ be a Brownian motion with values in $\mathbb{R}^2$. We assume that there exists a function $u$ in $\mathcal{C}^{1,2}([0,+\infty) \times \mathbb{R}^2)$, bounded with bounded derivatives, such that

$$(\star) \qquad \forall (t,x) \in [0,+\infty[\times \mathbb{R}^2,\ \frac{\partial u}{\partial t}(t,x) - \frac{1}{2}\Big(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\Big)(t,x) = 0\ ,\ u(0,x) = h(x)\ .$$

(1) Show that for any $T > 0$ any $x \in \mathbb{R}$, the process $(u(T-t, x+B_t))_{0\leq t\leq T}$ is a martingale.

(2) Deduce that $u(T,x) = \mathbb{E}(h(x+B_T))$.

(3) Deduce that $(\star)$ admits at most one solution $\mathcal{C}^{1,2}$, bounded with bounded derivatives, with $u(0,\cdot)$ as initial condition.

**Exercise 6.** Let $f$ be a deterministic locally admissible function.

(1) Show that

$$\forall t \geq 0,\ \mathbb{E}\Big[\exp\Big(\int_0^t f_s dB_s\Big)\Big] = \exp\Big(\frac{1}{2}\int_0^t f_s^2 ds\Big).$$

(2) Show that the process

$$\Big(\exp\Big(\int_0^t f_s dB_s - \frac{1}{2}\int_0^t f_s^2 ds\Big)\Big)_{t\geq 0}$$

is a martingale with respect to the natural filtration of $B$.

**Exercise 7.** Let $(B_t^1, B_t^2, B_t^3)_{t\geq 0}$ be a three dimensional Brownian motion w.r.t. some filtration $(\mathcal{F}_t)_{t\geq 0}$. For a given vector $(b_1, b_2, b_3) \in \mathbb{R}^3$, we consider the process

$$\forall t \geq 0, \quad X_t = \exp\big(\sum_{i=1}^3 b_i B_t^i - \frac{1}{2}\sum_{i=1}^3 b_i^2 t\big).$$

(1) Prove that $(X_t)_{t\geq 0}$ is a square integrable martingale.
(2) Prove that the process $((B_t^1 + B_t^2 - (b_1+b_2)t)X_t)_{t\geq 0}$ is also a martingale.

**Exercise 8.** Let $(B_t^1, B_t^2, B_t^3)_{t\geq 0}$ be a three dimensional Brownian motion w.r.t. some filtration $(\mathcal{F}_t)_{t\geq 0}$. For a given matrix $\sigma$ of size $3\times 3$, we consider the process

$$\forall t \geq 0, \quad X_t = \sigma \times \begin{pmatrix} B_t^1 \\ B_t^2 \\ B_t^3 \end{pmatrix}.$$

Show that the process $(M_t = \sum_{i=1}^3 (X_t^i)^2 - \mathrm{Trace}(\sigma\sigma^*)t)_{t\geq 0}$ is a martingale.

**Exercise 9.** Let $(B_t)_{t\geq 0}$ be a Brownian and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration. For $\mu \in \mathbb{R}$ et $\sigma > 0$, we set

$$\forall t \geq 0,\ Y_t = \exp\big(\mu t + \sigma B_t\big),$$

referred as Geometric Brownian motion.

(1) We set $r = \mu + \sigma^2/2$ and we define $\forall t \geq 0,\ \widetilde{B}_t = B_t + \sigma^{-1} r t$. What can you say of $(\widetilde{B}_t)_{0\leq t\leq 1}$ under the probability:

$$\forall A \in \mathcal{A},\ \mathbb{Q}(A) = \mathbb{E}\big[\exp\big(-\sigma^{-1} r B_1 - \frac{1}{2}\sigma^{-2} r^2\big)\mathbf{1}_A\big].$$

(2) Show that $(Y_t)_{0\leq t\leq 1}$ is, under the probability $\mathbb{Q}$, a martingale w.r.t. $(\mathcal{F}_t)_{0\leq t\leq 1}$.

**Exercise 10.** Let $(B_t)_{t\geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration, show that we can define a probabilty $\mathbb{Q}_1$ on $(\Omega, \mathcal{F}_1)$, equivalent to $\mathbb{P}$, such that $(B_t + B_t^3)_{0\leq t\leq 1}$ $(B_t^3 = (B_t)^3)$ be a martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ under $\mathbb{Q}_1$. Hint: apply first Itô's formula to $(B_t + B_t^3)_{t\geq 0}$.

**Exercise 11.** Let $(B_t)_{t\geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration, show that we can define a probabilty $\mathbb{Q}_1$ on $(\Omega, \mathcal{F}_1)$, equivalent to $\mathbb{P}$, such that $((2 + B_t^2)\exp(B_t))_{0\leq t\leq 1}$ $(B_t^2 = (B_t)^2)$ be a martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ under $\mathbb{Q}_1$. Hint: apply first Itô's formula to $((2 + B_t^2)\exp(B_t))_{t\geq 0}$.