Derivations on bounded pocrims and MV-algebras with product

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Derivations on rings

A **derivation** on a ring $(R; +, \cdot)$ is a map $f\colon R \to R$ satisfying

$$f(x+y) = f(x) + f(y) \text{ and } f(xy) = f(x)y + xf(y),$$

for all $x, y \in R$.

Papers about derivations on algebras:

- Lattices – Szász, G. (1975);
- MV-algebras – Alshehri (2010), Yazarli (2013), Ghorbani et al. (2013);
- BCI-algebras – Jun et al. (2004);
- Basic algebras – Krňávek and Kühr (2015);
- GMV-algebras – Rachůnek and Šalounová (2018).

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Bounded pocrims

A partially ordered commutative residuated integral monoid (pocrim) is a structure $(M; \leq, \odot, \to, 1)$ where:

- $(M; \leq, 1)$ is a poset with the greatest element;
- $(M; \odot, 1)$ is a commutative monoid;
- $x \odot y \leq z$ iff $x \leq y \to z$ for all $x, y, z \in M$.

Since $x \leq y$ iff $x \to y = 1$, pocrims may be defined as algebras $(M; \odot, \to, 1)$, and bounded pocrim as algebras $(M; \odot, \to, 0, 1)$.

Negation and addition are defined as follows:

$$x' = x \to 0 \quad \text{and} \quad x \oplus y = (x' \odot y')'.$$

In what follows, $\mathbf{M} = (M; \odot, \to, 1)$ is a bounded pocrim.

Nuclei and conuclei.

A nucleus on $\mathbf{M} = (M; \odot, \to, 1)$ is a closure operator f such that, for all $x, y \in M$,

$$f(x) \odot f(y) \leq f(x \odot y).$$

The f-image $\mathbf{M}_f = (M_f; \odot_f, \to, f(0), 1)$, where

$$x \odot_f y = f(x \odot y),$$

is a bounded pocrim.

A conucleus on is an interior operator satisfying

$$f(x) \odot f(y) \leq f(x \odot y) \quad \text{and} \quad f(1) \odot f(x) = f(x),$$

for all $x, y \in M$.

The f-image $\mathbf{M}_f = (M_f; \odot, \to_f, 0, f(1))$, where

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Derivations on bounded pocrims

A derivation on a bounded pocrim $\mathbf{M}$ is $f: M \to M$ satisfying

$$f(x \oplus y) = f(x) \oplus f(y) \quad \text{and} \quad f(x \odot y) = (f(x) \odot y) \oplus (x \odot f(y))$$

for all $x, y \in M$.

The set of derivations on $\mathbf{M}$ is denoted by $\mathcal{D}(\mathbf{M})$.

Simple examples:

- The zero map $o: x \mapsto 0$ is a trivial derivation.
- The identity map $id: x \mapsto x$ is a derivation iff $\mathbf{M}$ is (term-equvalent to) a Boolean algebra:
  - $\mathbf{M}$ satisfies the equation $x \oplus x = x$;
  - $\mathbf{M}$ satisfies the equations $x \odot x = x$ and $x'' = x$;
  - $x \odot y = x \wedge y$, $x \oplus y = x \vee y$ and $x \to x' \vee y$.
- Let $\mathbf{M} = \mathbf{K} \times \mathbf{L}$ where $\mathbf{K}$ is a Boolean algebra. The map $f: (x, y) \mapsto (x, 0)$ is a derivation on $\mathbf{M}$.

Derivations on bounded pocrims

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Derivations on bounded pocrims

The double negation $\delta : x \mapsto x''$ is nucleus on $\mathbf{M}$. The δ-image $\mathbf{M}_\delta = (M_\delta; \odot_\delta, \to, 0, 1)$ is an involutive pocrim, i.e., it satisfies the equation

$$x'' = x.$$

In general, δ is not homomorphism of $\mathbf{M}$ onto $\mathbf{M}_\delta$, and $\mathbf{M}_\delta$ is not a subalgebra of $\mathbf{M}$.

Further examples:

- δ is a derivation on $\mathbf{M}$ iff $\mathbf{M}_\delta$ is a Boolean algebra.
- Let $\mathbf{M} = \mathbf{K} \times \mathbf{L}$ where $\mathbf{K}_\delta$ is a Boolean algebra. The map $f : (x, y) \mapsto (x'', 0)$ is a derivation on $\mathbf{M}$.

Derivations on bounded pocrims

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Derivations on bounded pocrims

For every $f \in \mathcal{D}(\mathbf{M})$:

- $f(0) = 0$;
- $f(x) = f(x)'' = f(x'')$, so $f(x) \in M_\delta$, for all $x \in M$;
- $f(x) \leq x''$ and $f(x) \leq f(1)$ for all $x \in M$.

There is a bijection between $\mathcal{D}(\mathbf{M})$ and $\mathcal{D}(\mathbf{M}_\delta)$:

- for every $f \in \mathcal{D}(\mathbf{M})$, $f\restriction_{M_\delta} \in \mathcal{D}(\mathbf{M}_\delta)$;
- for every $f \in \mathcal{D}(\mathbf{M}_\delta)$, the map $\hat{f} : M \to M$ defined by $\hat{f}(x) = f(x'')$ is a derivation on $\mathbf{M}$ such that $\hat{f}\restriction_{M_\delta} = f$.

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Derivations on involutive pocrims

Let $\mathbf{M}$ be an involutive pocrim and $f \in \mathcal{D}(\mathbf{M})$. Then

$$f(x) = x \odot f(1)$$

for every $x \in M$. Moreover, f is a conucleus, the f-image $\mathbf{M}_f$ is a Boolean algebra, and f is a homomorphism of $\mathbf{M}$ onto $\mathbf{M}_f$.

Boolean elements

An element $a \in M$ is Boolean if $a \vee a'$ exists and $a \vee a' = 1$.
The set of Boolean elements of $\mathbf{M}$ is denoted by $\mathcal{B}(\mathbf{M})$.

Boolean elements $\longleftrightarrow$ direct product decompositions:

- If $a \in \mathcal{B}(\mathbf{M})$, then
  - $[\mathbf{0}, \mathbf{a}] = ([0, a]; \odot, \to_a, 0, a)$, where $x \to_a y = a \odot (x \to y)$, is a bounded pocrim,
  - $[\mathbf{a}, \mathbf{1}] = ([a, 1]; \odot, \to, a, 1)$ is a bounded pocrim,
  - $\mathbf{M} \cong [\mathbf{0}, \mathbf{a}] \times [\mathbf{a}, \mathbf{1}]$ under $\eta : \ x \mapsto (a \odot x, a' \to x) = (a \wedge x, a \vee x)$.
- If $\mathbf{M} \cong \mathbf{K} \times \mathbf{L}$ under $\theta : \ x \mapsto (x_{\mathbf{K}}, x_{\mathbf{L}})$, then $a = \theta^{-1}(1_{\mathbf{K}}, 0_{\mathbf{L}}) \in \mathcal{B}(\mathbf{M})$, $[\mathbf{0}, \mathbf{a}] \cong \mathbf{K}$ and $[\mathbf{a}, \mathbf{1}] \cong \mathbf{L}$.

Derivations on involutive pocrims

Let $\mathbf{M}$ be an involutive pocrim and $f \in \mathcal{D}(\mathbf{M})$.

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$$f(x) = x \odot f(1)$$

for every $x \in M$. Moreover, f is a conucleus, the f-image $\mathbf{M}_f$ is a Boolean algebra, and f is homomorphism of $\mathbf{M}$ onto $\mathbf{M}_f$.

In any involutive pocrim $\mathbf{M}$, there is a bijection between:

- the derivations $f \in \mathcal{D}(\mathbf{M})$ such that $f(1) \in \mathcal{B}(\mathbf{M})$;
- The Boolean elements $a \in \mathcal{B}(\mathbf{M})$ such that $[\mathbf{0}, \mathbf{a}]$ is a Boolean algebra ($\mathbf{M}_f = [\mathbf{0}, \mathbf{a}]$ if $a = f(1) \in \mathcal{B}(\mathbf{M})$);
- the direct product decompositions $\mathbf{M} \cong \mathbf{K} \times \mathbf{L}$ where $\mathbf{K}$ is a Boolean algebra ($\mathbf{M} \cong [\mathbf{0}, \mathbf{a}] \times [\mathbf{a}, \mathbf{1}]$ if $a = f(1) \in \mathcal{B}(\mathbf{M})$) .

Note: This applies to MV-algebras.

Derivations on involutive pocrims

Let $\mathbf{M}$ be an involutive pocrim and $f \in \mathcal{D}(\mathbf{M})$. If $f(1) \in \mathcal{B}(\mathbf{M})$, then

$$f(x) = x \odot f(1) = x \wedge f(1)$$

for every $x \in M$. Moreover, f is a conucleus, the f-image $\mathbf{M}_f$ is a Boolean algebra, and f is homomorphism of $\mathbf{M}$ onto $\mathbf{M}_f$.

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- the direct product decompositions $\mathbf{M} \cong \mathbf{K} \times \mathbf{L}$ where $\mathbf{K}$ is a Boolean algebra ($\mathbf{M} \cong [\mathbf{0}, \mathbf{a}] \times [\mathbf{a}, \mathbf{1}]$ if $a = f(1) \in \mathcal{B}(\mathbf{M})$) .

Note: This applies to MV-algebras.

Derivations on bounded pocrims

Let $\mathbf{M}$ be a bounded pocrim and $f \in \mathcal{D}(\mathbf{M})$. Then

$$f(x) = (x \odot f(a))''$$

for every $x \in M$. Moreover, $\mathbf{M}_f = (M_f; \odot_\delta, \to, 0, f(1))$ is a Boolean algebra.

Since $f{\restriction}_{M_\delta} \in \mathcal{D}(\mathbf{M}_\delta)$, we have

$$f(x) = f(x'') = x'' \odot_\delta f(1) = (x \odot f(1)))''$$

for $x \in M$.

Note: f is not a conucleus on $\mathbf{M}$, so $\mathbf{M}_f$ is the f-image of $\mathbf{M}_\delta$.

Derivations on bounded pocrims

A (bounded) pocrim is divisible if it satisfies the equation

$$x \odot (x \to y) = y \odot (y \to x).$$

Let $\mathbf{M}$ be a divisible pocrim and $f \in \mathcal{D}(\mathbf{M})$. Then

$$f(x) = (x \odot f(1))'' = x'' \odot f(1) = x'' \wedge f(1)$$

for every $x \in M$.

Derivations on bounded pocrims

Let **M** be a bounded pocrim and $f \in \mathcal{D}(\mathbf{M})$. If $f(1) \in \mathcal{B}(\mathbf{M})$, then

$$f(x) = (x \odot f(1))'' = x'' \odot f(1) = x'' \wedge f(1)$$

for every $x \in M$.

A (bounded) pocrim is prelinear if it satisfies the equation

$$((x \to y) \to z) \odot ((y \to x) \to z) \leq z.$$

If **M** is prelinear, then $f(1) \in \mathcal{B}(\mathbf{M})$ for every $f \in \mathcal{D}(\mathbf{M})$.

Derivations on bounded pocrims

Let **M** be a bounded pocrim and $f \in \mathcal{D}(\mathbf{M})$. If $f(1) \in \mathcal{B}(\mathbf{M})$, then

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for every $x \in M$.

A (bounded) pocrim is prelinear if it satisfies the equation

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If **M** is prelinear, then $f(1) \in \mathcal{B}(\mathbf{M})$ for every $f \in \mathcal{D}(\mathbf{M})$.

Derivations on bounded pocrims

Let $f \in \mathcal{D}(\mathbf{M})$. If $a = f(1) \in \mathcal{B}(\mathbf{M})$, then $\mathbf{M} \cong [\mathbf{0}, \mathbf{a}] \times [\mathbf{a}, \mathbf{1}]$, $f(x) = x'' \odot a = x'' \wedge a$ for all $x \in M$, and $\mathbf{M}_f = [\mathbf{0}, \mathbf{a}]_\delta$ is a Boolean algebra.

In any bounded pocrim $\mathbf{M}$, there is a bijection between:

- the derivations $f \in \mathcal{D}(\mathbf{M})$ such that $f(1) \in \mathcal{B}(\mathbf{M})$;
- the Boolean elements $a \in \mathcal{B}(\mathbf{M})$ such that $[\mathbf{0}, \mathbf{a}]_\delta$ is a Boolean algebra;
- the direct product decompositions $\mathbf{M} \cong \mathbf{K} \times \mathbf{L}$ where $\mathbf{K}_\delta$ is a Boolean algebra.

Note: This applies to prelinear bounded pocrims and, in particular, to BL-algebras.

Derivations and coderivations

Let $\mathbf{M}$ be involutive and $f \in \mathcal{D}(\mathbf{M})$. Since $f(x) = x \odot f(1)$, f is a residuated map, i.e., there exists a unique f^* such that

$$f(x) \leq y \quad \text{iff} \quad x \leq f^*(y)$$

for all $x, y \in M$, because

$$x \odot f(1) \leq y \quad \text{iff} \quad x \leq f(1) \to y,$$

by the residuation law.

Hence f^*, the residual of f, is given by

$$f^*(x) = f(1) \to x = (x' \odot f(1))' = f(x')'.$$

Characterization of the residuals f^* of derivations f?

Derivations and coderivations

A bounded pocrim $\mathbf{M}$ is normal if it satisfies the equation

$$(x \odot y)'' = x'' \odot y'',$$

or equivalently, if $\mathbf{M}_\delta$ is a subalgebra of $\mathbf{M}$.

Let $\mathbf{M}$ be normal. A coderivation on $\mathbf{M}$ is $f : M \to M$ satisfying

$$f(x \odot y) = f(x) \odot f(y) \quad \text{and} \quad f(x \oplus y) = (f(x) \oplus y) \odot (x \oplus f(y))$$

for all $x, y \in M$.

The set of coderivations is denoted by $\mathcal{C}(\mathbf{M})$.

Derivations and coderivations

For any map $f: M \to M$ we define $f^\sharp: M \to M$ by

$$f^\sharp(x) = f(x')'.$$

Let $\mathbf{M}$ be normal. Equip $\mathcal{D}(\mathbf{M})$ and $\mathcal{C}(\mathbf{M})$ with pointwise order.

- The map $\alpha: f \mapsto f^\sharp$ – formally (α, α) – is an antitone Galois connection between $\mathcal{D}(\mathbf{M})$ and $\mathcal{C}(\mathbf{M})$.
- All derivations $f \in \mathcal{D}(\mathbf{M})$ are closed, whereas a coderivation $f \in \mathcal{C}(\mathbf{M})$ is closed iff $f(x) = f(x'')$ for all $x \in M$.
- If $\mathbf{M}$ is involutive, then α is bijection.

PMV-algebras – MV-algebras with product

MV-algebras are term-equivalent with bounded pocrims satisfying the equation

$$(x \to y) \to y = (y \to x) \to x.$$

In the "standard" MV-algebra $[\mathbf{0}, \mathbf{1}]_{MV} = ([0, 1]; \odot, \to, 0, 1)$:

$$x \odot y = \max(x + y - 1, 0), \quad x \to y = \min(1 - x + y, 1),$$
$$x' = 1 - x, \quad x \oplus y = \min(x + y, 1).$$

PMV-algebras – MV-algebras with product

A PMV-algebra is an algebra $\mathbf{M} = (M;\ \odot, \to, \cdot, 0, 1)$ where

- $(M;\ \odot, \to, 0, 1)$ is (term-equivalent to) an MV-algebra,
- $M;\ \cdot, 1)$ is a commutative monoid, and
- $(x \odot y') \cdot z = (x \cdot z) \odot (y \cdot z)'$ for all $x,\ y,\ z \in M$.

In the “standard” PMV-algebra $[\mathbf{0}, \mathbf{1}]_{PMV} = ([0, 1];\ \odot, \to, \cdot, 0, 1)$:

$$x \odot y' = \max(x - y, 0).$$

The variety generated by $[\mathbf{0}, \mathbf{1}]_{MV}$ is the variety of MV-algebras, but the variety generated by $[\mathbf{0}, \mathbf{1}]_{PMV}$ is smaller than the variety of PMV-algebras.

Derivations on PMV-algebras

A derivation on a PMV-algebra $\mathbf{M}$ is $f : M \to M$ satisfying

$$f(x \oplus y) \quad \text{and} \quad f(x \cdot y) = (f(x) \cdot y) \oplus (x \cdot f(y))$$

for all $x, y \in M$.

Let $\mathbf{M}$ be a PMV-algebra. Then f is a derivation on $\mathbf{M}$ iff f is a derivation on the MV-algebra reduct of $\mathbf{M}$. For every derivation f, $f(x) = x \cdot f(1)$ for all $x \in M$.

Some difficulties:

- $x \odot y \leq x \cdot y \leq x \wedge y$ for all $x, y \in M$;
- $x' \odot x = 0$ for all $x \in M$, but $x' \cdot x = 0$ iff $x \in \mathcal{B}(\mathbf{M})$.

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