Characterization of Flat Polygonal Logics

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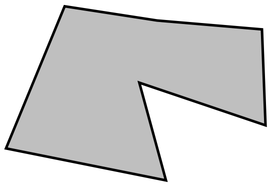
Polyhedra

Disclaimer: In this talk polyhedra are not necessarily compact

Polyhedra are:

- Finitely Boolean-generated from:
  - Compact polyhedra
  - Simplices
  - Half-hyperplanes
- Solution sets for systems of linear inequalities

Examples



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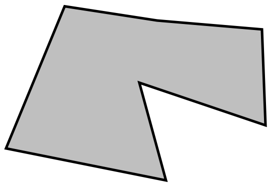

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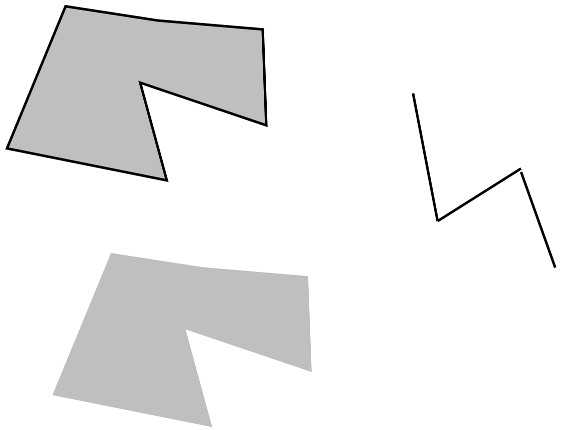

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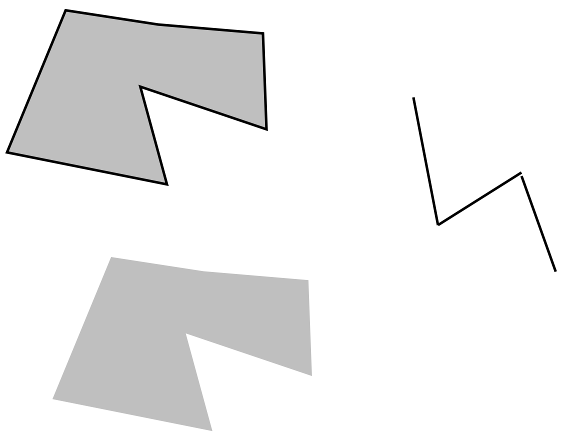

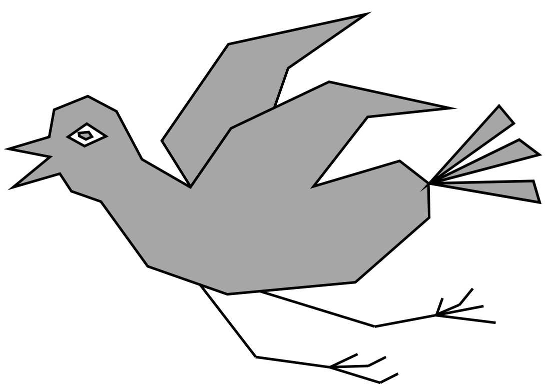

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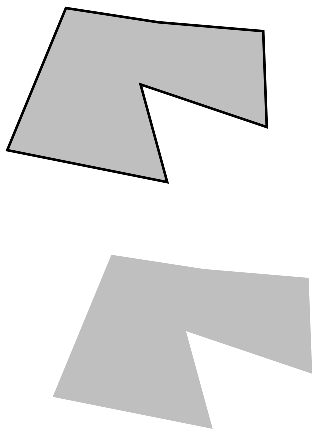

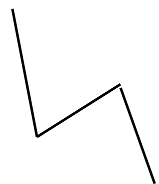

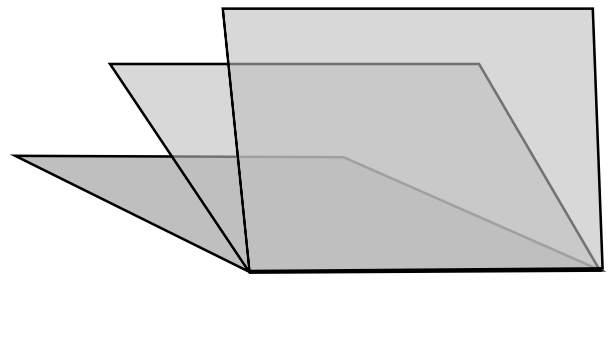

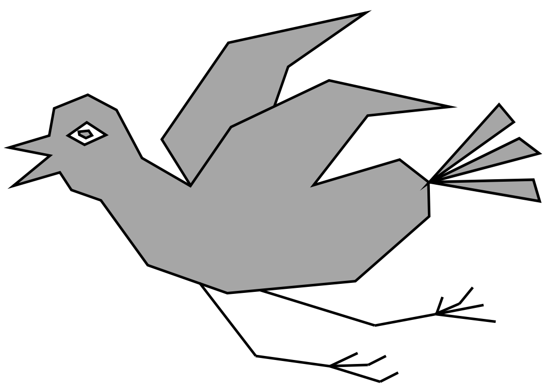

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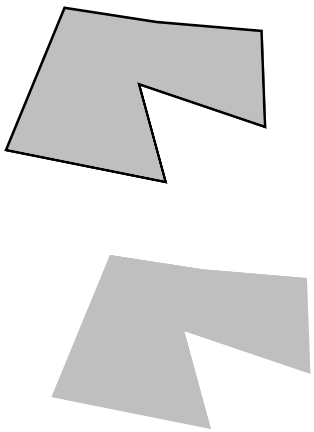

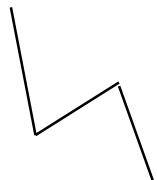

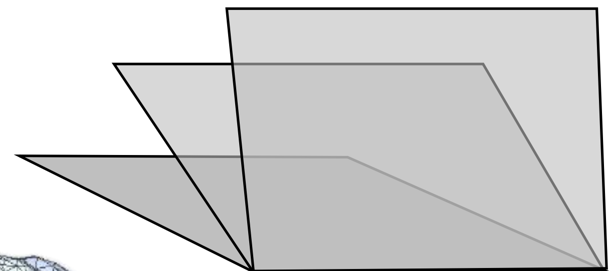

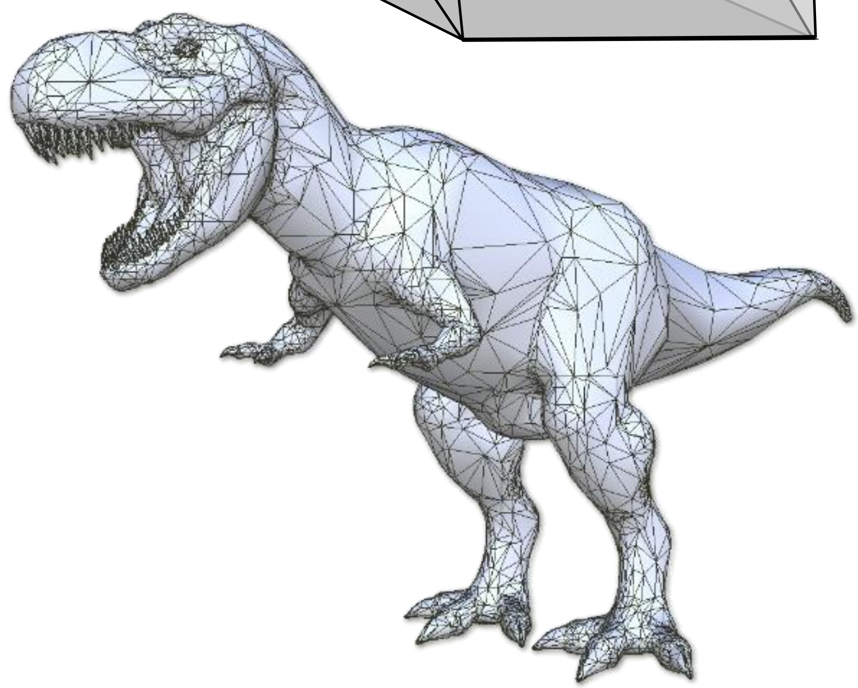

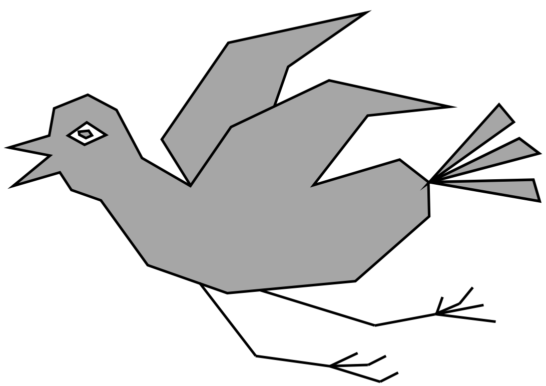

Polyhedral logics

- Given a polyhedron P, let $P^+ = (sub(P), \mathbb{C})$ be:

$sub(P)$	The Boolean algebra of all sub-polyhedra of P
$\mathbb{C}$	modal operator (topological closure)

Theorem: P^+ is a locally finite **S4.Grz**-algebra

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A modal logic $\mathcal{L}$ is **polyhedral**, if $\mathcal{L}$ is generated by $\{P_i^+ | i \in I\}$ for some collection of polyhedra $\{P_i | i \in I\}$

Polyhedral logics are extensions of **S4.Grz** and are determined by their finite (s.i.) algebras

Kripke semantics for S4.Grz

- **S4.Grz**-frames are Noetherian posets
  - Quasiorders with no strictly ascending chains
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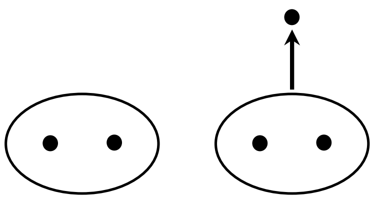
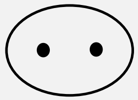
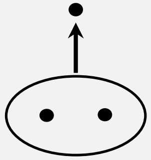
In terms of forbidden configurations:

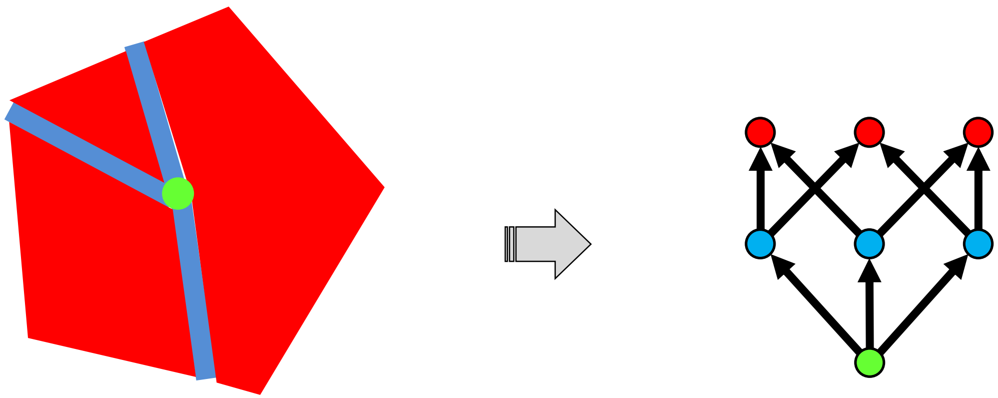
The **S4.Grz**-frames are precisely the **S4**-frames that are not up-reducible to any of the two frames



S4.Grz = S4 + $\chi($$) + \chi($$)$

Open continuous maps

- Connection between polyhedra and posets – open-continuous maps



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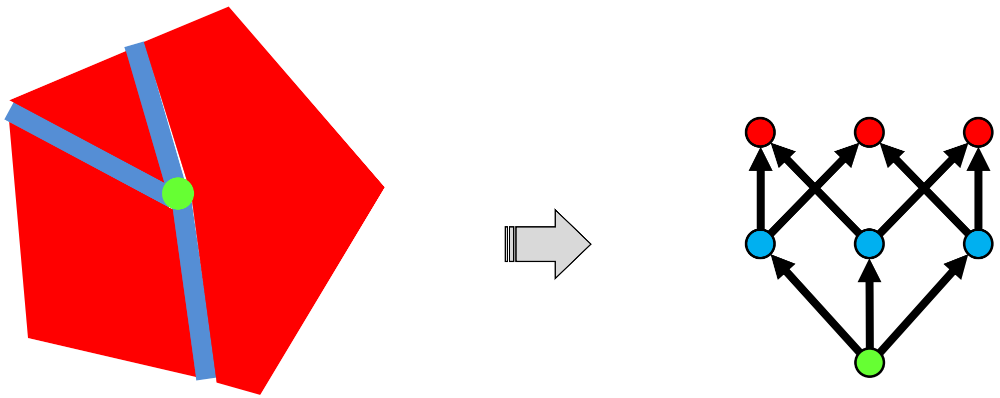

Main question – which logics are polyhedral?
Let us go by dimension. The logic of all polyhedra of dim $\leq$ n is $S4.Grz_{n+1} = S4.Grz.\chi(C_{n+1})$

Dimension 1 ("Linear" logics)

Extensions of $S4.Grz_2$ Posets of height ≤ 2

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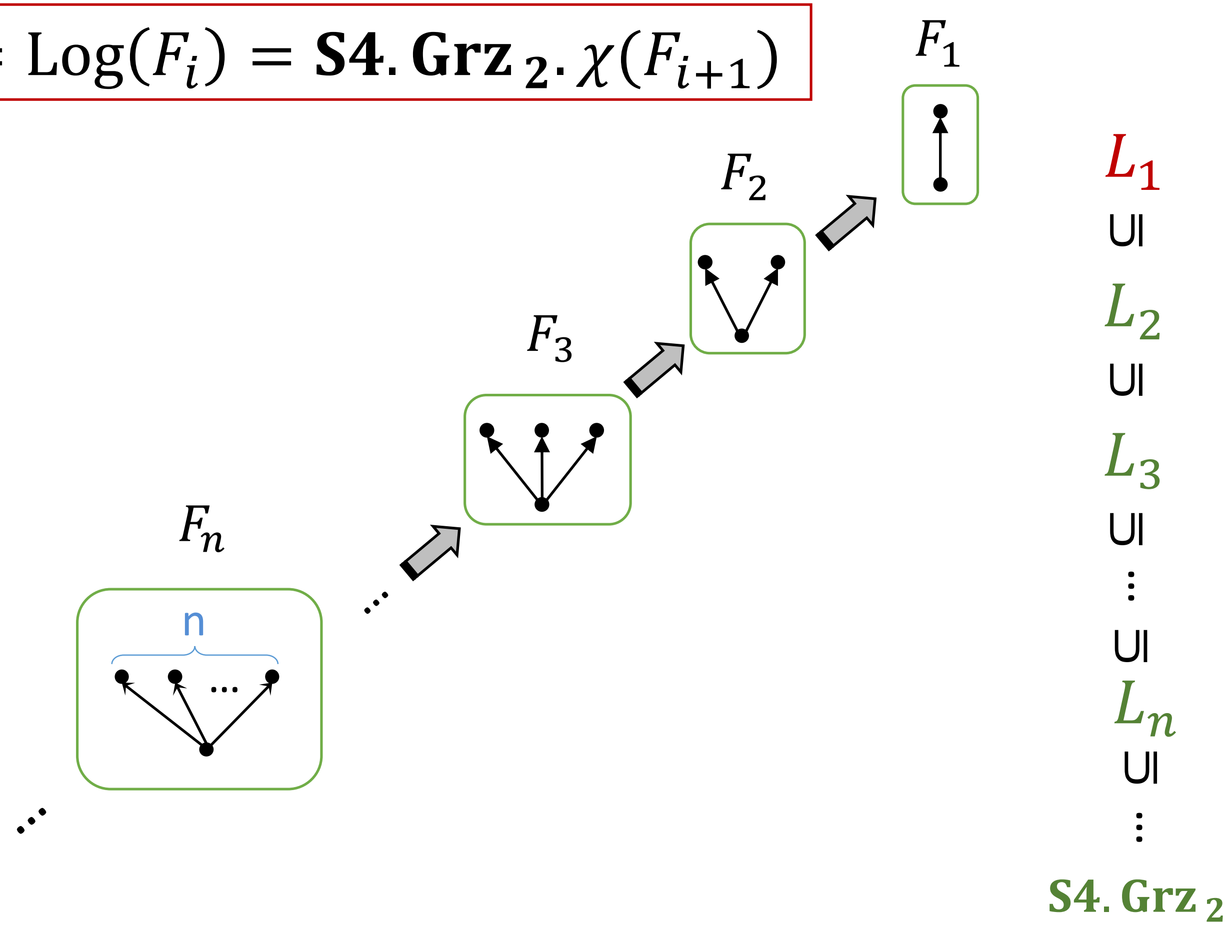
$$L_i = \mathrm{Log}(F_i) = \mathbf{S4.Grz}_{\mathbf{2}}.\chi(F_{i+1})$$



DIMENSION 2 (POLYGONAL LOGICS)

- Extensions of $\mathbf{S4.Grz_3}$
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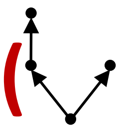
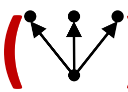
In terms of forbidden frames: $\mathbf{PL_2 = S4.Grz_3 + \chi(}$  $\mathbf{) + \chi(}$  $\mathbf{)}$

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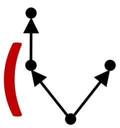
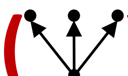
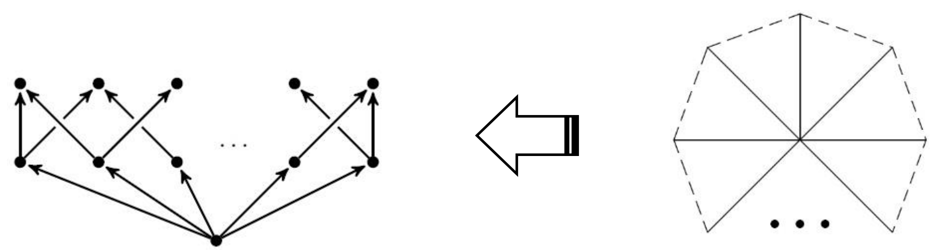
In terms of forbidden frames: $\mathbf{PL_2 = S4.Grz_3 + \chi(}$  $\mathbf{) + \chi(}$  $\mathbf{)}$

In terms of admitted frames:



Flat polygons

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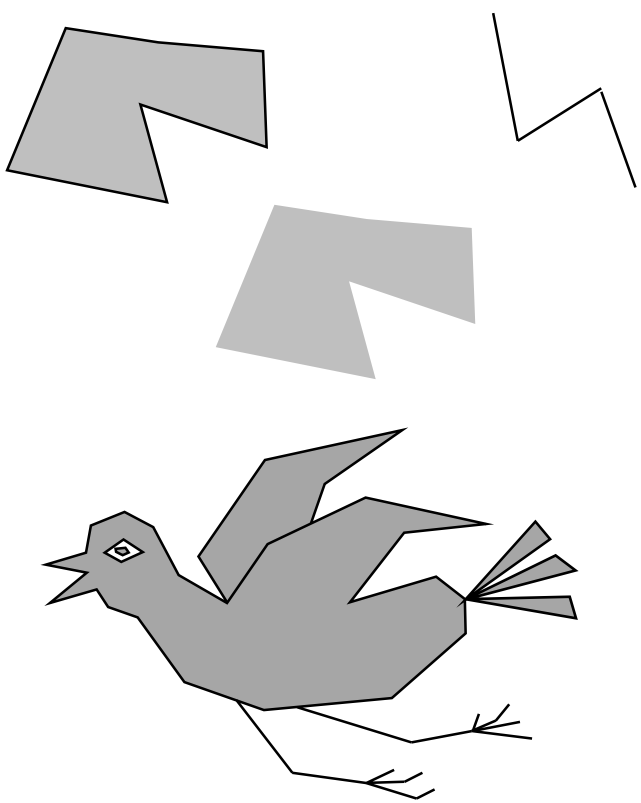

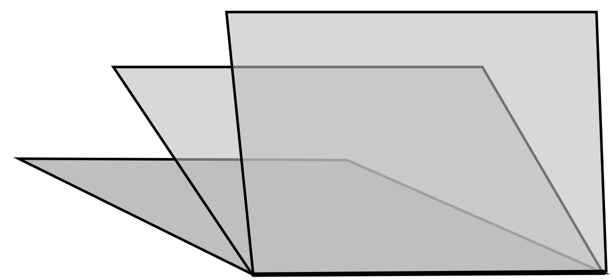

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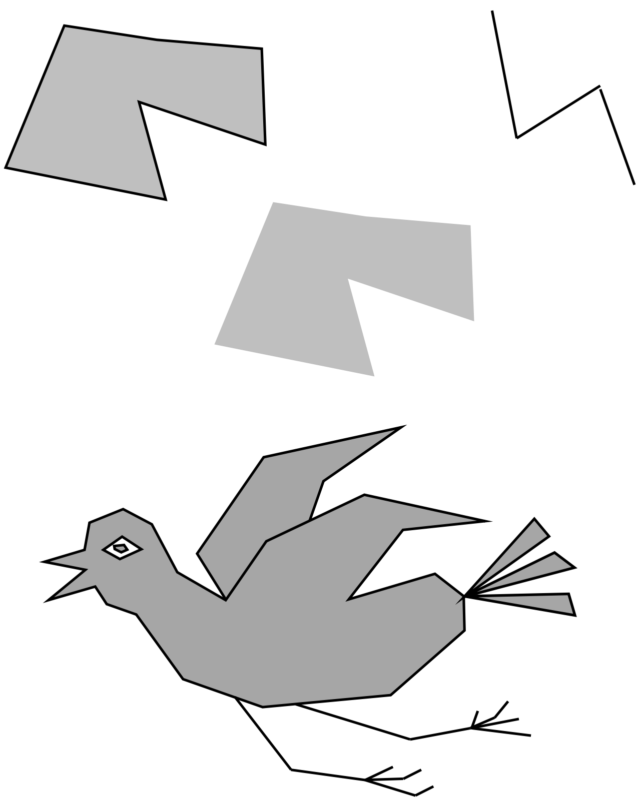
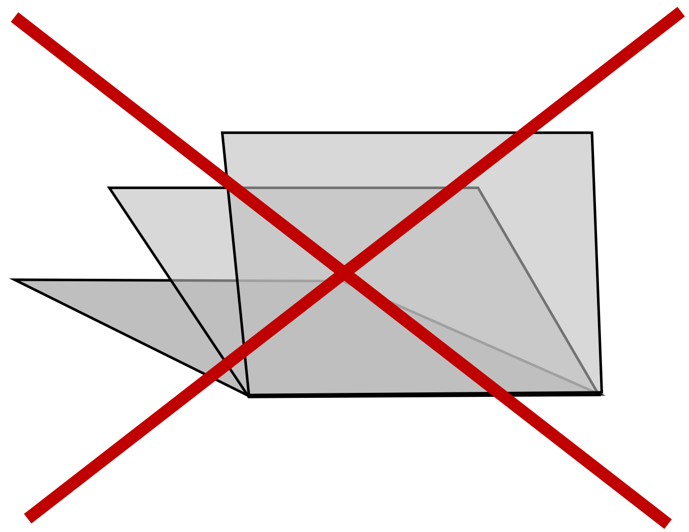
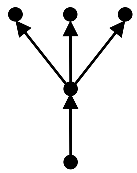




FLAT POLYGONS

- Polygonal subsets of the plane
- Which finite posets can we obtain from flats?
- Which finite posets cannot be obtained from flats?

$\mathrm{Flat}_2 = \mathrm{S4.Grz}_3 + \sigma($  $)$

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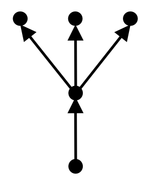
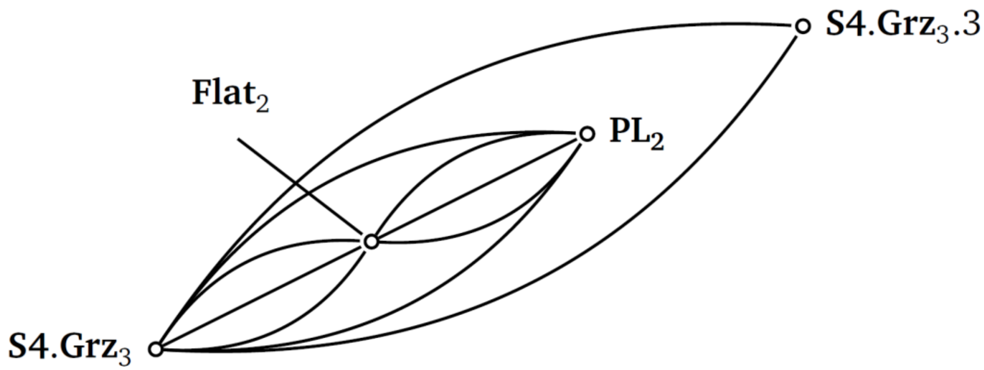
$\mathrm{Flat}_2 = \mathrm{S4.Grz}_3 + \sigma($  $)$



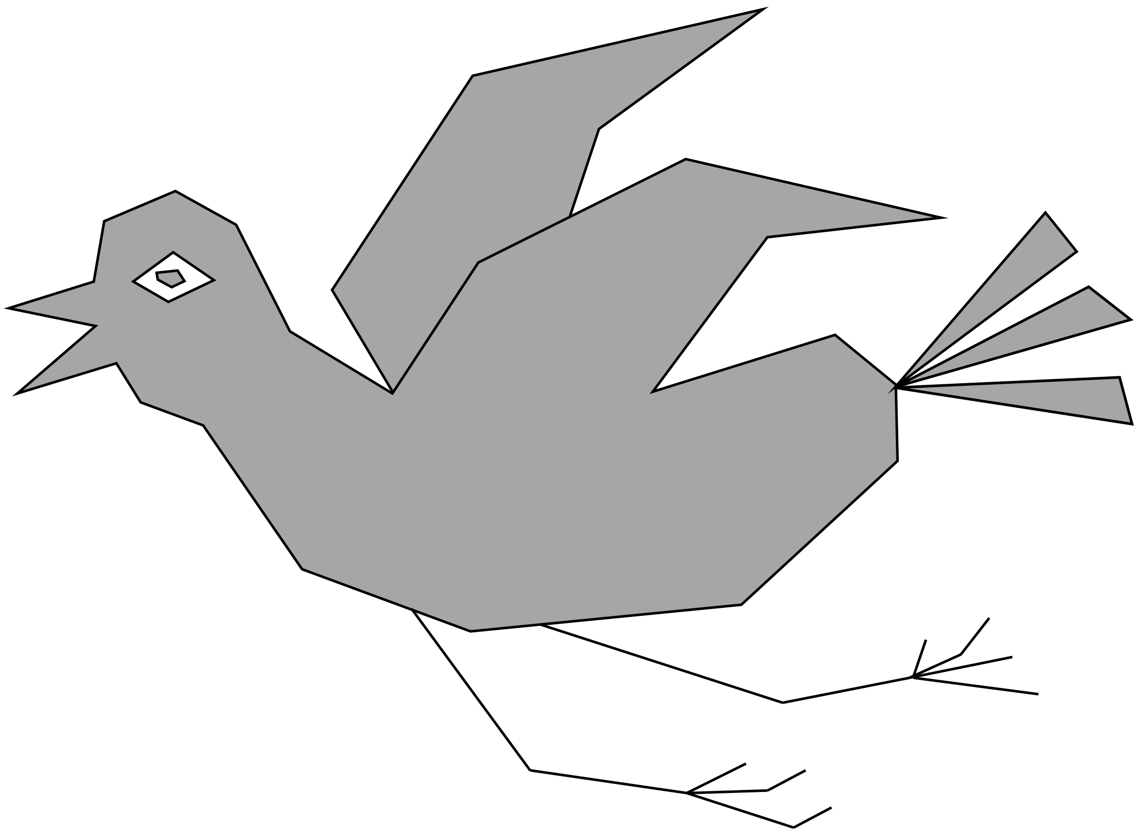
Flat polygonal logics

- Take any collection of flat polygons, generate the modal logic

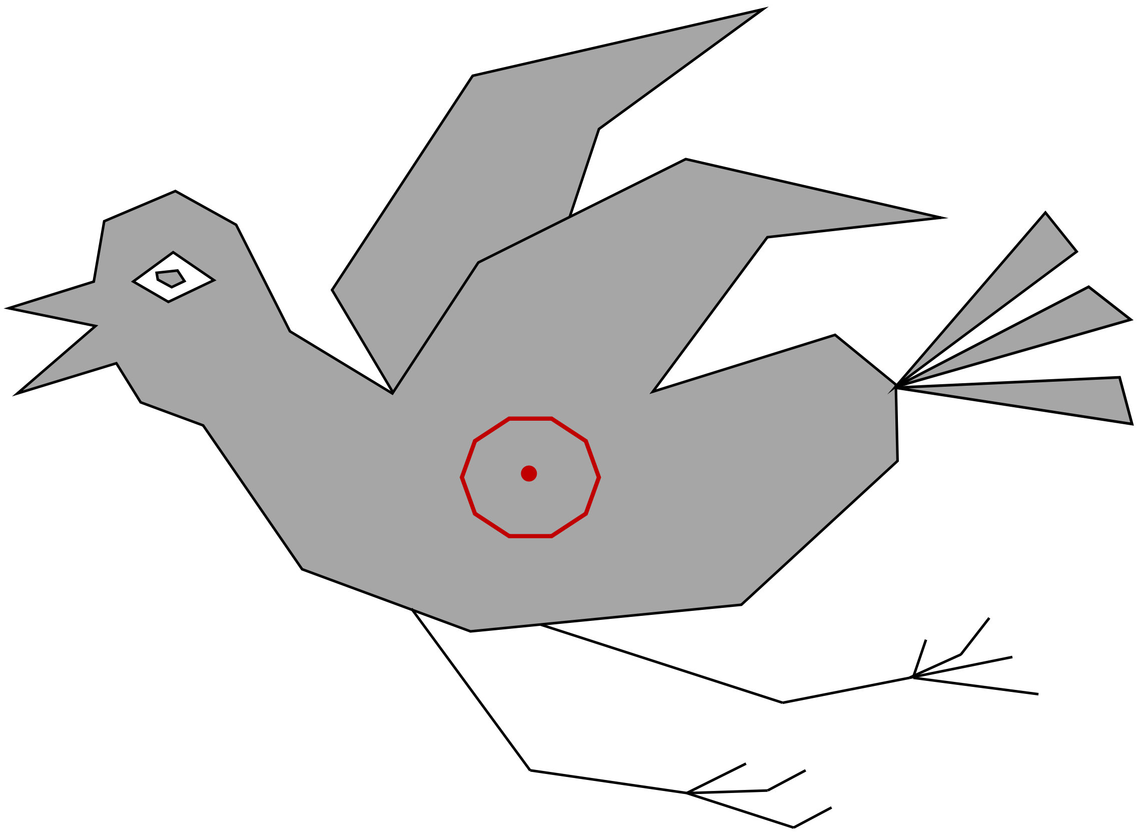
- Question: how can we characterize the flat polygonal logics?

Let us take a single flat polygon and see which posets can we get

SINGLE FLAT POLYGON



Single flat polygon



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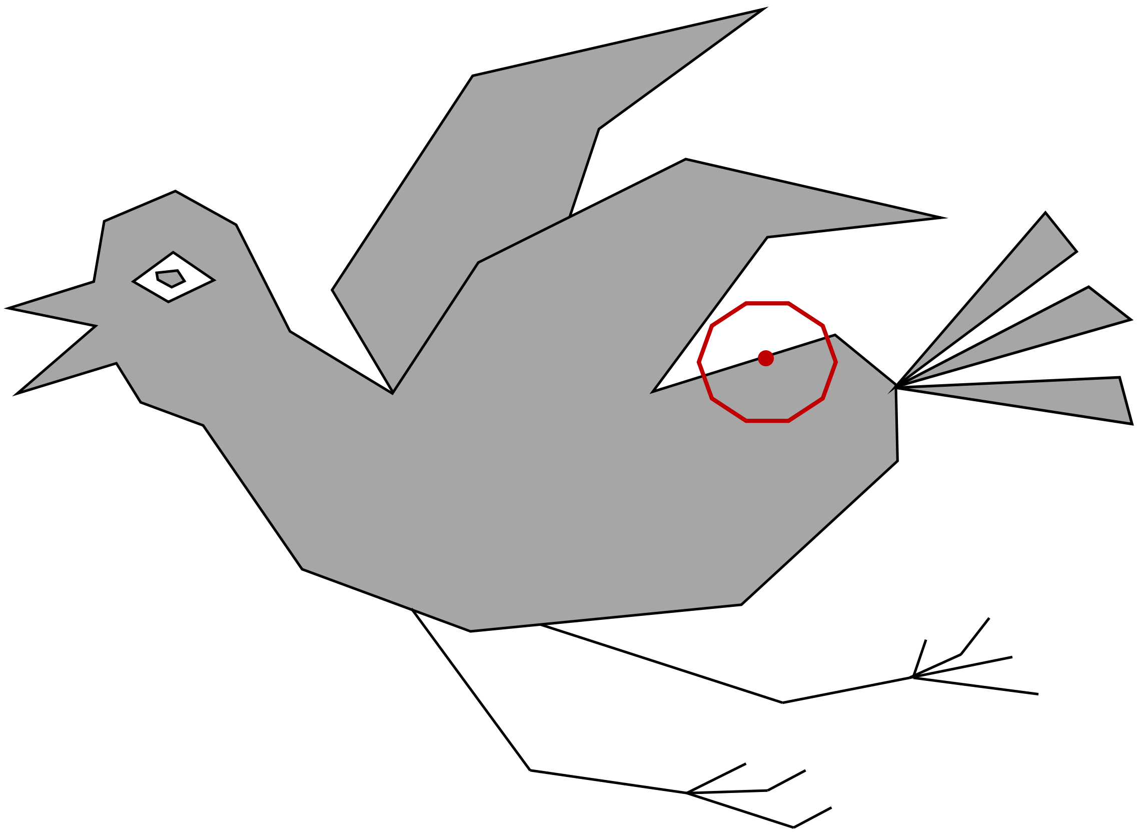

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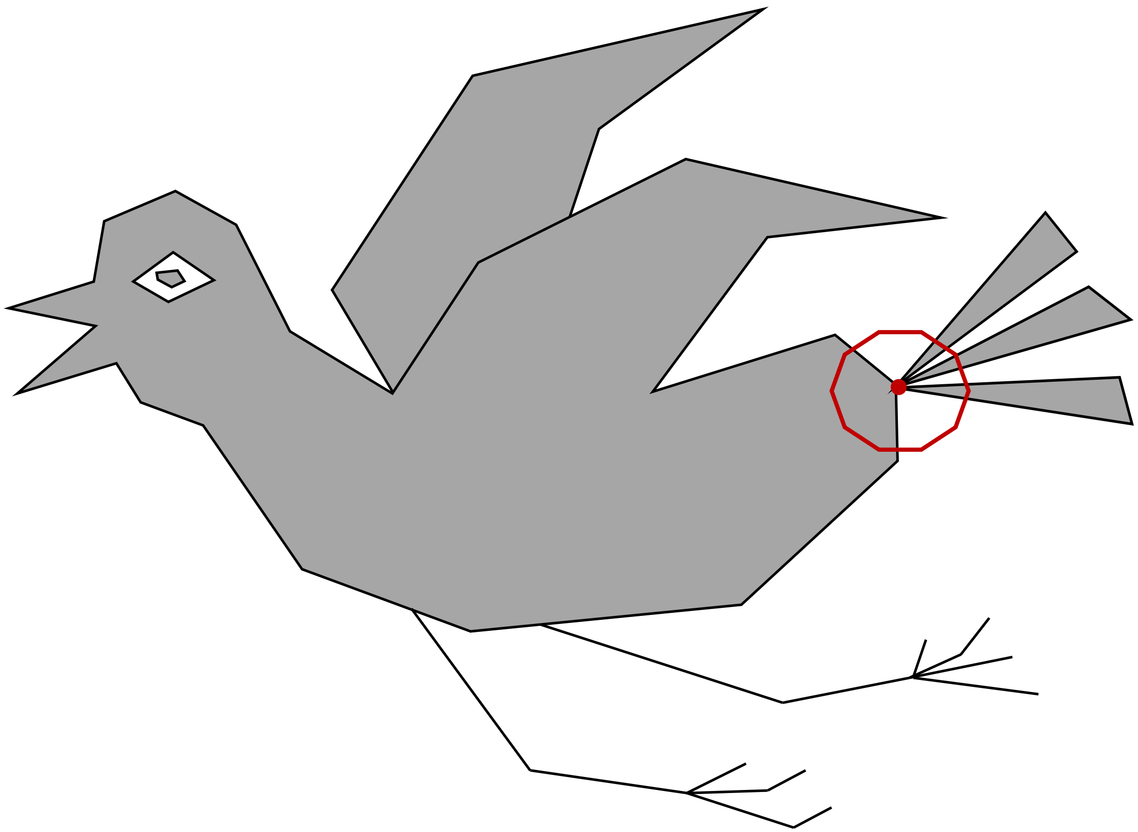

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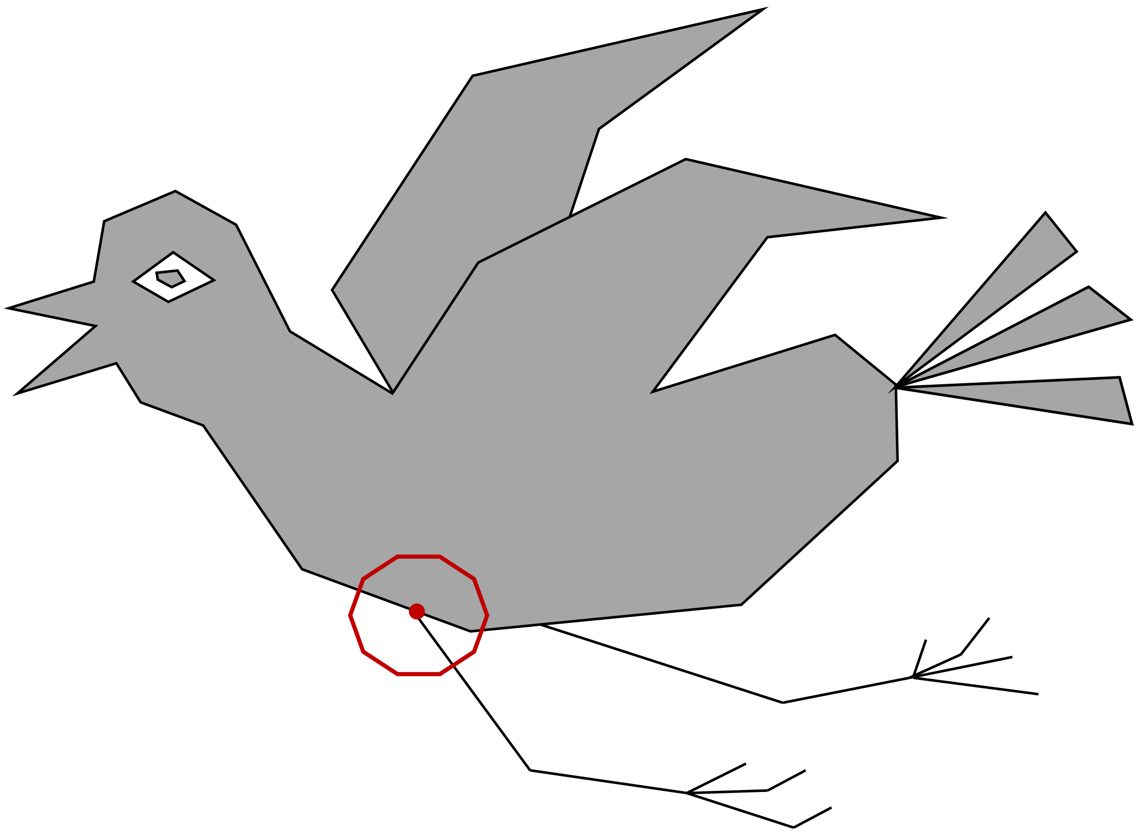

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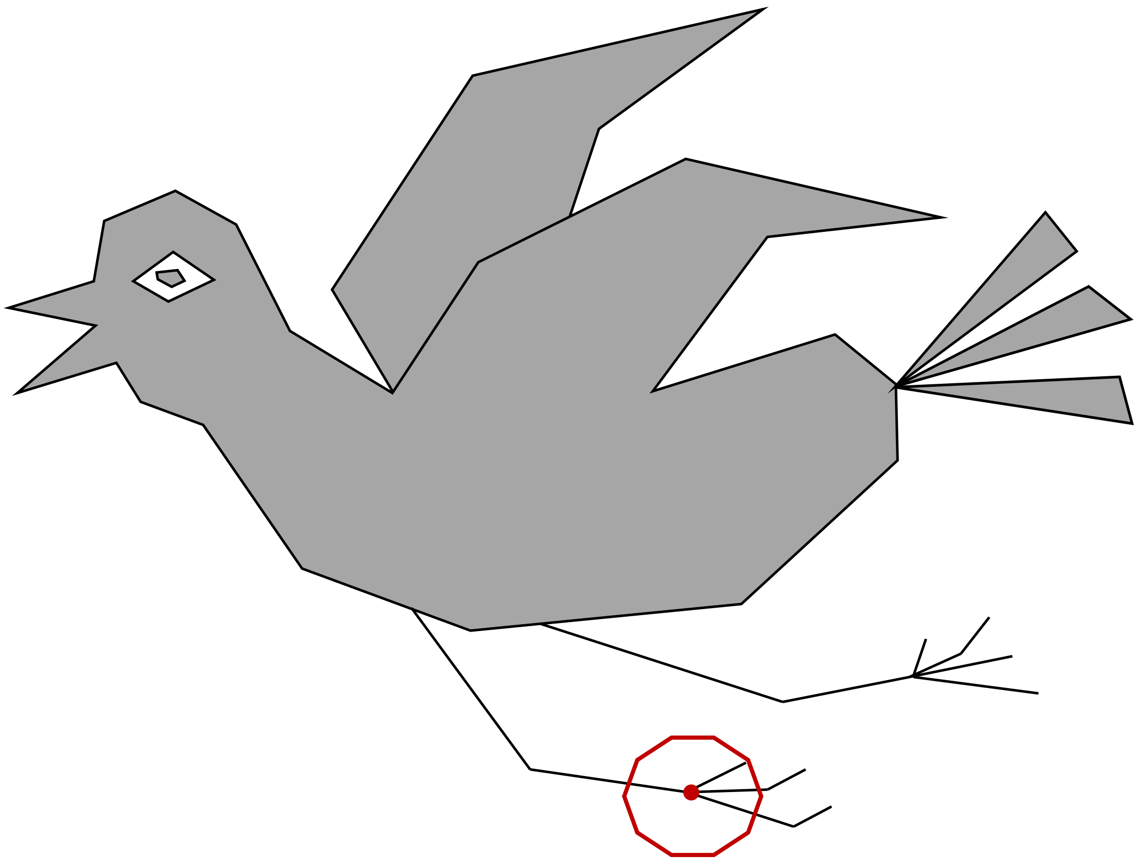

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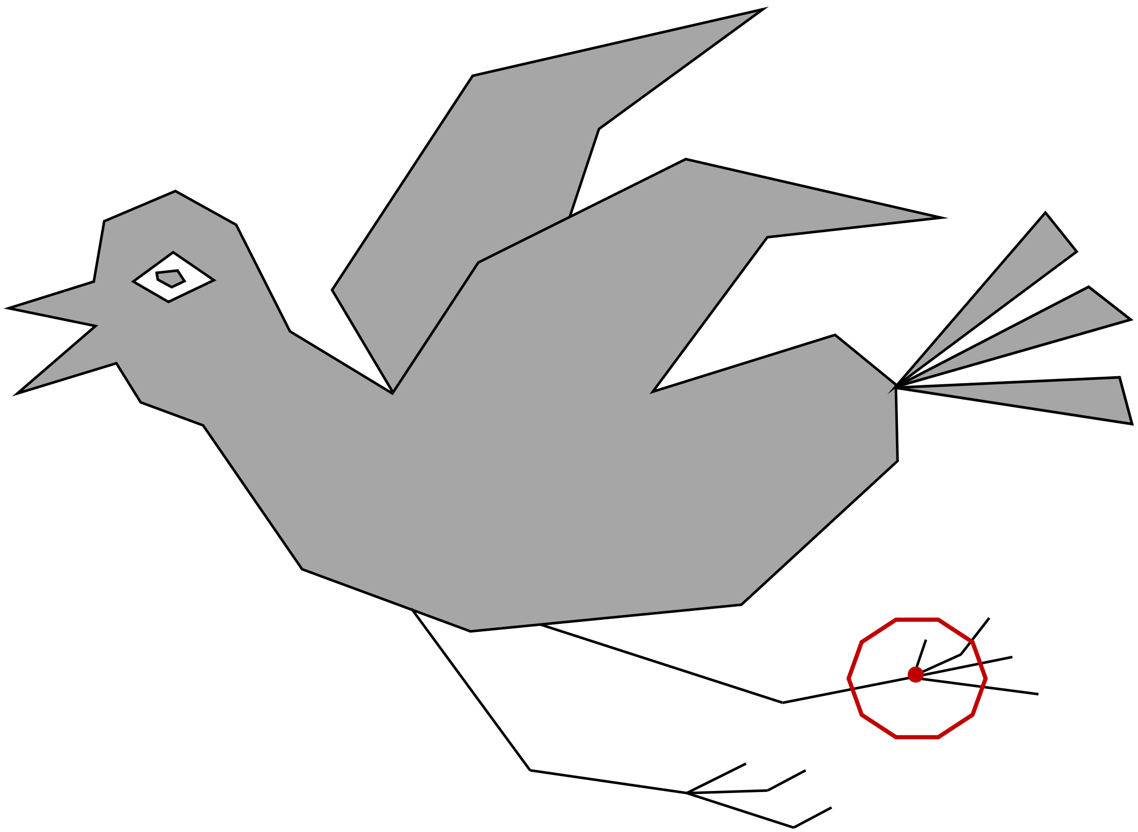

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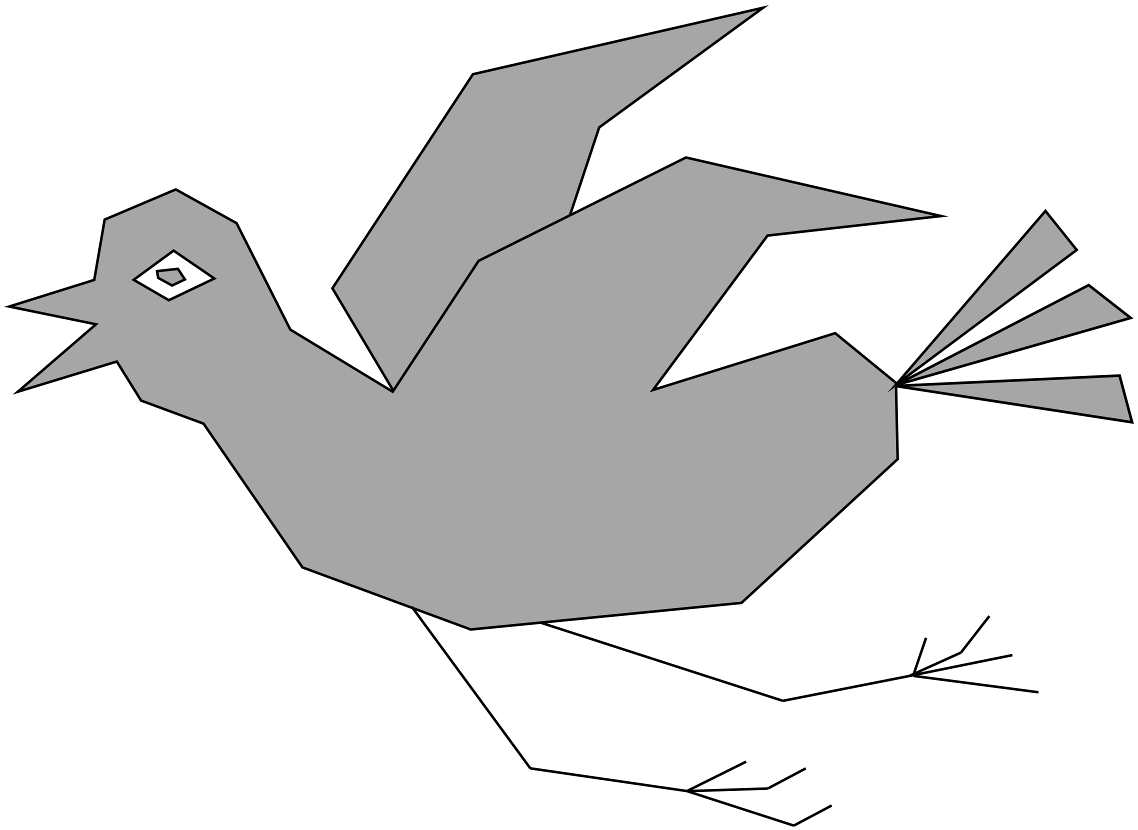

Characterizing flat polygonal logics

Let α be an antichain in the ordering Q. Let L_α be the extension of $S4.Grz_3$ by $\chi(F)$, with F in α.

Lemma: Each α is finite.
Theorem: The flat polygonal logics are precisely the L_α.
Corollary: There are countably many flat polygonal logics, each finitely axiomatizable.

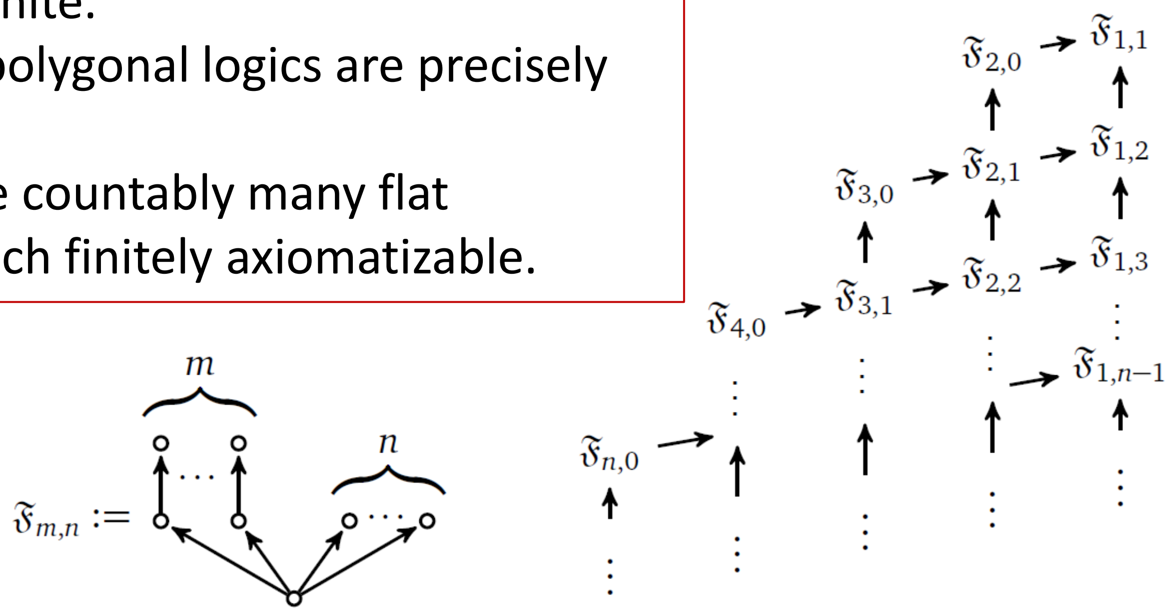

Figure 2: Poset Q of the frames $\mathfrak{F}_{m,n}$ ordered by reducibility

Further work

- Characterize all dim 2 polygonal logics
- Characterize flat polyhedral logics of higher dimension
- All polyhedral logics?

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Thank You