Université Nice-Sophia Antipolis
SMEMP302 - ECUE Probabilitic computational methods, 2022-2023
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Home Project

We consider on the one hand the sequence of random variables recursively defined by

$$Y_{k+1} = Y_k(1+\mu\Delta) + \sigma\sqrt{\Delta}Z_{k+1}\,,\ k \geq 0\,,\ Y_0 = 0\,,$$

where $\mu > 0$, $1 \geq \Delta > 0$, are positive real numbers, and the Ornstein-Uhlenbeck process solution to the SDE

$$dX_t = \mu X_t dt + \sigma dW_t\,,\ X_0 = x\,, \tag{0.1}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian Motion in $\mathbb{R}$. The exact strong solution of Equation (0.1) is:

$$X_t = xe^{\mu t} + \sigma\int_0^t e^{\mu(t-s)}dW_s\,. \tag{0.2}$$

Set $t_k = k\Delta$, $k \geq 0$ and $Z_k = \frac{W_{t_k}-W_{t_{k-1}}}{\sqrt{\Delta}}$, $k \geq 1$.

(1) Show that, for every $k > 0$,

$$\mathbb{E}(Y_k^2) = \frac{\sigma^2}{\mu} \times \frac{(1+\mu\Delta)^{2k}-1}{2+\mu\Delta}\,.$$

(2) Show that, for every $k \geq 0$, $X_{t_{k+1}} = e^{\mu\Delta}X_{t_k} + \sigma e^{\mu t_{k+1}}\int_{t_k}^{t_{k+1}} e^{-\mu s}dW_s$.

(3) Show that, for every $k \geq 0$, for every $\Delta > 0$,

$$\begin{aligned}\mathbb{E}((Y_{k+1} - X_{t_{k+1}})^2 &\leq (1+\Delta)e^{2\mu\Delta}\mathbb{E}((Y_k - X_{t_k})^2) \\ &\quad + \left(1+\frac{1}{\Delta}\right)\mathbb{E}((Y_k)^2)(e^{\mu\Delta}-1-\mu\Delta)^2 + \sigma^2\int_0^\Delta (e^{\mu u}-1)^2du\,.\end{aligned}$$

(Hint: show that for all $a, b \geq 0$, $2ab \leq \Delta a^2 + b^2/\Delta$.) In what follows, we assume that $\Delta = \Delta_n = T/n$ where T is a positive real number and n is in $\mathbb{N}^*$ (n may vary). However, we keep on using the notation Y_k rather then $Y_k^{(n)}$.

(4) Show that, for every $k \in \{0,\dots,n\}$, $\mathbb{E}((Y_k)^2) \leq \frac{\sigma^2 e^{2\mu T}}{2\mu}$.

(5) Deduce the existence of a real constant $C = C_{\mu,\sigma,T} > 0$ such that, for every $k \in \{0,1,\dots,n-1\}$,

$$\mathbb{E}((Y_{k+1} - X_{t_{k+1}})^2) \leq (1+\Delta_n)e^{2\mu\Delta_n}\mathbb{E}((Y_k - X_{t_k})^2) + C\Delta_n^3\,.$$

We suppose that $\Delta_n < 1$. Conclude that $\mathbb{E}((Y_{k+1} - X_{t_{k+1}})^2) \leq C'\left(\frac{T}{n}\right)^2$ for some real constant $C' = C'_{\mu,\sigma,T}$.

(6) From now, we take $\mu = -1$, $\sigma = 1$, $T = 1$, $x = 1$. For all function f , we set

$$\overline{f(Y_n)}^M := \frac{1}{M}\sum_{j=1}^M f(Y_n^j)$$

)(where $(Y_n^j)_{j\geq 1}$ are i.i.d. copies of Y_n). We are interested in the error $\mathbb{E}(f(X_T)) - \mathbb{E}(f(\overline{Y}^M))$ for some Lipshitz function f (Lipshitz constant named f_{Lip}). This error can be decomposed into

$$\begin{aligned}|\mathbb{E}(f(X_T)) - \overline{f(Y_n)}^M| &\leq |\mathbb{E}(f(X_T) - f(Y_n))| + \left|f(Y_n) - \overline{f(Y_n)}^M\right| \\ &\leq f_{\text{Lip}}^{1/2}\mathbb{E}((X_T - Y_n)^2)^{1/2} + \left|f(Y_n) - \overline{f(Y_n)}^M\right|\end{aligned}$$

(We admit this computation .) For a fixed n, we will choose $M = M_n = n^2$ so that the two terms in the error above are of the same order. Write a `python` function that simulate a variable Y_n^j for any given n and plot the trajectory $(Y_n^j)_{0\leq j\leq n}$ (add the plot for $n = 1000$ in your report).

(7) Write a `python` function that compute $\overline{f(Y_n)}^{M_n}$ for any given n.

(8) Write a `python` code that compute a Monte-Carlo approximation of $|\mathbb{E}(f(X_T)) - \overline{f(Y_n)}^{M_n}|$ for any given n (and $f = \mathrm{Id}$).

(9) Repeat the above code for various n's and plot $|\mathbb{E}(f(X_T)) - \overline{f(Y_n)}^{M_n}|$ vs n, in log-log scale (and $f = \mathrm{Id}$).

(10) Use a linear regression to find the slope of the above plot (the result does not have to be very precise).