Twist products arising from residuated bimodules

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Introduction

We investigate a certain construction of an “algebra of fractions” arising originally in ring theory (Nagata 1962) in the context of residuated lat’s.

This will allow us to build a residuated lattice from a (residuated) module.

In particular, it will naturally embed a res. lattice into an involutive one.

We build on the results of Tsinakis & Wille (2006) and Busaniche & Cignoli (2014). A related construction also studied by Ono & Rivieccio (2014).

Obligatory definitions

Definition. A **residuated lattice** is an algebra $\langle L, \vee, \wedge, \cdot, \backslash, /, 1\rangle$ such that

- $\langle L, \vee, \wedge\rangle$ is a lattice,
- $\langle L, \cdot, 1\rangle$ is a monoid, and
- the binary operations $x\backslash y$ and x/y are the **residuals** of multiplication:

$$x \leq z/y \iff x \cdot y \leq z \iff y \leq x\backslash z.$$

An **involutive** (more precisely, cyclic involutive) residuated lattice is a residuated lattice with a constant 0 such that $x\backslash 0 = 0/x$ (denoted $\overline{x}$) and

$$x\backslash y = \overline{\overline{y} \cdot x} \qquad x/y = \overline{y \cdot \overline{x}}.$$

Semigroup action

Let us fix a **semigroup of scalars** $\mathbf{S} = \langle S, \cdot \rangle$. Scalars will be denoted a, b, c. The elements on which the scalars act will be denoted x, y, z.

Definition. A **(two-sided) action** of **S** on a set X is a pair of maps $*\colon S \times X \to X$ and $*\colon X \times S \to X$ such that for each $x \in X$

$$a * (b * x) = (a \cdot b) * x,$$
$$(x * a) * b = x * (a \cdot b),$$
$$(a * x) * b = a * (x * b).$$

If S has a unit 1, a **unital action** moreover satisfies

$$1 * x = x = x * 1.$$

(No confusion threatens: $a * x$ is the left action, $x * a$ is the right action.)

Semigroup modules

Definition. A **semigroup (bi)module** is a two-sorted algebra consisting of a semigroup of scalars **S** acting on a semigroup $\mathbf{M} = \langle M, + \rangle$ so that

$$a * (x + y) = (a * x) + (a * y),$$
$$(x + y) * a = (x * a) + (y * a).$$

For most of this talk, **M** will be a join semilattice $\mathbf{M} = \langle M, \vee \rangle$.

A **partially ordered semigroup (posemigroup)** is a semigroup with a partial order w.r.t. which multiplication is isotone. Same for **pomonoids**.

Accordingly, in a **posemigroup module** a posemigroup **S** has an isotone action on a posemigroup **M** satisfying the above equalities.

Residuated modules

Definition. A **residuated** posemigroup module is one equipped with the residuals of the actions, that is, with maps

$$ {}_*\backslash : S \times M \to M, \qquad {}^*/ : M \times M \to S, $$
$$ \backslash^* : M \times M \to S, \qquad /_* : M \times S \to S, $$

which satisfy the equivalences

$$ x \leq a \,{}_*\backslash\, y \iff a * x \leq y \iff a \leq y \,{}^*/\, x, $$
$$ a \leq x \backslash^* y \iff x * a \leq y \iff x \leq y /_* a. $$

Unital modules

There are two ways in which a posemigroup module can be unital:

- **S** has a unit 1 and $1 * x = x = x * 1$ (**S**-unitality, or unitality *simpliciter*),
- **M** has a unit 0 and $a * 0 = 0 = 0 * a$ (**M**-unitality).

We shall generally embrace the first and avoid the second: **M** is going to be join semilattice, so **M**-unitality will amount to having a lower bound.

For the most part we be interested in unital residuated modules where a residuated **lattice** acts on a **lattice**. These will be called **urℓ-modules**.

Examples of residuated modules

Trivial example. A residuated lattice $\mathbf{L} = \langle L, \vee, \wedge, \cdot, \backslash, /, 1\rangle$ has a unital residuated module action on itself, i.e. on $\langle L, \vee\rangle$:

$$a * x = a \cdot x, \qquad a\,{}_*\backslash x = a \backslash x, \qquad y\,{}^*/ x = y / x,$$
$$x * a = x \cdot a, \qquad x \backslash^* y = x \backslash y, \qquad x /_* a = x / a.$$

Important example. A residuated lattice $\mathbf{L} = \langle L, \vee, \wedge, \cdot, \backslash, /, 1\rangle$ has a unital residuated module action on its order dual, i.e. on $\mathbf{L}^\partial := \langle L, \wedge\rangle$:

$$a * x = x / a, \qquad y\,{}^*/ x = y \backslash x, \qquad a\,{}_*\backslash x = x \cdot a,$$
$$x * a = a \backslash x, \qquad x \backslash^* y = x / y, \qquad x /_* a = a \cdot x.$$

This will be called the **canonical action** of **L** on its dual $\mathbf{L}^\partial$.

Modules arising from residuated lattices

We now give a construction which will yield most urℓ-modules starting from a one-sorted algebra. Some definitions will be needed first...

Definition. A **conucleus** on a residuated lattice $\mathbf{L}$ is an interior operator σ such that $\sigma x \cdot \sigma y \leq \sigma(x \cdot y)$ and $\sigma(1) = 1$.

The image of σ (the set of σ-open elements of $\mathbf{L}$) is a residuated lattice $\mathbf{L}_\sigma = \langle \sigma[L], \vee, \wedge_\sigma, \cdot, 1, \backslash_\sigma, /_\sigma \rangle$, where $x \circ_\sigma y = \sigma(x \circ y)$ for $\circ \in \{\wedge, \backslash, /\}$.

Example. A conucleus on a Boolean algebra is precisely a topological interior operator: $\Box(x \wedge y) = \Box x \wedge \Box y$ and $\Box\Box x = \Box x \leq x$.

Similarly, if γ is a closure operator on a (residuated) lattice $\mathbf{L}$, then the γ-closed elements of $\mathbf{L}$ form a lattice $\mathbf{L}_\gamma = \langle \gamma[L], \wedge, \vee_\gamma \rangle$.

Modules arising from residuated lattices

Fact. Suppose that a conucleus σ and a closure operator γ on a residuated lattice $\mathbf{L}$ satisfy the following conditions (called **structurality**):

$$\sigma a \cdot \gamma x \leq \gamma(a \cdot x),$$
$$\gamma x \cdot \sigma a \leq \gamma(x \cdot a).$$

Then we obtain a urℓ-module where $\mathbf{L}_\sigma$ acts on $\mathbf{L}_\gamma$ as:

$$a * x = \gamma(a \cdot x), \qquad x \backslash^* y = \sigma(x \backslash y), \qquad a_* \backslash x = a \backslash x,$$
$$x * a = \gamma(x \cdot a), \qquad x^* / y = \sigma(x / y), \qquad x /_* a = x / a.$$

We now show that almost every urℓ-module in fact arises in this way: given a urlℓ-module, we construct a suitable res. lattice with σ and γ.

Semigroups of fractions

Consider a posemigroup module with **S** acting on **M**. We define the posemigroup of fractions (the **Nagata product**) $\mathbf{S} \rtimes \mathbf{M}$ as follows:

$$\langle a,x\rangle \leq \langle b,y\rangle \iff a \leq b \text{ and } x \leq y,$$
$$\langle a,x\rangle \circ \langle b,y\rangle = \langle a \cdot b, x * b + a * y\rangle.$$

This is nothing but the ordinary rule for adding fractions:

$$\frac{x}{a} + \frac{y}{b} = \frac{xb + ay}{ab}$$

If the module is both **S**-unital and **M**-unital, then $\mathbf{S} \rtimes \mathbf{M}$ has a unit $\langle 1, 0\rangle$.

This construction was introduced by Nagata (1962) for modules over rings, but as observed by Schmidt (1975) it is in essence a semigroup construction. Later rediscovered by Tsinakis & Wille (2006) in the context of residuated lattices.

Semigroups of fractions: example

Example. The multiplicative monoid $\{3^i \mid i \in \mathbb{N}\}$ (isomorphic to the add. monoid $\mathbb{N}$) acts on the additive monoid $\mathbb{N}$, resulting in a url-module:

$$3^i * n = 3^i \cdot n, \qquad 3^i {}_*\backslash n = \lfloor \frac{n}{3^i} \rfloor, \qquad n^*/m = 3^{\lfloor \log_3 \frac{n}{m} \rfloor}.$$

The Nagata product is the additive monoid of *formal* fractions of form $\frac{n}{3^i}$.

As in the construction of the group of fractions, to obtain the additive monoid of *actual* fractions of the form $\frac{n}{3^i}$ we factor by the congruence:

$$\langle a,x\rangle \sim \langle b,y\rangle \iff a*y = b*x.$$

Moreover, this monoid of fractions can be equipped with the multiplication $\langle a,x\rangle \circ \langle b,y\rangle = \langle a\cdot b, x\cdot y\rangle$. This leads to **semirings** (not pursued here).

Semigroup of fractions: residuation and unitality

Fact. For each urℓ-module the ℓ-semigroup $\mathbf{S} \rtimes \mathbf{M}$ is residuated:

$$\langle a,x\rangle \backslash \langle b,y\rangle = \langle a\backslash b \wedge x\backslash^{*} y, a_{*}\backslash y\rangle,$$
$$\langle a,x\rangle / \langle b,y\rangle = \langle a/b \wedge x^{*}/y, x/_{*} b\rangle.$$

Moreover, it is has a unit $\langle 1, \bot\rangle$ whenever $\mathbf{M}$ has a lower bound $\bot$.

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Moreover, it is has a unit $\langle 1, \bot\rangle$ whenever $\mathbf{M}$ has a lower bound $\bot$.

We would like to avoid assuming that $\mathbf{M}$ has a lower bound, though. Take any $0 \in \mathbf{M}$ and consider the subalgebra $\mathbf{S} \rtimes_0 \mathbf{M}$ of $\mathbf{S} \times \mathbf{M}$ with the universe

$$\{\langle a,x\rangle \in \mathbf{S} \rtimes \mathbf{M} \mid 0 * a \leq x \text{ and } a * 0 \leq x\}.$$

Fact. $\mathbf{S} \rtimes_0 \mathbf{M}$ is a residuated lattice with unit $\langle 1, 0\rangle$ for each urℓ-module.

Remark. The two constructions coincide if $\mathbf{M}$ has a bottom $\bot$ and $0 = \bot$.

Semigroup of fractions: embedding $\mathbf{S}$ and $\mathbf{M}$ into $\mathbf{S} \rtimes_0 \mathbf{M}$

From now on we assume that 0 is a **cyclic** element of **M**: $a * 0 = 0 * a$.
A module equipped with such a point will be called **cyclic-pointed**.

Fact. There are embeddings $\iota_{\mathbf{S}} \colon \mathbf{S} \to \mathbf{S} \rtimes_0 \mathbf{M}$ and $\iota_{\mathbf{M}} \colon \mathbf{M} \to \mathbf{S} \rtimes_0 \mathbf{M}$:

$$\iota_{\mathbf{S}} \colon a \mapsto \langle a, 0 * a \rangle = \langle a, a * 0 \rangle,$$
$$\iota_{\mathbf{M}} \colon x \mapsto \langle 0 \backslash^{*} x, x \rangle = \langle x^{*}/0, x \rangle.$$

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$$\iota_{\mathbf{S}} \colon a \mapsto \langle a, 0 * a \rangle = \langle a, a * 0 \rangle,$$
$$\iota_{\mathbf{M}} \colon x \mapsto \langle 0 \backslash^* x, x \rangle = \langle x^* / 0, x \rangle.$$

Similarly, there is a conucleus σ and a closure operator γ on $\mathbf{S} \rtimes_0 \mathbf{M}$:

$$\sigma \colon \langle a, x \rangle \mapsto \langle a, 0 * a \rangle = \langle a, a * 0 \rangle,$$
$$\gamma \colon \langle a, x \rangle \mapsto \langle 0 \backslash^* x, x \rangle = \langle x^* / 0, x \rangle.$$

Clearly the images of $\iota_{\mathbf{S}}$ and σ coincide, as do the images of $\iota_{\mathbf{M}}$ and γ.
The maps σ and γ therefore allow us to recover $\mathbf{S}$ and $\mathbf{M}$ from $\mathbf{S} \rtimes_0 \mathbf{M}$.

(The algebra $\mathbf{S} \rtimes_0 \mathbf{M}$ also comes with the constant $\iota_{\mathbf{M}}(0)$ denoted just 0.)

Recovering the original module from $\mathbf{S} \rtimes_0 \mathbf{M}$

The maps σ and γ satisfy the compatibility conditions that we called structurality: $\sigma a \circ \gamma x \leq \gamma(a \circ x)$ and $\gamma x \circ \sigma a \leq \gamma(x \circ a)$.

Therefore $(\mathbf{S} \rtimes_0 \mathbf{M})_\sigma \cong \mathbf{S}$ acts on $(\mathbf{S} \rtimes_0 \mathbf{M})_\gamma \cong \mathbf{M}$ as follows:

$$
\begin{aligned}
a * x &= \gamma(a \circ x), & x \backslash^* y &= \sigma(x \backslash y), & a \,{}_*\backslash\, x &= a \backslash x, \\
x * a &= \gamma(x \circ a), & x \,{}^*/\, y &= \sigma(x/y), & x /_* a &= x/a.
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We have in fact recovered our original module from $\mathbf{S} \rtimes_0 \mathbf{M}$ and σ and γ!

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Corollary. The conuclear images of (cyclic) involutive residuated lattices are precisely residuated lattices which contain a cyclic point 0 ($a \backslash 0 = 0 / a$).

Proof. Witnessed by $\mathbf{L} \rtimes_0 \mathbf{L}^\partial$ and σ for the canonical action of $\mathbf{L}$ on $\mathbf{L}^\partial$.

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We have in fact recovered our original module from $\mathbf{S} \rtimes_0 \mathbf{M}$ and σ and γ!

Corollary. The conuclear images of (cyclic) involutive residuated lattices are precisely residuated lattices which contain a cyclic point 0 ($a \backslash 0 = 0 / a$).

Proof. Witnessed by $\mathbf{L} \rtimes_0 \mathbf{L}^\partial$ and σ for the canonical action of $\mathbf{L}$ on $\mathbf{L}^\partial$.

(Compare Montagna & Tsinakis (2010): the conuclear images of Abelian ℓ-groups are precisely the commutative cancellative residuated lattices.

Also: the conuclear images of Boolean algebras are Heyting algebras.)

One-sorted and two-sorted structures

Let us take stock of where we are right now.

We have two kinds of structures:

- Two-sorted: cyclic-pointed urℓ-modules, i.e. **S**, **M**, $*$ and residuals, 0.
- One-sorted: residuated lattices with (structural) σ, γ, and (cyclic) 0.

We can pass from one side to the other:

- From one- to two-sorted: use the action defined by σ and γ.
- From two- to one-sorted: construct the Nagata product.

We have just seen that each cyclic-pointed urℓ-module arises this way.
Conversely, which pointed res. lattices with σ and γ arise this way?

Subalgebras of Nagata products

Theorem. A residuated lattice $\mathbf{L}$ equipped with a conucleus σ, a closure operator γ, and a constant $0 = \gamma(0)$ embeds into a Nagata product if and only if it satisfies the following equations:

$$\sigma x \backslash y \wedge x \backslash \gamma y = x \backslash y \qquad \sigma(x \cdot y) = \sigma x \cdot \sigma y$$
$$x / \sigma y \wedge \gamma x / y = x / y \qquad \sigma(x \vee y) = \sigma x \vee \sigma y$$
$$\sigma x \cdot y \vee x \cdot \sigma y = x \cdot y \qquad \gamma(x \wedge y) = \gamma x \wedge \gamma y$$

$$\sigma x \backslash \gamma y = \gamma(x \backslash y)$$
$$\gamma x / \sigma y = \gamma(x / y)$$

$$\gamma(0 \cdot \sigma x) = \gamma(\sigma x) \qquad \sigma(0 \backslash \gamma x) = \sigma(\gamma x)$$
$$\gamma(\sigma x \cdot 0) = \gamma(\sigma x) \qquad \sigma(\gamma x / 0) = \sigma(\gamma x)$$

In that case $x \mapsto \langle \sigma x, \gamma x \rangle$ is a map embedding the algebra into $\mathbf{L}_\sigma \rtimes_0 \mathbf{L}_\gamma$.

We have axiomatizated the variety of subalgebras of Nagata products. The two constructions in fact yield a categorical adjunction between varieties.

Nagata twist product

What if we restrict to urℓ-modules where a res. lattice $\mathbf{L}$ acts canonically on $\mathbf{L}^{\partial}$? That is, to the **Nagata twist products** $\mathbf{L}_{0}^{\bowtie} := \mathbf{L} \rtimes_{0} \mathbf{L}^{\partial}$ for cyclic 0.

(A point $0 \in \mathbf{L}$ being cyclic with respect to this action means: $a \backslash 0 = 0 / a$.)

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(A point $0 \in \mathbf{L}$ being cyclic with respect to this action means: $a \backslash 0 = 0/a$.)

Fact. The Nagata twist product is an **involutive** residuated lattice, where the dualizing element is $\langle 0, 1 \rangle$ and the involution is the natural one:

$$\langle a, b \rangle \backslash \langle 0, 1 \rangle = \langle b, a \rangle = \langle a, b \rangle / \langle 0, 1 \rangle.$$

The maps σ and γ are dual in a Nagata twist product ($\overline{\langle a, b \rangle} = \langle b, a \rangle$):

$$\sigma(x) = \overline{\gamma(\overline{x})} \text{ and } \gamma(x) = \overline{\sigma(\overline{x})}, \text{ where } \overline{\langle a, b \rangle} = \langle b, a \rangle.$$

Studied by Tsinakis & Wille (2006). Similar but distinct twist product construction also studied by Ono & Rivieccio (2014).

Subalgebras of Nagata twist products

Theorem. A 0-pointed residuated lattice with σ and γ embeds into a Nagata **twist** product if and only if it satisfies the previous equations and moreover it is (cyclic) **involutive** and $\gamma(x) = \overline{\sigma(\overline{x})}$.

This construction again yields a cat. adjunction between two varieties.

(Note that since γ is term definable, we can drop it from the signature.)

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A **special case** of this adjunction for (Nagata twist products of) integral commutative residuated lattices is due to Busaniche & Cignoli (2014).

Their axiomatization looks somewhat different, in particular they did not need to add the conucleus σ to the signature: in their case $\sigma(x) = 1 \wedge x$.

Application to Brouwerian algebras

Finally, let us mention an application of the Nagata twist product.

Theorem. Each Brouwerian algebras is the negative cone of a commutative idempotent involutive residuated lattice.

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Proof sketch. Take the twist product $\mathbf{L}_1^{\bowtie}$ with the embbeding $\iota_{\mathbf{L}}\colon \mathbf{L} \to \mathbf{L}_0^{\bowtie}$.

Call elements of the form $\iota_{\mathbf{L}}(a) \cdot \overline{\iota_{\mathbf{L}}(b)}$ fractions, $\iota_{\mathbf{L}}(a) + \overline{\iota_{\mathbf{L}}(b)}$ co-fractions.

Fractions = image of the conucleus $\langle a, b \rangle \mapsto \langle a, a \to b \rangle$ on $\mathbf{L}_0^{\bowtie}$.

Fractions $\cap$ co-fractions = image of nucleus $\langle a, b \rangle \mapsto \langle b \to a, b \rangle$ on fractions.

Fractions $\cap$ co-fractions form the desired involutive residuated lattice.

(Moreover, its elements have the form $\iota(a) \cdot \overline{\iota(b)}$ as well as $\iota(a) + \overline{\iota(b)}$.)

Conclusion

To conclude with, let us mention some connections between residuated lattice modules and other areas:

- constructing semirings of fractions
- triple constructions (Chen & Grätzer for Stone lattices, ...):
  restrict to elements of the form $\langle a, a * x \rangle$ (Schmidt 1975)
- many-valued modal logic:
  a many-valued accessibility relation may take values in **S**,
  with $u(\Diamond a) = \bigvee_{v \in W} (uRv * v(a))$ and $u(\Box a) = \bigwedge_{v \in W} (uRv \,{}_{*}\backslash\, v(a))$

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Thank you for your attention!