Product of neighborhood frames with additional modality

Rajab Aghamov [1] Andrey Kudinov [1 2 3]

[1]Higher School of Economics

[2]Institute for Information Transmission Problems

[3]Moscow Institute of Physics and Technology

June 17, 2019

Introduction and history: semantics

Semantics for modal logic

Topological semantics

- A. Tarski (1938)
- J. C. C. McKinsey and A. Tarski (1944)

Kripke semantics

- S. Kripke (1963)

Neighborhood semantics

- D. Scott (1970)
- R. Montague (1970)

Introduction and history: products

Product of Kripke frames

- V.Shehtman (1978) [in russian]
- D. Gabbay and V. Shehtman (1998)

Product of topological spaces.

- J. van Benthem et al. (2006)

Product of neighborhood frames.

- K. Sano (2011)

For logics L_1 and L_2 we define

- $L_1 \times L_2$ is the logic of products of L_1- and L_2- Kripke frames.
- $L_1 \times_t L_2$ is the logic of products of L_1- and L_2- topological spaces.
- $L_1 \times_n L_2$ is the logic of products of L_1- and L_2- neighbourhood frames.

Introduction and history: products

Product of Kripke frames

- V.Shehtman (1978) [in russian]
- D. Gabbay and V. Shehtman (1998)

Product of topological spaces.

- J. van Benthem et al. (2006)

Product of neighborhood frames.

- K. Sano (2011)

For logics L_1 and L_2 we define

- $L_1 \times L_2$ is the logic of products of L_1- and L_2- Kripke frames.
- $L_1 \times_t L_2$ is the logic of products of L_1- and L_2- topological spaces.
- $L_1 \times_n L_2$ is the logic of products of L_1- and L_2- neighbourhood frames.

Introduction and history: products

Product of Kripke frames

- V.Shehtman (1978) [in russian]
- D. Gabbay and V. Shehtman (1998)

Product of topological spaces.

- J. van Benthem et al. (2006)

Product of neighborhood frames.

- K. Sano (2011)

For logics L_1 and L_2 we define

- $L_1 \times L_2$ is the logic of products of L_1- and L_2- Kripke frames.
- $L_1 \times_t L_2$ is the logic of products of L_1- and L_2- topological spaces.
- $L_1 \times_n L_2$ is the logic of products of L_1- and L_2- neighbourhood frames.

The product of topological spaces

(van Benthem et al, 2006)
For two topological space $\mathfrak{X}_1 = (X_1, T_1)$ and $\mathfrak{X}_2 = (X_2, T_2)$

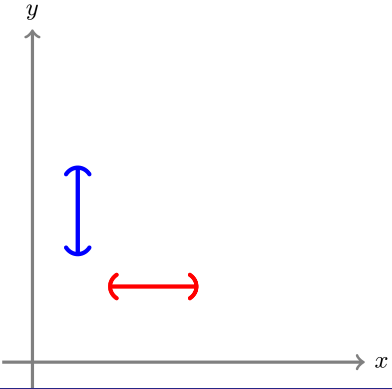
$$\mathfrak{X}_1 \times \mathfrak{X}_2 = (X_1 \times X_2, T_1^*, T_2^*), \text{ where } T_1^* \text{ has base } \{U_1 \times \{x_2\} \mid U_1 \in T_1 \ \& \ x_2 \in X_2\}$$
$$T_2^* \text{ has base } \{\{x_1\} \times U_2 \mid x_1 \in X_1 \ \& \ U_2 \in T_2\}$$

The product of topological spaces

(van Benthem et al, 2006)
For two topological space $\mathfrak{X}_1 = (X_1, T_1)$ and $\mathfrak{X}_2 = (X_2, T_2)$

$\mathfrak{X}_1 \times \mathfrak{X}_2 = (X_1 \times X_2, T_1^*, T_2^*)$, where T_1^* has base $\{U_1 \times \{x_2\} \mid U_1 \in T_1 \ \& \ x_2 \in X_2\}$

T_2^* has base $\{\{x_1\} \times U_2 \mid x_1 \in X_1 \ \& \ U_2 \in T_2\}$



Neighborhood frames

A (normal) neighborhood frame (or an n-frame) is a pair $\mathfrak{X} = (X, \tau)$, where

- $X \neq \varnothing$;
- $\tau : X \to 2^{2^X}$

τ — neighborhood function of $\mathfrak{X}$,
$\tau(x)$ — a family of neighborhoods of x.

Filter on X: nonempty $\mathcal{F} \subseteq 2^X$ such that
1) $U \in \mathcal{F}$ & $U \subseteq V \Rightarrow V \in \mathcal{F}$
2) $U, V \in \mathcal{F} \Rightarrow U \cap V \in \mathcal{F}$ (filter base)

The neighborhood model (n-model) is a pair $(\mathfrak{X}, V)$, where $\mathfrak{X} = (X, \tau)$ is a n-frame and $V : PV \to 2^X$ is a valuation.
Similar: neighborhood k-frame (n-k-frame) is $(X, \tau_1, \ldots \tau_k)$ such that τ_i is a neighborhood function on X for each i.

Neighborhood frames

A (normal) neighborhood frame (or an n-frame) is a pair $\mathfrak{X} = (X, \tau)$, where

- $X \neq \varnothing$;
- $\tau : X \to 2^{2^X}$, such that $\tau(x)$ is a filter on X;

τ — neighborhood function of $\mathfrak{X}$,
$\tau(x)$ — a family of neighborhoods of x.

Filter on X: nonempty $\mathcal{F} \subseteq 2^X$ such that
1) $U \in \mathcal{F} \;\&\; U \subseteq V \Rightarrow V \in \mathcal{F}$
2) $U, V \in \mathcal{F} \Rightarrow U \cap V \in \mathcal{F}$ (filter base)

The neighborhood model (n-model) is a pair $(\mathfrak{X}, V)$, where $\mathfrak{X} = (X, \tau)$ is a n-frame and $V : PV \to 2^X$ is a valuation.
Similar: neighborhood k-frame (n-k-frame) is $(X, \tau_1, \dots \tau_k)$ such that τ_i is a neighborhood function on X for each i.

Neighborhood frames

A (normal) neighborhood frame (or an n-frame) is a pair $\mathfrak{X} = (X, \tau)$, where

- $X \neq \varnothing$;
- $\tau : X \to 2^{2^X}$, such that $\tau(x)$ is a filter on X;

τ — neighborhood function of $\mathfrak{X}$,
$\tau(x)$ — a family of neighborhoods of x.

Filter on X: nonempty $\mathcal{F} \subseteq 2^X$ such that
1) $U \in \mathcal{F}$ & $U \subseteq V \Rightarrow V \in \mathcal{F}$
2) $U, V \in \mathcal{F} \Rightarrow U \cap V \in \mathcal{F}$ (filter base)

The neighborhood model (n-model) is a pair $(\mathfrak{X}, V)$, where $\mathfrak{X} = (X, \tau)$ is a n-frame and $V : PV \to 2^X$ is a valuation.
Similar: neighborhood k-frame (n-k-frame) is $(X, \tau_1, \ldots \tau_k)$ such that τ_i is a neighborhood function on X for each i.

Product of neighborhood frames

Definition

Let $\mathcal{X}_1 = (X_1, \tau_1)$ and $\mathcal{X}_2 = (X_2, \tau_2)$ be two n-frames. Then the product of these n-frames is an n-frame defined as follows

$$\mathcal{X}_1 \times \mathcal{X}_2 = (X_1 \times X_2, \tau_1', \tau_2'),$$
$$\tau_1'(x_1, x_2) = \{U \subseteq X_1 \times X_2 |\ \exists V (V \in \tau_1(x_1)\ \&\ V \times \{x_2\} \subseteq U)\},$$
$$\tau_2'(x_1, x_2) = \{U \subseteq X_1 \times X_2 |\ \exists V (V \in \tau_2(x_2)\ \&\ \{x_1\} \times V \subseteq U)\},$$

Fusion of logics

Definition

Let L_1 and L_2 be two modal logics with one modality $\Box$ then the fusion of these logics is

$$L_1 \otimes L_2 = K_2 + L_{1(\Box \to \Box_1)} + L_{2(\Box \to \Box_2)};$$

where $L_{i(\Box \to \Box_i)}$ is the set of all formulas from L_i where all $\Box$ replaced by $\Box_i$.

Logics

K is the minimal logic.
We will use the following logics:
$\mathbf{T} = \mathbf{K} + \Box p \to p$,
$\mathbf{D} = \mathbf{K} + \Box p \to \Diamond p$,
$\mathbf{D4} = \mathbf{D} + \Box p \to \Box\Box p$,
$\mathbf{S4} = \mathbf{T} + \Box p \to \Box\Box p$.

Known products of logics

Theorem (Shehtman and Gabbay, 1998)

If L_1, L_2 are Horn logics then

$$L_1 \times L_2 = L_1 \otimes L_2 + \Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p + \Diamond_1\Box_2 p \rightarrow \Box_2\Diamond_1 p.$$

Theorem (van Benthem, 2006)

$S4 \times_t S4 = S4 \otimes S4.$

Theorem (Kudinov, 2012)

*Let $L_1, L_2 \in \{$**D4**, **D**, **T**, **S4**$\}$, then*

$$L_1 \times_n L_2 = L_1 \otimes L_2.$$

Epistemic logic

$\Box_i \phi$ is reading as "agent i knows ϕ".
The logic for one agent is usually **S5**, but can be others: **S4**, **D4**, **K**, **T**, ...
If the logic for each agent is **S4** then the logic of two agents is the fusion **S4** $\otimes$ **S4**.
And **S4** $\times_t$ **S4** = **S4** $\otimes$ **S4**.
An open neighborhood of a possible world x is all the worlds that indistinguishable from x with certain information.
With two agents we have two sets of information for each agent.

Epistemic logic

$\Box_i \phi$ is reading as "agent i knows ϕ".
The logic for one agent is usually **S5**, but can be others: **S4**, **D4**, **K**, **T**, ...
If the logic for each agent is **S4** then the logic of two agents is the fusion **S4** $\otimes$ **S4**.
And **S4** $\times_t$ **S4** = **S4** $\otimes$ **S4**.
An open neighborhood of a possible world x is all the worlds that indistinguishable from x with certain information.
With two agents we have two sets of information for each agent.

Adding the standard product topology

In topology there is a different product of topologies.

Definition

Let $\mathfrak{X}_1 = (X_1, T_1)$ and $\mathfrak{X}_2 = (X_2, T_2)$ we define the plus-product:

$$\mathfrak{X}_1 \times^+ \mathfrak{X}_2 = (X_1 \times X_2, T_1', T_2', T),$$

where $\{U_1 \times U_2 \,|\, U_1 \in T_1, U_2 \in T_2\}$ is the base for T.
For two unimodal logics L_1 and L_2 we define t-plus-product of them as

$$L_1 \times_t^+ L_2 = L(\mathfrak{X}_1 \times^+ \mathfrak{X}_2 \mid \mathfrak{X}_1 \models L_1 \;\&\; \mathfrak{X}_2 \models L_2).$$

Products with additional modality

Definition

Let $\mathcal{X}_1 = (X_1, \tau_1)$ and $\mathcal{X}_2 = (X_2, \tau_2)$ be two n-frames. Then the product of these n-frames with additional modality is an n-3-frame defined as follows

$$\mathcal{X}_1 \times^+ \mathcal{X}_2 = (X_1 \times X_2, \tau_1', \tau_2', \tau),$$
$$\tau_1'(x_1, x_2) = \{U \subseteq X_1 \times X_2 |\ \exists V (V \in \tau_1(x_1) \ \&\ V \times \{x_2\} \subseteq U)\}\ ,$$
$$\tau_2'(x_1, x_2) = \{U \subseteq X_1 \times X_2 |\ \exists V (V \in \tau_2(x_2) \ \&\ \{x_1\} \times V \subseteq U)\},$$
$$\tau(x_1, x_2) = \{U | \exists V_1 \in \tau_1(x) \ \&\ \exists V_2 \in \tau_2(y)(V_1 \times V_2 \subseteq U)\}.$$

For two unimodal logics L_1 and L_2 we define n-plus-product of them as

$$L_1 \times_n^+ L_2 = L(\mathfrak{X}_1 \times^+ \mathfrak{X}_2 \mid \mathfrak{X}_1 \models L_1 \ \&\ \mathfrak{X}_2 \models L_2).$$

Adding the standard-product-topology-like modal operator

Definition

$$\mathbf{LS4} = \mathbf{S4} \otimes \mathbf{S4} \otimes \mathbf{S4} + \Box p \to \Box_1 p \wedge \Box_2 p;$$
$$\mathbf{LD4} = \mathbf{D4} \otimes \mathbf{D4} \otimes \mathbf{D4} + \Box p \to \Box_1 p \wedge \Box_2 p;$$
$$\mathbf{LD} = \mathbf{D} \otimes \mathbf{D} \otimes \mathbf{D} + \Box p \to \Box_1 p \wedge \Box_2 p;$$
$$\mathbf{LT} = \mathbf{T} \otimes \mathbf{T} \otimes \mathbf{T} + \Box p \to \Box_1 p \wedge \Box_2 p.$$

Theorem (Benthem, J., G. Bezhanishvili, B. Cate and D. Sarenac, 2006)

$$Log(\mathbb{Q} \times^{+} \mathbb{Q}) = \textbf{\textit{LS4}} = \textbf{\textit{S4}} \times_t^{+} \textbf{\textit{S4}}$$

Theorem (Kudinov A., 2013)

$$Log_d(\mathbb{Q} \times^{+} \mathbb{Q}) = \textbf{\textit{LD4}} = \textbf{\textit{D4}} \times_n^{+} \textbf{\textit{D4}}$$

Adding the standard-product-topology-like modal operator

Definition

$$\mathbf{LS4} = \mathbf{S4} \otimes \mathbf{S4} \otimes \mathbf{S4} + \Box p \to \Box_1 p \wedge \Box_2 p;$$
$$\mathbf{LD4} = \mathbf{D4} \otimes \mathbf{D4} \otimes \mathbf{D4} + \Box p \to \Box_1 p \wedge \Box_2 p;$$
$$\mathbf{LD} = \mathbf{D} \otimes \mathbf{D} \otimes \mathbf{D} + \Box p \to \Box_1 p \wedge \Box_2 p;$$
$$\mathbf{LT} = \mathbf{T} \otimes \mathbf{T} \otimes \mathbf{T} + \Box p \to \Box_1 p \wedge \Box_2 p.$$

Theorem (Benthem, J., G. Bezhanishvili, B. Cate and D. Sarenac, 2006)

$Log(\mathbb{Q} \times^{+} \mathbb{Q}) = \textbf{\textit{LS4}} = \textbf{\textit{S4}} \times^{+}_{t} \textbf{\textit{S4}}$

Theorem (Kudinov A., 2013)

$Log_d(\mathbb{Q} \times^{+} \mathbb{Q}) = \textbf{\textit{LD4}} = \textbf{\textit{D4}} \times^{+}_{n} \textbf{\textit{D4}}$

Adding the standard-product-topology-like modal operator

Definition

$$\mathbf{LS4} = \mathbf{S4} \otimes \mathbf{S4} \otimes \mathbf{S4} + \Box p \to \Box_1 p \wedge \Box_2 p;$$
$$\mathbf{LD4} = \mathbf{D4} \otimes \mathbf{D4} \otimes \mathbf{D4} + \Box p \to \Box_1 p \wedge \Box_2 p;$$
$$\mathbf{LD} = \mathbf{D} \otimes \mathbf{D} \otimes \mathbf{D} + \Box p \to \Box_1 p \wedge \Box_2 p;$$
$$\mathbf{LT} = \mathbf{T} \otimes \mathbf{T} \otimes \mathbf{T} + \Box p \to \Box_1 p \wedge \Box_2 p.$$

Theorem (Kudinov A., 2013)

$Log_d(\mathbb{Q} \times^+ \mathbb{Q}) = \textbf{\textit{LD4}} = \textbf{\textit{D4}} \times_n^+ \textbf{\textit{D4}}$

Theorem

$\textbf{\textit{LD}} = \textbf{\textit{D}} \times_n^+ \textbf{\textit{D}}$
$\textbf{\textit{LT}} = \textbf{\textit{T}} \times_n^+ \textbf{\textit{T}}$

Back to epistemic logic

This additional modality is similar to common knowledge (or belief) operator. It contains all the agents' knowledges, and it is transitive (in case of **S4** and **D4**).
In case of logics **T** and **D** we should consider the following logics:

$$\mathbf{L4D} = \mathbf{D} \otimes \mathbf{D} \otimes \mathbf{D4} + \Box p \to \Box_1 p \wedge \Box_2 p;$$
$$\mathbf{L4T} = \mathbf{T} \otimes \mathbf{T} \otimes \mathbf{S4} + \Box p \to \Box_1 p \wedge \Box_2 p.$$

The corresponding completeness theorems can be proved using similar methods.

3 and more agents

Another way to generalize the results of [van Benthem et al., 2006] is to consider 3 and more agents:
If we have 3 agents ($\Box_1, \Box_2$ and $\Box_3$) then there can be 4 additional modalities: $\Box_{1,2}, \Box_{2,3}, \Box_{1,3}, \Box_{1,2,3}$.

We also can consider more then 3 agents.

The corresponding completeness theorems also can be proved by same methods.

3 and more agents

Another way to generalize the results of [van Benthem et al., 2006] is to consider 3 and more agents:
If we have 3 agents ($\Box_1, \Box_2$ and $\Box_3$) then there can be 4 additional modalities: $\Box_{1,2}, \Box_{2,3}, \Box_{1,3}, \Box_{1,2,3}$.

We also can consider more then 3 agents.

The corresponding completeness theorems also can be proved by same methods.

3 and more agents

Another way to generalize the results of [van Benthem et al., 2006] is to consider 3 and more agents:
If we have 3 agents ($\Box_1, \Box_2$ and $\Box_3$) then there can be 4 additional modalities: $\Box_{1,2}, \Box_{2,3}, \Box_{1,3}, \Box_{1,2,3}$.

We also can consider more then 3 agents.

The corresponding completeness theorems also can be proved by same methods.

Conclusion

We can try and extend the technique to non-serial logics such as $K, K4$ and so on.

THANK YOU!

$T_{\omega,\omega,\omega}$

$T_{2,2}, T_{2,2,6}$ (Benthem, J., G. Bezhanishvili, B. Cate and D. Sarenac, 2006).
$T_{\omega,\omega,\omega}$

$T_{\omega,\omega,\omega}$

Lemma

LD *is complete with respect to* $T_{\omega,\omega,\omega}$.

Bounded morphism

Definition

Let $\mathcal{X} = (X, \tau_1, \ldots)$ and $\mathcal{Y} = (Y, \sigma_1, \ldots)$ be n-frames. Then function $f : X \to Y$ is a bounded morphism if

1. f is surjective;
2. For any $x \in X$ and $U \in \tau_i(x)$ we have $f(U) \in \sigma_i(f(x))$;
3. For any $x \in X$ and $V \in \sigma_i(f(x))$ there exists $U \in \tau_i(x)$, such that $f(U) \subseteq V$.

In notation $f : \mathcal{X} \twoheadrightarrow \mathcal{Y}$.

Lemma

If $f : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ then $L(\mathcal{X}) \subseteq L(\mathcal{Y})$, where f is a bound morphism.

Bounded morphism

Definition

Let $\mathcal{X} = (X, \tau_1, \dots)$ and $\mathcal{Y} = (Y, \sigma_1, \dots)$ be n-frames. Then function $f : X \to Y$ is a bounded morphism if

1. f is surjective;
2. For any $x \in X$ and $U \in \tau_i(x)$ we have $f(U) \in \sigma_i(f(x))$;
3. For any $x \in X$ and $V \in \sigma_i(f(x))$ there exists $U \in \tau_i(x)$, such that $f(U) \subseteq V$.

In notation $f : \mathcal{X} \twoheadrightarrow \mathcal{Y}$.

Lemma

If $f : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ then $L(\mathcal{X}) \subseteq L(\mathcal{Y})$, where f is a bound morphism.

Bounded morphism

Definition

Let $\mathcal{X} = (X, \tau_1, \dots)$ and $\mathcal{Y} = (Y, \sigma_1, \dots)$ be n-frames. Then function $f : X \to Y$ is a bounded morphism if

1. f is surjective;
2. For any $x \in X$ and $U \in \tau_i(x)$ we have $f(U) \in \sigma_i(f(x))$;
3. For any $x \in X$ and $V \in \sigma_i(f(x))$ there exists $U \in \tau_i(x)$, such that $f(U) \subseteq V$.

In notation $f : \mathcal{X} \twoheadrightarrow \mathcal{Y}$.

Lemma

If $f : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ then $L(\mathcal{X}) \subseteq L(\mathcal{Y})$, where f is a bound morphism.

Bounded morphism

Definition

Let $\mathcal{X} = (X, \tau_1, \dots)$ and $\mathcal{Y} = (Y, \sigma_1, \dots)$ be n-frames. Then function $f : X \to Y$ is a bounded morphism if
1. f is surjective;
2. For any $x \in X$ and $U \in \tau_i(x)$ we have $f(U) \in \sigma_i(f(x))$;
3. For any $x \in X$ and $V \in \sigma_i(f(x))$ there exists $U \in \tau_i(x)$, such that $f(U) \subseteq V$.

In notation $f : \mathcal{X} \twoheadrightarrow \mathcal{Y}$.

Lemma

If $f : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ then $L(\mathcal{X}) \subseteq L(\mathcal{Y})$, where f is a bound morphism.

Pseudo-infinite paths

Definition

For a nonempty set A, such that $0 \notin A$ we define $f_F : X_A \to A^\star$ which "forgets" all zeros, where $A^\star$ is the set of all finite sequences of elements from A, including the empty sequence Λ and

$$X_A = \{a_1, a_2 \ldots \mid a_i \in A \cup \{0\} \ \& \ \exists N \ \forall k \geq N (a_k = 0)\}.$$

Pseudo-infinite paths

For $\alpha \in X_A$ such that $\alpha = a_1 a_2 \ldots$ we define

$$st(\alpha) = min\{N \mid \forall k > N(a_k = 0)\};$$
$$\alpha|k = a_1 \ldots a_k;$$
$$U_k(\alpha) = \{\beta \mid \alpha|_m = \beta|_m \ \& \ f_F(\alpha) R f_F(\beta), \ m = max(k, st(\alpha))\},$$

where $\overline{a} R \overline{b} \Leftrightarrow \exists c \in A \ (\overline{b} = \overline{a} \cdot c)$.

Results

Theorem

There is a function f*, such that* $f : \mathcal{N}_\omega(D) \times^+ \mathcal{N}_\omega(D) \twoheadrightarrow T_{\omega,\omega,\omega}$.

Corollary

$D \times_n^+ D = LD$.

Results

Theorem

There is a function f*, such that* $f : \mathcal{N}_\omega(D) \times^+ \mathcal{N}_\omega(D) \twoheadrightarrow T_{\omega,\omega,\omega}$.

Corollary

$D \times_n^+ D = LD$.