A Glivenko Theorem for Lattice-Ordered Groups

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Glivenko's Theorem

Theorem (Glivenko 1929)

Let s, t be terms in the language of Heyting algebras. Then

$$\mathcal{BA} \models s \leq t \quad \textit{if, and only if,} \quad \mathcal{HA} \models \neg\neg s \leq \neg\neg t.$$

We will present a version of Glivenko's theorem in the setting of residuated lattices, with the variety of *ℓ-groups* taking the place of $\mathcal{BA}$ and the variety of *integrally closed* residuated lattices taking the place of $\mathcal{HA}$.

We will then use this Glivenko theorem to construct a *non-standard sequent calculus* for the equational theory of integrally closed residuated lattices.

Finally, we will discuss connections with *pseudo BCI-algebras* and *sirmonoids*.

Residuated lattices

Definition

A *residuated lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e}\rangle$ of type $\langle 2, 2, 2, 2, 2, 0\rangle$ such that

1. the structure $\langle A, \wedge, \vee\rangle$ is a lattice,
2. the structure $\langle A, \cdot, \mathrm{e}\rangle$ is a monoid,
3. for all $a, b, c \in A$

$$a \cdot b \leq c \iff b \leq a\backslash c \iff a \leq c/b.$$

Integrally closed residuated lattices

Definition

Following FUCHS 1963 we call a residuated lattice satisfying the equations

$$x\backslash x \approx \mathrm{e} \quad \text{and} \quad x/x \approx \mathrm{e},$$

integrally closed.

Proposition

Let $\mathbf{A}$ *be a residuated lattice. Then the following are equivalent.*

1. *The residuated lattice* $\mathbf{A}$ *is integrally closed.*
2. *The residuated lattice* $\mathbf{A}$ *satisfies the quasi-equations:*

$$xy \le x \implies y \le \mathrm{e} \quad \textit{and} \quad yx \le x \implies y \le \mathrm{e}.$$

3. *The monoidal unit* e *is the largest idempotent element of* $\mathbf{A}$.

Examples

Example

The following residuated lattices are integrally closed.

1. Integral residuated lattices: $x \leq \mathrm{e}$.
2. Cancellative residuated lattices: $x\backslash xy \approx y$ and $yx/x \approx y$
3. Lattice-ordered groups (ℓ-groups): $x(x\backslash e) \approx \mathrm{e}$.
4. GBL-algebras: $((x \wedge y)\backslash y)y \approx x \wedge y \approx y(y/(x \wedge y))$.

Negations and e-cyclicity

In any residuated lattice $\mathbf{A}$ we have two notions of negation:

$$\sim a := a\backslash \mathrm{e} \qquad \text{and} \qquad - a := \mathrm{e}/a.$$

A residuated lattice is called *e-cyclic* if it satisfies the equation

$$\sim x \approx -x.$$

Proposition

Any integrally closed residuated lattice $\mathbf{A}$ *is* e*-cyclic.*

A double-negation nucleus

Definition

A nucleus on a residuated lattice $\mathbf{A}$ is a closure operator $\gamma\colon A \to A$ such that $\gamma(a)\gamma(b) \leq \gamma(ab)$, for all $a, b \in A$.

Proposition

If $\mathbf{A}$ *is an* e*-cyclic residuated lattice, then the map* $\alpha\colon A \to A$ *given by*

$$a \mapsto {\sim}{\sim}\, a$$

is a nucleus on $\mathbf{A}$*.We therefore obtain a residuated lattice*

$$\mathbf{A}_{\sim\sim} = \langle \alpha[A], \wedge, \vee_{\sim\sim}, \cdot_{\sim\sim}, \backslash, /, \mathrm{e} \rangle,$$

with $a \vee_{\sim\sim} b = {\sim}{\sim}(a \vee b)$ *and* $a \cdot_{\sim\sim} b = {\sim}{\sim}(a \cdot b)$*, for* $a, b \in \alpha[A]$*.*

e-principal homomorphisms

An *e-principal* homomorphism will be a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ of residuated lattices satisfying $h^{-1}(\downarrow \mathrm{e}^{\mathbf{B}}) \subseteq \downarrow \mathrm{e}^{\mathbf{A}}$.

Proposition

Let $\mathbf{A}$ be a integrally closed residuated lattice.

1. *The map $\alpha\colon \mathbf{A} \twoheadrightarrow \mathbf{A}_{\sim\sim}$ is an* e*-principal homomorphism.*
2. *The image $\mathbf{A}_{\sim\sim}$ is an ℓ-group.*
3. *Every $h\colon \mathbf{A} \to \mathbf{G}$, with $\mathbf{G}$ an ℓ-group factors through $\mathbf{A}_{\sim\sim}$.*
4. *The residuated lattice $\mathbf{A}_{\sim\sim}$ is up to isomorphism the unique* e*-principal homomorphic image of $\mathbf{A}$ which is an ℓ-group.*

Corollary

A residuated lattice $\mathbf{A}$ is integrally closed if, and only if, there is an e*-principal homomorphism $\mathbf{A} \twoheadrightarrow \mathbf{G}$ onto an ℓ-group.*

An interior operator on the lattice of subvarieties

1. Let $\mathcal{I}c\mathcal{RL}$ denote the variety of integrally closed residuated lattices.
2. Let $\mathcal{LG}$ denote the variety of ℓ-groups.
3. Evidently, $\mathcal{LG} \subseteq \mathcal{I}c\mathcal{RL}$.
4. For $\mathcal{K} \subseteq \mathcal{I}c\mathcal{RL}$, we define $\mathcal{K}_{\sim\sim} := \{\mathbf{A}_{\sim\sim} \mid \mathbf{A} \in \mathcal{K}\} \subseteq \mathcal{LG}$.

Proposition

1. *If $\mathcal{V} \subseteq \mathcal{I}c\mathcal{RL}$ is a variety, then $\mathcal{V}_{\sim\sim}$ is a variety of ℓ-groups.*
2. *The map $\mathcal{V} \mapsto \mathcal{V}_{\sim\sim}$ is an interior operator on the lattice of subvarieties of the variety $\mathcal{I}c\mathcal{RL}$.*
3. *In particular, $\mathcal{I}c\mathcal{RL}_{\sim\sim} = \mathcal{LG}$.*

The Glivenko property

Definition

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of residuated lattice. Following Galatos & Ono 2006, we say that $\mathcal{V}$ admits the (*equational*) *Glivenko property* with respect to $\mathcal{W}$ if

$$\mathcal{V} \models -\sim s \leq -\sim t \iff \mathcal{W} \models s \leq t \iff \mathcal{V} \models \sim - s \leq \sim - t.$$

In case $\mathcal{V}$ is a variety of e-cyclic residuated lattices, this simplifies to

$$\mathcal{W} \models s \leq t \iff \mathcal{V} \models \sim\sim s \leq \sim\sim t.$$

A Glivenko theorem for varieties of ℓ-groups

Theorem

Let $\mathcal{V} \subseteq \mathcal{IcRL}$ be a variety of integrally closed residuated lattices.

1. *The variety $\mathcal{V}$ admits the Glivenko property with respect to the variety of ℓ-groups $\mathcal{V}_{\sim\sim}$. That is,*

$$\mathcal{V}_{\sim\sim} \models s \leq t \iff \mathcal{V} \models {\sim\sim}\, s \leq {\sim\sim}\, t.$$

2. *If the variety $\mathcal{V}$ is axiomatized by equations of the form $s \leq \mathrm{e}$, then $\mathcal{V}$ is the largest variety of residuated lattices admitting the Glivenko property with respect to $\mathcal{V}_{\sim\sim}$.*

In particular, $\mathcal{IcRL}$ is the is the largest variety of residuated lattices admitting the Glivenko property with respect to $\mathcal{LG}$.

ℓ-group integrality

Corollary

Let $\mathcal{V}$ be a variety of integrally closed residuated lattices. Then

$$\mathcal{V} \models s \leq \mathrm{e} \iff \mathcal{V}_{\sim\sim} \models s \leq \mathrm{e}.$$

In particular,

$$\mathcal{LG} \models s \leq \mathrm{e} \implies \mathcal{IcRL} \models s \leq \mathrm{e}.$$

This fact can also be phrased as an inference rule

$$\frac{\mathcal{IcRL} \models s_1 s_2 \leq t \quad \mathcal{LG} \models u \leq \mathrm{e}}{\mathcal{IcRL} \models s_1 u s_2 \leq t}$$

Sequents

A *sequent* to be an expression of the form $\Gamma \Rightarrow t$ where Γ is a finite sequence of terms and t a term in the language of residuated lattices.

We say that a sequent $s_1, \ldots, s_n \Rightarrow t$ is *valid* in a class of residuated lattices $\mathcal{K}$, denoted by $\models_{\mathcal{K}} \Gamma \Rightarrow t$, if

$$\mathcal{K} \models s_1 \cdots s_n \leq t,$$

where the empty product is understood as e.

A sequent calculus for $\mathcal{I}c\mathcal{RL}$

Let IcRL denote the sequent calculus obtained adding the rule

$$\frac{\Gamma, \Pi \Rightarrow t \quad \models_{\mathcal{LG}} \Delta \Rightarrow \mathrm{e}}{\Gamma, \Delta, \Pi \Rightarrow t} \; (\mathcal{LG}\text{-w})$$

to the standard sequent calculus for residuated lattices.

Theorem

A sequent is valid in all integrally closed residuated lattices, if and only if, it is derivable in the calculus IcRL.

1. The condition $\models_{\mathcal{LG}} \Delta \Rightarrow \mathrm{e}$ is decidable (Holland & McCleary 1979), indeed co-NP-complete (Galatos & Metcalfe 2016).
2. The rule $(\mathcal{LG}\text{-w})$ can be seen as a generalized version of the *balanced weakening rule* due to Kashima & Komori 1992.
3. The rule $(\mathcal{LG}\text{-w})$ can also be seen as a version of the abelian "mix rule" (M_A) introduced by Metcalfe 2006.

Decidability

Theorem

The sequent calculus IcRL *admits cut-elimination.*

Theorem

The equational theory of the variety of integrally closed residuated lattices is decidable, indeed PSPACE-complete.

Note that the quasi-equational theory of $\mathcal{I}c\mathcal{RL}$ is not decidable.

Pseudo BCI-algebras and sirmonoids I

Definition

1. An algebra $\mathbf{A} = \langle A, \backslash, /, \mathrm{e}\rangle$ of type $\langle 2, 2, 0\rangle$ satisfying:
   (i) $((x\backslash z)/(y\backslash z))/(x\backslash y) \approx \mathrm{e}$,
   (ii) $(y/x)\backslash((z/y)\backslash(z/x)) \approx \mathrm{e}$,
   (iii) $\mathrm{e}\backslash x \approx x$,
   (iv) $x/\mathrm{e} \approx x$,
   (v) $x\backslash y \approx \mathrm{e} \;\;\&\;\; y\backslash x \approx \mathrm{e} \implies x \approx y$.

   is called a *pseudo BCI-algebra*.

2. An algebra $\mathbf{S} = \langle S, \cdot, \backslash, /, \mathrm{e}\rangle$ of type $\langle 2, 2, 2, 0\rangle$ such that
   (i) The reduct $\langle S, \backslash, /, \mathrm{e}\rangle$ is a pseudo BCI-algebra.
   (ii) $\mathbf{S}$ satisfies the equation $(x \cdot y)\backslash z \approx y\backslash(x\backslash z)$,

   is called a *semi-integral residuated monoid* or (*sirmonoid*).

$$a \preceq b \text{ if, and only if, } a\backslash b = \mathrm{e}.$$

$$a \preceq b \text{ if, and only if, } b/a = \mathrm{e}.$$

Pseudo BCI-algebras and sirmonoids II

Theorem (Emanovský & Kühr 2018, Raftery & van Alten 2000)

Any pseudo BCI-algebra is a subreduct of a sirmonoid.

Corollary

The quasi-equational theory of sirmonoids is a conservative extension of the quasi-equational theory of pseudo BCI-algebras.

Sirmonoids and integrally closed residuated lattices

Proposition

1. *Any* $\{\cdot, \backslash, /, \mathrm{e}\}$*-reduct of an integrally closed residuated lattice is a sirmonoid.*
2. *There are sirmonoids which are not subreducts of any integrally closed residuated lattice.*

Proposition

The equational theory of the variety $\mathcal{I}c\mathcal{RL}$ *is a conservative extension of the equational theory of sirmonoids.*

Corollary

The equational theories of pseudo BCI-algerbas and sirmonoids are decidable.

Concluding remarks

1. Can we give examples of other non-standard versions of structural rules, such as *exchange* or *contraction*?
2. Can we axiomatize the quasi-variety of sirmonoids which are subreducts of integrally closed residuated lattices?

Thank you very much for your time and attention

http://arxiv.org/abs/1902.08144