Enriched distributivity over finite commutative residuated lattices

Adriana Balan*[1]
joint work with Peter Jipsen‡ and Alexander Kurz‡

*University Politehnica of Bucharest, ‡Chapman University

TACL 2019, Nice

[1]Supported by a mobility grant of the Romanian Ministry of Research and Innovation, CNCS-UEFISCDI, project number PN-III-P1-1.1-MC-2019-1083, within PNCDI III

What is this talk about?

Many-valued complete distributivity, equationally

What is this talk about?

Many-valued complete distributivity, equationally

What does many-valued mean?

This talk: quantale-enriched

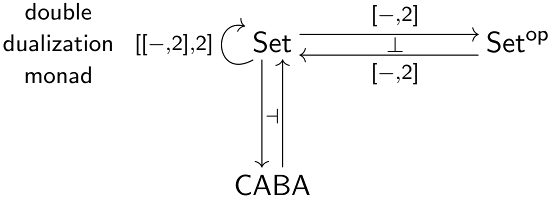
Origins: Stone duality

$$\mathsf{Set} \underset{[-,2]}{\overset{[-,2]}{\rightleftarrows}} \mathsf{Set}^{\mathsf{op}} \quad (\bot)$$

Origins: Stone duality

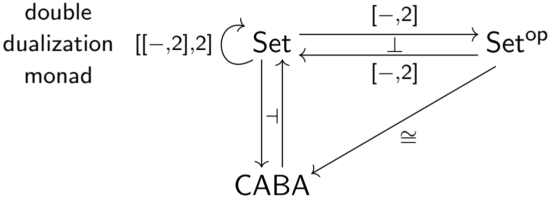
$$\begin{array}{c}\text{double}\\ \text{dualization}\\ \text{monad}\end{array} \quad [[-,2],2] \circlearrowright \mathsf{Set} \underset{[-,2]}{\overset{[-,2]}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Set}^{op}$$

Origins: Stone duality



- The algebras for the double dualization monad: complete atomic Boolean algebras (CABA)

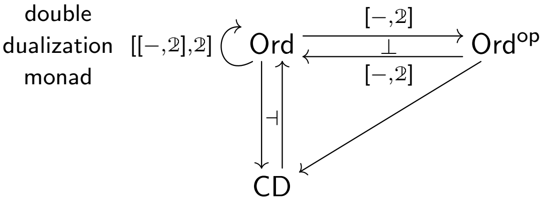
Origins: Stone duality



- The algebras for the double dualization monad: complete atomic Boolean algebras (CABA)

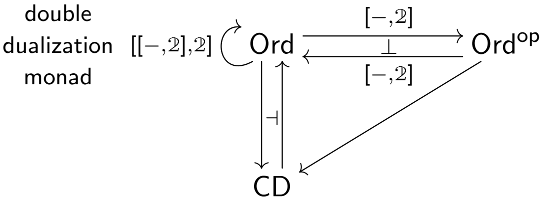
- $[-,2] : \mathsf{Set}^{\mathsf{op}} \to \mathsf{Set}$ monadic

Ordered Stone duality



- The algebras for the double dualization monad: completely distributive lattices (CD)
- $[-, \mathbb{2}] : \mathsf{Ord}^{\mathsf{op}} \to \mathsf{Ord}$ not monadic
- CD monadic over Ord (ordered variety)

Ordered Stone duality



- The algebras for the double dualization monad: completely distributive lattices (CD)
- $[-, \mathbb{2}] : \mathsf{Ord}^{op} \to \mathsf{Ord}$ not monadic
- CD monadic over Ord (ordered variety)
- CD also monadic over Set (variety)

  Each completely distributive lattice A is a complete lattice satisfying

$$\bigwedge_{k \in K} \bigvee \{a \mid a \in S_k\} = \bigvee_{f \in \mathcal{F}} \bigwedge \{a \mid a \in f(A)\}$$

  for every family of subsets $(S_k)_{k \in K}$ of A, with $\mathcal{F}$ the set of choice functions

From order (two-valued) to quantale-enriched (multi-valued)

Let $\mathcal{Q} = (\mathcal{Q}, \otimes, e, [-,-])$ be a commutative quantale

- a sup-lattice $(\mathcal{Q}, \vee)$
- a commutative monoid $(\mathcal{Q}, \otimes, e)$

such that $x \otimes -$ preserves all suprema, hence it has a right adjoint $[x, -]$

$$x \otimes y \leq z \quad \iff \quad y \leq [x, z]$$

Examples

- $\mathcal{Q} = (2, \wedge, 1)$
- $\mathcal{Q} = ([0, \infty]^{\mathrm{op}}, +, 0)$

From order (two-valued) to quantale-enriched (multi-valued)

Let $\mathcal{Q} = (\mathcal{Q}, \otimes, e, [-,-])$ be a commutative quantale

- a sup-lattice $(\mathcal{Q}, \vee)$
- a commutative monoid $(\mathcal{Q}, \otimes, e)$

such that $x \otimes -$ preserves all suprema, hence it has a right adjoint $[x, -]$

$$x \otimes y \leq z \quad \iff \quad y \leq [x, z]$$

Examples

- $\mathcal{Q} = (\mathbb{2}, \wedge, 1)$
- $\mathcal{Q} = ([0, \infty]^{\mathsf{op}}, +, 0)$
- There are three possible quantale structures on $\mathbb{3} = \{0 < 1/2 < 1\}$

From order (two-valued) to quantale-enriched (multi-valued)

Let $\mathcal{Q} = (\mathcal{Q}, \otimes, e, [-,-])$ be a commutative quantale

- a sup-lattice $(\mathcal{Q}, \bigvee)$
- a commutative monoid $(\mathcal{Q}, \otimes, e)$

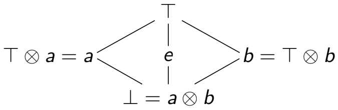
such that $x \otimes -$ preserves all suprema, hence it has a right adjoint $[x, -]$

$$x \otimes y \leq z \qquad \Longleftrightarrow \qquad y \leq [x, z]$$

Examples

- $\mathcal{Q} = (\mathbb{2}, \wedge, 1)$
- $\mathcal{Q} = ([0, \infty]^{\mathrm{op}}, +, 0)$
- There are three possible quantale structures on $\mathbb{3} = \{0 < 1/2 < 1\}$
- There are also non-distributive quantales:

e.g. M_3 idempotent tensor



Quantales and quantale-enriched categories

- $\mathcal{Q}$-category $\mathscr{A} = (A, \mathscr{A} : A \times A \to \mathcal{Q})$

$$e \leq \mathscr{A}(a,a) \quad \text{and} \quad \mathscr{A}(a,b) \otimes \mathscr{A}(b,c) \leq \mathscr{A}(a,c)$$

- $\mathcal{Q}$-functor $f : \mathscr{A} \to \mathscr{A}'$

$$\mathscr{A}(a,b) \leq \mathscr{A}'(fa, fb)$$

Examples

- ordered sets

$$a \leq a, \quad (a \leq b) \wedge (b \leq c) \Rightarrow (a \leq c)$$

- quasi-metric spaces

$$0 \geq \mathscr{A}(a,a), \quad \mathscr{A}(a,b) + \mathscr{A}(b,c) \geq \mathscr{A}(a,c)$$

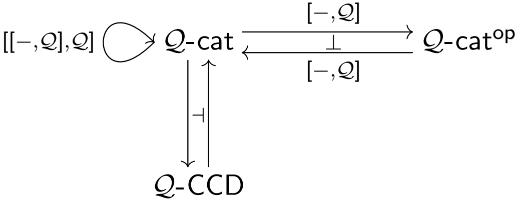
- In particular, each $\mathcal{Q}$-category $\mathscr{A}$ carries an order

$$a \leq b \quad \iff \quad e \leq \mathscr{A}(a,b)$$

Completely distributive quantale-enriched-categories

$$[[-,\mathcal{Q}],\mathcal{Q}] \circlearrowright \mathcal{Q}\text{-cat} \underset{[-,\mathcal{Q}]}{\overset{[-,\mathcal{Q}]}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{Q}\text{-cat}^{\mathsf{op}}$$

Completely distributive quantale-enriched-categories



- $\mathcal{Q}$-CCD: the category of algebras

  These are (complete and) cocomplete $\mathcal{Q}$-categories, such that taking $\mathcal{Q}$-suprema is a continuous $\mathcal{Q}$-functor. Analogous to the ordered case, we call them **completely distributive** $\mathcal{Q}$-categories ($\mathcal{Q}$-ccd)

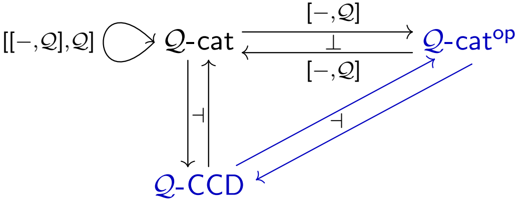
  Homomorphisms: continuous and cocontinuous $\mathcal{Q}$-functors.

  Stubbe. *Towards "dynamic domains": Totally continuous cocomplete $\mathcal{Q}$-categories* (2007)

  Stubbe. *The double power monad is the composite power monad* (2017)

  Băbuș&Kurz. *On the Logic of Generalised Metric Spaces* (2016)

Completely distributive quantale-enriched-categories



- $\mathcal{Q}$-CCD: the category of algebras

These are (complete and) cocomplete $\mathcal{Q}$-categories, such that taking $\mathcal{Q}$-suprema is a continuous $\mathcal{Q}$-functor. Analogous to the ordered case, we call them **completely distributive** $\mathcal{Q}$-categories ($\mathcal{Q}$-ccd)

Homomorphisms: continuous and cocontinuous $\mathcal{Q}$-functors.

Stubbe. *Towards "dynamic domains": Totally continuous cocomplete $\mathcal{Q}$-categories* (2007)

Stubbe. *The double power monad is the composite power monad* (2017)

Băbuș&Kurz. *On the Logic of Generalised Metric Spaces* (2016)

Completely distributive quantale-enriched-categories

Examples

- $\mathcal{Q}$, seen as a $\mathcal{Q}$-category with $[-,-]$, is $\mathcal{Q}$-completely distributive
- For any $\mathcal{Q}$-category $\mathscr{X}$, $[\mathscr{X}^{\mathsf{op}}, \mathcal{Q}]$ is $\mathcal{Q}$-completely distributive
  In particular, for any set X, the $\mathcal{Q}$-powerset $[X, \mathcal{Q}]$ is $\mathcal{Q}$-ccd.
- For a cocomplete $\mathcal{Q}$-category $\mathscr{A}$, the following are equivalent:
  - $\mathscr{A}$ is projective as a cocomplete $\mathcal{Q}$-category
  - $\mathscr{A}$ is $\mathcal{Q}$-completely distributive
  - $\mathscr{A}$ is the $\mathcal{Q}$-category of regular presheaves on a regular $\mathcal{Q}$-semicategory

Stubbe. *Towards "dynamic domains": Totally continuous cocomplete $\mathcal{Q}$-categories* (2007)

Completely distributive $\mathcal{Q}$-categories

Remarks

- $\mathcal{Q}$-complete distributivity does not necessarily entail complete distributivity!

  For example, $\mathcal{Q}$ itself is $\mathcal{Q}$-ccd but not necessarily distributive as a lattice

- However, every $\mathcal{Q}$-completely distributive $\mathcal{Q}$-category $\mathscr{A}$ is completely distributive as a lattice $\iff$ $\mathcal{Q}$ is a completely distributive lattice

  Lai&Zhang. *Many-Valued Complete Distributivity.* (2006)

Completely distributive $\mathcal{Q}$-categories

- $\mathcal{Q}$-CCD is monadic over Set – in particular, the free $\mathcal{Q}$-ccd over a set X is $[[X, \mathcal{Q}]^{\mathrm{op}}, \mathcal{Q}]$

  Pu&Zhang. *Categories enriched over a quantaloid: algebras.* (2015)

- Therefore $\mathcal{Q}$-CCD must have an **equational axiomatisation**

Completely distributive $\mathcal{Q}$-categories

- $\mathcal{Q}$-CCD is monadic over Set – in particular, the free $\mathcal{Q}$-ccd over a set X is $[[X, \mathcal{Q}]^{\text{op}}, \mathcal{Q}]$

  Pu&Zhang. *Categories enriched over a quantaloid: algebras.* (2015)

- Therefore $\mathcal{Q}$-CCD must have an **equational axiomatisation**

  Recall that a $\mathcal{Q}$-completely distributive $\mathcal{Q}$-category is an algebra, i.e. a complete and cocomplete $\mathcal{Q}$-category, such that such that taking $\mathcal{Q}$-suprema is a continuous $\mathcal{Q}$-functor. Completeness and cocompleteness can be expressed by **operations and equations**:

$$\mathscr{A} = (A, \bigsqcup, \bigsqcap, (v * -)_{v \in \mathcal{Q}}, (v \rhd -)_{v \in \mathcal{Q}})$$

  such that

  - $(A, \bigsqcup, \bigsqcap)$ is a complete lattice
  - $v * -$ and $v \rhd -$ are **adjoint** unary operators satisfying

$$e * a = a \qquad v * (w * a) = (v \otimes w) * a \qquad \left(\bigvee_i v_i\right) * a = \bigsqcup_i (v_i * a)$$

$$e \rhd a = a \qquad v \rhd (w \rhd a) = (v \otimes w) \rhd a \qquad \left(\bigvee_i v_i\right) \rhd a = \bigsqcap_i (v_i \rhd a)$$

Completely distributive $\mathcal{Q}$-categories

- **What about $\mathcal{Q}$-complete distributivity?**
- Let $\sup : [\mathscr{A}^{\mathsf{op}}, \mathcal{Q}] \to \mathscr{A}$ be the $\mathcal{Q}$-functor taking $\mathcal{Q}$-suprema

  Recall that being $\mathcal{Q}$-ccd means that $\sup$ preserves weighted limits:

  $$\sup\,(\lim_w G) = \lim_w(\sup \circ G)$$

  for every $\mathcal{Q}$-functors $w : \mathscr{K}^{\mathsf{op}} \to \mathcal{Q}$ and $G : \mathscr{K} \to [\mathscr{A}^{\mathsf{op}}, \mathcal{Q}]$

Completely distributive $\mathcal{Q}$-categories

- **What about $\mathcal{Q}$-complete distributivity?**
- Let $\sup : [\mathscr{A}^{\mathsf{op}}, \mathcal{Q}] \to \mathscr{A}$ be the $\mathcal{Q}$-functor taking $\mathcal{Q}$-suprema

  Recall that being $\mathcal{Q}$-ccd means that $\sup$ preserves weighted limits:

  $$\sup\,(\lim_w G) = \lim_w(\sup \circ G)$$

  for every $\mathcal{Q}$-functors $w : \mathscr{K}^{\mathsf{op}} \to \mathcal{Q}$ and $G : \mathscr{K} \to [\mathscr{A}^{\mathsf{op}}, \mathcal{Q}]$
- Expressing $\sup$ by tensors and joins, and likely the weighted limits above by cotensors and meets in $\mathscr{A}$, the above rewrites as

  $$\bigsqcup_a \left(\bigwedge_k [w(k), G(k)(a)] \right) * a = \bigsqcap_k w(k) \triangleright (\bigsqcup_a G(k)(a) * a)$$

  for every $\mathcal{Q}$-functors $w : \mathscr{K}^{\mathsf{op}} \to \mathcal{Q}$ and $G : \mathscr{K} \to [\mathscr{A}^{\mathsf{op}}, \mathcal{Q}]$

Completely distributive $\mathcal{Q}$-categories

- **What about $\mathcal{Q}$-complete distributivity?**
- Let $\sup : [\mathscr{A}^{\mathsf{op}}, \mathcal{Q}] \to \mathscr{A}$ be the $\mathcal{Q}$-functor taking $\mathcal{Q}$-suprema

  Recall that being $\mathcal{Q}$-ccd means that $\sup$ preserves weighted limits:

$$\sup\,(\lim_w G) = \lim_w(\sup \circ G)$$

  for every $\mathcal{Q}$-functors $w : \mathscr{K}^{\mathsf{op}} \to \mathcal{Q}$ and $G : \mathscr{K} \to [\mathscr{A}^{\mathsf{op}}, \mathcal{Q}]$
- Expressing $\sup$ by tensors and joins, and likely the weighted limits above by cotensors and meets in $\mathscr{A}$, the above rewrites as

$$\bigsqcup_a \left(\bigwedge_k [w(k), G(k)(a)] \right) * a = \bigsqcap_k w(k) \rhd (\bigsqcup_a G(k)(a) * a)$$

  for every $\mathcal{Q}$-functors $w : \mathscr{K}^{\mathsf{op}} \to \mathcal{Q}$ and $G : \mathscr{K} \to [\mathscr{A}^{\mathsf{op}}, \mathcal{Q}]$
- One can always without loss of generality **replace $\mathscr{K}$ by a discrete $\mathcal{Q}$-category (a set)**

  Hence $w : \mathscr{K}^{\mathsf{op}} \to \mathscr{A}$ will just be a function $K \to A$

Equational presentation of $\mathcal{Q}$-ccd

- Also, **replace the $\mathcal{Q}$-functor $G : \mathscr{K} \to [\mathscr{A}^{\mathrm{op}}, \mathcal{Q}]$ by a function $G : K \to [A, \mathcal{Q}]$**

  But there is a price to pay: the passage from a family of $\mathcal{Q}$-downsets G to a family of $\mathcal{Q}$-subsets forces the appearance of the $\mathcal{Q}$-down-closure of each "$\mathcal{Q}$-subset" $G(k) \in [A, \mathcal{Q}]$

$$\bigsqcup_a \left(\bigwedge_k [w(k), {\downarrow} G(k)(a)] \right) * a = \bigsqcap_k w(k) \rhd \left(\bigsqcup_a G(k)(a) * a \right)$$

Equational presentation of $\mathcal{Q}$-ccd

- Also, **replace the $\mathcal{Q}$-functor $G : \mathscr{K} \to [\mathscr{A}^{\mathrm{op}}, \mathcal{Q}]$ by a function $G : K \to [A, \mathcal{Q}]$**

  But there is a price to pay: the passage from a family of $\mathcal{Q}$-downsets G to a family of $\mathcal{Q}$-subsets forces the appearance of the $\mathcal{Q}$-down-closure of each "$\mathcal{Q}$-subset" $G(k) \in [A, \mathcal{Q}]$

$$\bigsqcup_a \left(\bigwedge_k [w(k), \downarrow G(k)(a)] \right) * a = \bigsqcap_k w(k) \rhd \left(\bigsqcup_a G(k)(a) * a \right)$$

- Expressing $\downarrow G(k)$ by tensors and joins in $\mathscr{A}$ produces

$$\bigsqcup_a \left(\bigwedge_k [w(k), \bigvee_b G(k)(b) \otimes \mathscr{A}(a, b)] \right) * a = \bigsqcap_k w(k) \rhd \left(\bigsqcup_a G(k)(a) * a \right)$$

Equational presentation of $\mathcal{Q}$-ccd

- Also, **replace the $\mathcal{Q}$-functor $G : \mathscr{K} \to [\mathscr{A}^{\mathrm{op}}, \mathcal{Q}]$ by a function $G : K \to [A, \mathcal{Q}]$**

  But there is a price to pay: the passage from a family of $\mathcal{Q}$-downsets G to a family of $\mathcal{Q}$-subsets forces the appearance of the $\mathcal{Q}$-down-closure of each "$\mathcal{Q}$-subset" $G(k) \in [A, \mathcal{Q}]$

$$\bigsqcup_a \left(\bigwedge_k [w(k), \downarrow G(k)(a)] \right) * a = \prod_k w(k) \rhd \left(\bigsqcup_a G(k)(a) * a \right)$$

- Expressing $\downarrow G(k)$ by tensors and joins in $\mathscr{A}$ produces

$$\bigsqcup_a \left(\bigwedge_k [w(k), \bigvee_b G(k)(b) \otimes \mathscr{A}(a,b)] \right) * a = \prod_k w(k) \rhd \left(\bigsqcup_a G(k)(a) * a \right)$$

- Unfortunately, the $\mathcal{Q}$-category structure of $\mathscr{A}$

$$\mathscr{A}(a,b) = \bigvee \{ v \in \mathcal{Q} \mid v * a \leq b \},$$

  depends on the condition $v * a \leq b$

Equational presentation of $\mathcal{Q}$-ccd

What can it be done about the $\mathcal{Q}$-complete distributivity relation?

$$\bigsqcup_a \left(\bigwedge_k [w(k), \bigvee_b G(k)(b) \otimes \mathscr{A}(a,b)] \right) * a = \bigsqcap_k w(k) \rhd \left(\bigsqcup_a G(k)(a) * a \right)$$

- Look for a formulation of the distributive law above which translates $[w(k), \bigvee_b G(k)(b) \otimes \mathscr{A}(a,b)]$ to a more traditional formulation using choice functions (as in the case $\mathcal{Q} = \mathbb{2}$)
- This may require additional conditions on the quantale $\mathcal{Q}$ (but ones which are satisfied in the case $\mathcal{Q} = \mathbb{2}$ and thus do generalise it)

Equational presentation of $\mathcal{Q}$-ccd

Let $\mathcal{Q}$ be a commutative unital quantale. Assume that

- $\mathcal{Q}$ is completely distributive as a lattice, and
- all powers $[v, -] : \mathcal{Q} \to \mathcal{Q}$, for $v \in \mathcal{Q}$, preserve non-empty joins

Let $\mathscr{A} = (A, \bigsqcup, \bigsqcap, (v * -)_{v \in \mathcal{Q}}, (v \rhd -)_{v \in \mathcal{Q}})$ be a cocomplete (and complete) $\mathcal{Q}$-category.

Then $\mathscr{A}$ is $\mathcal{Q}$-ccd **iff** for every functions $w : K \to A$, $G : K \to [A, \mathcal{Q}]$, the following holds

$$\bigsqcap_{k \in K} w(k) \rhd \left(\bigsqcup_{a \in A} G(k)(a) * a \right) = \bigsqcup_{f \in \mathcal{F}} \bigsqcap_{k \in K} w(k) \rhd (G(k)(fk) * fk)$$

where $\mathcal{F}$ is the set of functions $K \to A$

Equational presentation of $\mathcal{Q}$-ccd

Remarks

- Finite commutative MTL-algebras are quantales satisfying previous conditions
- We already know that the assumption $\mathcal{Q}$ completely distributive entails that each $\mathcal{Q}$-ccd is also completely distributive
- Hence, we may recover complete distributivity by choosing trivial weights $w(k) = e$ and discrete $\mathcal{Q}$-subsets $G(k)$ corresponding to a family of ordinary subsets $(A_k)_{k \in K}$ of A

$$\bigsqcap_{k \in K} \bigsqcup_{a \in A_k} a = \bigsqcup_{\{f : K \to A \mid fk \in A_k\}} \bigsqcap_{k \in K} fk$$

Equational presentation of $\mathcal{Q}$-ccd

Remarks

- The particular case $K = \{0\}$, $w(0) = v$, $G(k)(-) = e$ gives

$$v \rhd \bigsqcup_{a \in A} a = \bigsqcup_{a \in A} v \rhd a$$

  hence $v \rhd -$ distributes over (non-empty) joins[2], for each $v \in \mathcal{Q}$

- In fact, each $\mathcal{Q}$-ccd is a quotient of a subalgebra of a product of copies of $\mathcal{Q}$

  Lai&Zhang. *Many-Valued Complete Distributivity.* (2006)

- That is, $\mathcal{Q}$ generates the variety of $\mathcal{Q}$-ccd.

  Hence an equation holds in a $\mathcal{Q}$-completely distributive $\mathcal{Q}$-category $\mathscr{A}$ iff it holds in $\mathcal{Q}$

[2] Observe that the empty $\mathcal{Q}$-category cannot be $\mathcal{Q}$-ccd

Equational presentation of $\mathcal{Q}$-ccd

Remarks

- Looking at the constructive/non-constructive $\mathcal{Q}$-ccd equations

$$\bigsqcap_{k\in K} w(k) \triangleright \left(\bigsqcup_{a\in A} G(k)(a) * a\right) = \bigsqcup_{a\in A}\left(\bigwedge_{k\in K}[w(k), \downarrow G(k)(a)]\right) * a$$

$$\bigsqcap_{k\in K} w(k) \triangleright \left(\bigsqcup_{a\in A} G(k)(a) * a\right) = \bigsqcup_{f\in\mathcal{F}} \bigsqcap_{k\in K} w(k) \triangleright (G(k)(fk) * fk)$$

  we see that the lhs coincide

- The inequality

$$\bigsqcup_{a\in A}\left(\bigwedge_{k\in K}[w(k), \downarrow G(k)(a)]\right) * a \geq \bigsqcup_{f\in\mathcal{F}} \bigsqcap_{k\in K} w(k) \triangleright (G(k)(fk) * fk)$$

  always holds for $\mathcal{Q}$-ccd, but it can be strict (e.g for non-distributive quantale $\mathcal{Q}$)

Conclusions and open questions

Conclusions and open questions

- The distributive law arising from enriching over a commutative quantale $\mathcal{Q}$ **can be expressed in terms of operations and equations**, similar to the familiar distributive law of lattices, under suitable hypotheses – completely distributive quantale $\mathcal{Q}$ with the property that powers preserve non-empty joins (in particular, for **finite MTL-algebras**)

$$\bigsqcap_{k \in K} w(k) \triangleright \left(\bigsqcup_{a \in A} G(k)(a) * a \right) = \bigsqcup_{f \in \mathcal{F}} \bigsqcap_{k \in K} w(k) \triangleright (G(k)(fk) * fk)$$

Conclusions and open questions

- The distributive law arising from enriching over a commutative quantale $\mathcal{Q}$ **can be expressed in terms of operations and equations**, similar to the familiar distributive law of lattices, under suitable hypotheses – completely distributive quantale $\mathcal{Q}$ with the property that powers preserve non-empty joins (in particular, for **finite MTL-algebras**)

$$\bigsqcap_{k\in K} w(k) \triangleright \left(\bigsqcup_{a\in A} G(k)(a) * a \right) = \bigsqcup_{f\in\mathcal{F}} \bigsqcap_{k\in K} w(k) \triangleright (G(k)(fk) * fk)$$

- What about other distributivity-like laws, e.g. $v \triangleright (w * a) = [v, w] * a$?

Conclusions and open questions

- The distributive law arising from enriching over a commutative quantale $\mathcal{Q}$ **can be expressed in terms of operations and equations**, similar to the familiar distributive law of lattices, under suitable hypotheses – completely distributive quantale $\mathcal{Q}$ with the property that powers preserve non-empty joins (in particular, for **finite MTL-algebras**)

$$\bigsqcap_{k \in K} w(k) \rhd \left(\bigsqcup_{a \in A} G(k)(a) * a \right) = \bigsqcup_{f \in \mathcal{F}} \bigsqcap_{k \in K} w(k) \rhd (G(k)(fk) * fk)$$

- What about other distributivity-like laws, e.g. $v \rhd (w * a) = [v, w] * a$?

- More important: to obtain an equational axiomatisation of $\mathcal{Q}$-ccd even for non-distributive quantales

Conclusions and open questions

- The distributive law arising from enriching over a commutative quantale $\mathcal{Q}$ **can be expressed in terms of operations and equations**, similar to the familiar distributive law of lattices, under suitable hypotheses – completely distributive quantale $\mathcal{Q}$ with the property that powers preserve non-empty joins (in particular, for **finite MTL-algebras**)

$$\bigsqcap_{k \in K} w(k) \rhd \left(\bigsqcup_{a \in A} G(k)(a) * a \right) = \bigsqcup_{f \in \mathcal{F}} \bigsqcap_{k \in K} w(k) \rhd (G(k)(fk) * fk)$$

- What about other distributivity-like laws, e.g. $v \rhd (w * a) = [v, w] * a$?

- More important: to obtain an equational axiomatisation of $\mathcal{Q}$-ccd even for non-distributive quantales

- What about a finitary version of $\mathcal{Q}$-ccd (see my talk at TACL2017)?

Thank you for your attention!