Université Nice-Sophia Antipolis
SMEMP302 - ECUE Probabilitic Computational Methods, 2022-2023
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# Final exam (duration : 2h)

*Authorized documents: course notes only.*

**Exercise 1.** We want to compute $p_l = \mathbb{P}(X \in [l, l+1])$ where $X$ has an exponential law of parameter 1 and $l \geq 0$.

(1)
  (a) Propose a Monte-Carlo method to compute $p_l$ (based the simulation of $M$ variables of exponential law).
  (b) Compute the asymptotic variance of this method (when $M \to +\infty$).

(2)
  (a) Propose an importance sampling method.
  (b) Compute the asymptotic variance of the new method.

(3) Compare the two variances computed above when $l \to +\infty$.

**Exercise 2.** One considers the geometric Brownian motion $X_t = e^{-\frac{t}{2}+W_t}$ ($(W_t)$ being a standard Brownian motion). The process $(X_t)$ is solution to

$$dX_t = X_t dW_t \, , \, X_0 = 1 \, .$$

For all $n \geq 1$, we introduce the Euler scheme (of order $n$) associated to the above equation: $(\overline{X}_{t_k^n})_{k\geq 0}$ on the interval $[0,T]$ ($T = 1$, $t_{k+1}^n - t_k^n = \frac{1}{n}$ for all $k$). Show that, for every $n \geq 1$ and every $k \geq 1$,

$$\overline{X}_{t_k^n} = \prod_{l=1}^{k} (1 + \Delta W_{t_l^n}) \, ,$$

where $t_l^n = \frac{lT}{n}$, $\Delta W_{t_l^n} = W_{t_l^n} - W_{t_{l-1}^n}$ ($l \geq 1$).

**Exercise 3.** We are interested in the computation of $\mathbb{E}(f(X_T))$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a $\mathcal{C}^4$ and Lipschitz function with derivatives "at most polynomial" and $(X_t)_{t\in[0,T]}$ is solution of the following EDS

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt \, , \, X_0 = x_0 \, ,$$

where $(W_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^d$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ and $b : \mathbb{R}^n \to \mathbb{R}^n$ are $\mathcal{C}^4$ functions with bounded derivatives and $x_0 \in \mathbb{R}^n$. We write $\overline{X}_T^N$ the approximation of $X_T$ obtained by a Euler scheme with $N$ discretization steps and whose computational time is proportional to $N$. We suppose the strong and weak rate of convergence are the following ($\alpha, \beta \geq 1$):

$$\exists C < \infty \, , \, \forall N \in \mathbb{N}^* \, , \, \mathbb{E}(|X_T - \overline{X}_T^N|^2) \leq \frac{C}{N^\alpha} \, , \, |\mathbb{E}(f(X_T)) - \mathbb{E}(f(\overline{X}_T^N))| \leq \frac{C}{N^\beta} \, .$$

(1) Let $Y$ be an estimator of $\mathbb{E}(f(X_T))$ such that $\mathbb{E}(Y^2) < \infty$ (remember this means that $Y$ can be just any random variable). Show the bias/variance decomposition

$$\mathbb{E}[(\mathbb{E}(f(X_T)) - Y)^2] = (\mathbb{E}(f(X_T)) - \mathbb{E}(Y))^2 + \text{Var}(Y)$$

of the quadratic error.

(2) We suppose that $Y = \frac{1}{M} \sum_{i=1}^{M} f(\overline{X}_T^{i,N})$ is the empirical mean of $M$ independent copies of $f(\overline{X}_T^N)$.
  (a) How many steps $N$ should we choose in order to have $[\mathbb{E}(f(X_T)) - \mathbb{E}(Y)]^2$ of size $\epsilon$, where $\epsilon$ is a fixed precision level (small)?

(b) Show that $\lim_{N\to+\infty} \mathbb{E}[(f(X_T) - f(\overline{X}_T^N))^2] = 0$. Deduce from this that

$$\lim_{N\to+\infty} \mathrm{Var}(f(\overline{X}_T^N)) = \mathrm{Var}(f(X_T))\,.$$

We suppose $\mathrm{Var}(f(X_T))$ is a known constant. How many copies $M$ should we choose in order to have $\mathrm{Var}(Y) = \epsilon^2$ (approximatively, for $N$ very big)?

(c) Conclude that the computational time we need to attain the quadratic error $2\epsilon^2$ is proportional to $\epsilon^{-(2+1/\beta)}$.