Lifting Functors from **Pos** to **Pries**

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Contents

- Coalgebraic positive logic
  - Predicate liftings
- Connection between **Pos**- and **Pries**-functors
  - Lifting via semantics (predicate liftings)
  - Lifting via (cofiltered) limits
  - Comparison

Coalgebras

Definition

- A coalgebra for a functor $\mathsf{T} : \mathbf{C} \to \mathbf{C}$ is a pair (X, γ) where $X \in C$ and $\gamma : X \to \mathsf{T}X$.

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For a set Λ of predicate liftings, define $\mathbb{L}(\mathsf{T}, \Lambda)$ by

$$\varphi ::= \bot \mid \top \mid p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n).$$

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Interpretation in (X, γ) with valuation $V : \mathrm{Prop} \to \mathsf{Up}X$:

$$x \Vdash \heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n) \iff \gamma(x) \in \lambda([\![\varphi_1]\!], \ldots, [\![\varphi_n]\!]).$$

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Example: Convex powerset functor on **Pos**

$\mathcal{P}_c : \mathbf{Pos} \to \mathbf{Pos}$

- $X \mapsto$ convex subsets of X ordered by $\sqsubseteq$:

$$a \sqsubseteq b \quad \text{iff} \quad a \subseteq {\downarrow} b \text{ and } b \subseteq {\uparrow} a$$

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$\mathcal{V}_c : \mathbf{Pries} \to \mathbf{Pries}$

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$$\boxplus a = \{ b \in \mathcal{V}_c\mathcal{X} \mid b \subseteq a \}, \qquad \otimes a = \{ b \in \mathcal{V}_c\mathcal{X} \mid b \cap a \neq \varnothing \},$$

  where $a \in \mathrm{ClpUp}\mathcal{X}$.
- For a morphism $f : \mathcal{X} \to \mathcal{X}'$ let $(\mathcal{V}_c f)(b) = {\downarrow} f[b] \cap {\uparrow} f[b]$.

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Semantic functor lift

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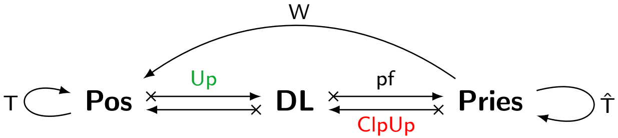

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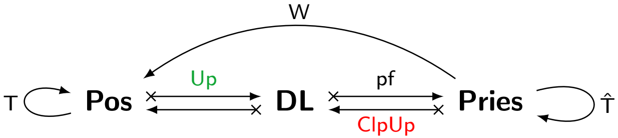

Definition

For a set Λ of predicate liftings, define $\mathsf{D}_{\mathsf{T},\Lambda} : \textbf{Pries} \to \textbf{DL}$ by:

- Let $\mathsf{D}_{\mathsf{T},\Lambda}\mathcal{X}$ be the sub-DL of $\mathsf{Up}(\mathsf{T}(\mathsf{W}\mathcal{X}))$ generated by

$$\{\lambda_{\mathsf{W}\mathcal{X}}(a_1, \ldots, a_n) \mid \lambda \in \Lambda, a_i \in \mathsf{ClpUp}\mathcal{X}\}.$$

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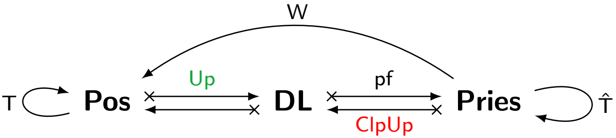

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- For $f : \mathcal{X} \to \mathcal{Y}$ let $\mathsf{D}_{\mathsf{T},\Lambda}f : \mathsf{D}_{\mathsf{T},\Lambda}\mathcal{Y} \to \mathsf{D}_{\mathsf{T},\Lambda}\mathcal{X}$ be the restriction of $\mathsf{Up}(\mathsf{T}(\mathsf{W}f)) = (\mathsf{T}f)^{-1}$.

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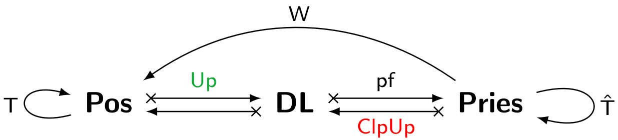

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Define the semantic lift of T w.r.t. Λ by

$$\hat{\mathsf{T}} = \mathsf{pf} \circ \mathsf{D}_{\mathsf{T},\Lambda} : \mathbf{Pries} \to \mathbf{Pries}.$$

Example

Consider $\mathcal{P}_c$ and $\Lambda = \{\lambda^\Box, \lambda^\Diamond\}$. Then

$$\widehat{(\mathcal{P}_c)}_\Lambda = \mathcal{V}_c.$$

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Proof idea

For a Priestley space $\mathcal{X}$, we have $\mathrm{D}_{\mathcal{P}_c,\Lambda}\mathcal{X} = \mathrm{ClpUp}(\mathcal{V}_c\mathcal{X})$ via $\varphi : \mathrm{ClpUp}(\mathcal{V}_c\mathcal{X}) \to \mathrm{D}_{\mathcal{P}_c,\Lambda}\mathcal{X}$ generated by

$$\varphi(\boxplus a) = \lambda^{\Box}(a), \qquad \varphi(\otimes a) = \lambda^{\Diamond}(a).$$

Functor lift using cofiltered limits

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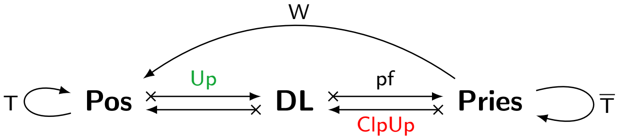

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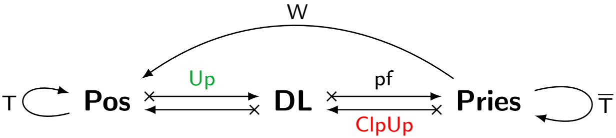

Remarks

1. $\mathbf{Pos}_f \cong \mathbf{Pries}_f$
2. For $\mathcal{X} \in \mathbf{Pries}_f$ we have $\mathsf{ClpUp}\mathcal{X} = \mathsf{Up}(\mathsf{W}\mathcal{X})$.
3. For $\mathcal{X} \in \mathbf{Pries}$, let $\mathcal{X} \downarrow \mathbf{Pries}_f$ be the coslice category and $\mathsf{U}_{\mathcal{X}} : (\mathcal{X} \downarrow \mathbf{Pries}_f) \to \mathbf{Pries}$ the obvious forgetful functor. Then $\mathcal{X} = \lim \mathsf{U}_{\mathcal{X}}$.

Functor lift using cofiltered limits

Objects

Suppose $\mathsf{T} : \mathbf{Pos} \to \mathbf{Pos}$ restricts to an endofunctor on $\mathbf{Pos}_f$. For $\mathcal{X} \in \mathbf{Pries}$ define

$$\overline{\mathsf{T}}\mathcal{X} = \lim(\mathsf{T}\mathsf{U}_{\mathcal{X}}).$$

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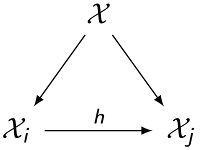

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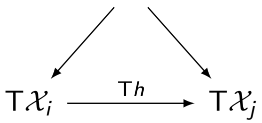

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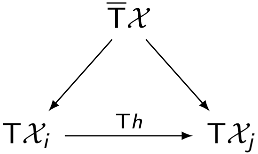

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Morphisms

If $f : \mathcal{X} \to \mathcal{Y}$, every object in the diagram $\mathsf{U}_{\mathcal{Y}}$ is also in $\mathsf{U}_{\mathcal{X}}$.

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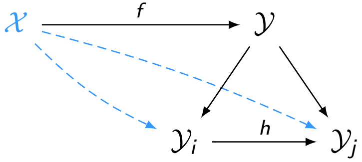

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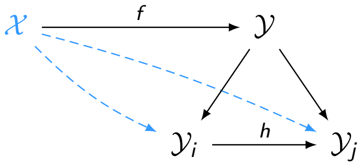

Therefore $\overline{\mathsf{T}}\mathcal{X}$ is a cone for $\mathsf{T}\mathsf{U}_{\mathcal{Y}}$. By the limit property we get $\overline{\mathsf{T}}f : \overline{\mathsf{T}}\mathcal{X} \to \lim(\mathsf{T}\mathsf{U}_{\mathcal{Y}}) = \overline{\mathsf{T}}\mathcal{Y}$.

Comparison of functor lifts

View $\mathsf{ClpUp}(\overline{\mathsf{T}}\mathcal{X})$ as sub-DL of $\mathsf{Up}(\mathsf{TW}\mathcal{X})$.

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Lemma

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2. If T preserves epis and cofiltered limits then $s_{\mathcal{X}}$ is injective.
3. $\mathsf{D}_{\mathsf{T},\wedge}\mathcal{X}$ is a subframe of $\mathrm{im}\, s_{\mathcal{X}}$.
4. If T is embedding-preserving then $\mathsf{D}_{\mathsf{T},\wedge}\mathcal{X} \cong \mathrm{im}\, s_{\mathcal{X}}$.

Comparison of functor lifts

Theorem

Let T be an endofunctor on **Pos** which

- restricts to $\mathbf{Pos}_f$; and
- preserves epis, embeddings and cofiltered limits.

Let Λ be the set of all positive predicate liftings for T. Then there is a natural isomorphism $\overline{\mathsf{T}} \to \hat{\mathsf{T}}$.

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Example

Finitely generated convex powerset functor on **Pos**.

Generalization?

- Similar methods and result for lifting **Set**-functors to **Stone**-functors.
- More general approach?

Thank you