The van Benthem Characterisation Theorem for Descriptive Models

TACL 2019

Nick Bezhanishvili & Tim Henke

19th June 2019

Van Benthem Characterisation Theorem

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- *For any $\mathfrak{M}, w$ and $\mathfrak{N}, v$, if $\mathfrak{M}, w \underline{\leftrightarrow} \mathfrak{N}, v$ then*

$$\mathfrak{M} \models \alpha[w] \iff \mathfrak{N} \models \alpha[v].$$

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Theorem (Van Benthem Characterisation Theorem over $\mathcal{C}$)

Modal logic is the bisimulation-invariant fragment of first-order logic **over** *$\mathcal{C}$:*
Let $\alpha(x)$ be a first-order formula in one free variable. Then the following are equivalent:

- *There is a modal formula φ such that* **over** *$\mathcal{C}$*

$$\alpha(x) \equiv_{\mathcal{C}} \mathsf{ST}_x(\varphi).$$

- *For any $\mathfrak{M}, w$ and $\mathfrak{N}, v$,* **both in** *$\mathcal{C}$, if $\mathfrak{M}, w \,\underline{\leftrightarrow}\, \mathfrak{N}, v$ then*

$$\mathfrak{M} \models \alpha[w] \iff \mathfrak{N} \models \alpha[v].$$

Historical results

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All Kripke models (classical)	Benthem (1976)
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Rooted, transitive Kripke models rooted finite Kripke models, etc.	Dawar and Otto (2009)
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Now: models on descriptive frames.

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Valuations $V : \text{Prop} \to A$.

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No infinite, irreflexive linear order can be a made a descriptive frame.

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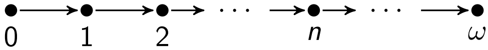

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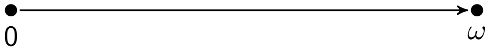

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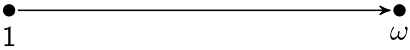

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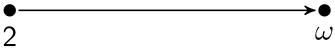

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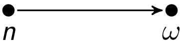

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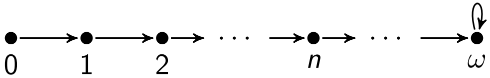

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- Compactness Theorem for first-order logic
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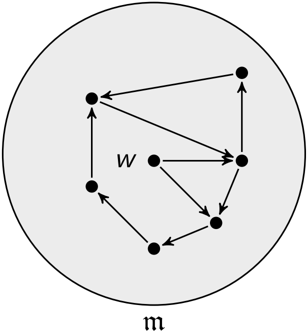

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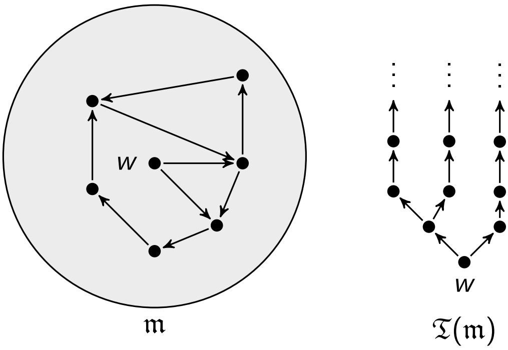

Descriptive unravelling

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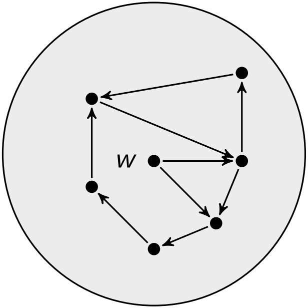

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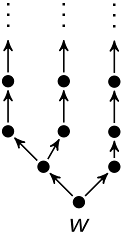

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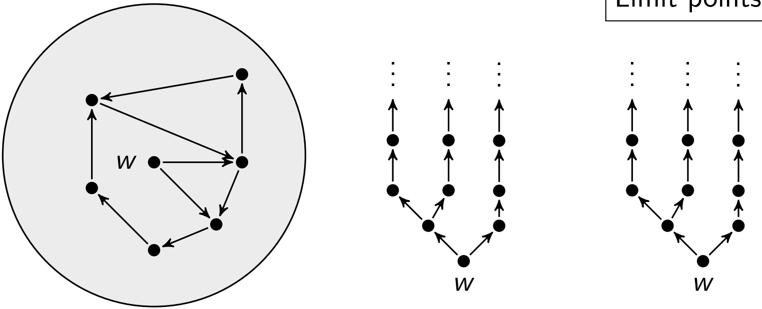
Limit points



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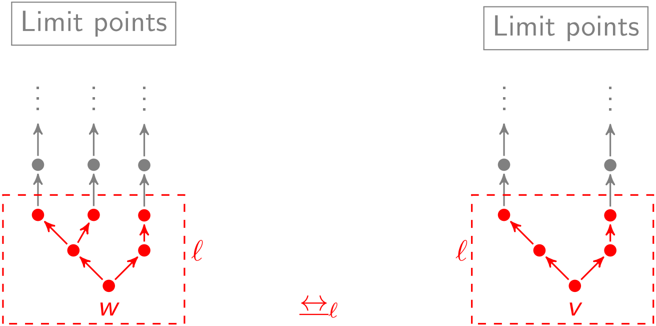

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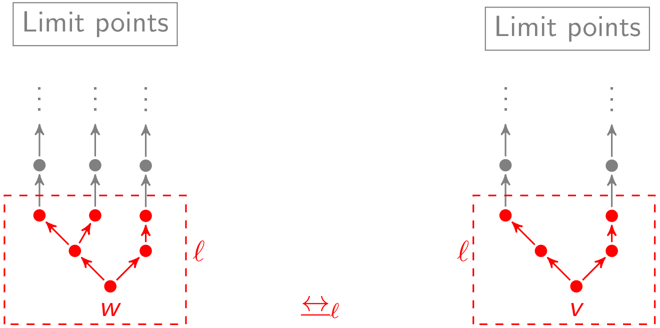

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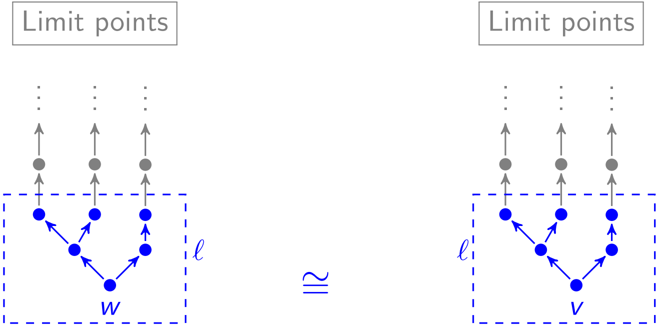

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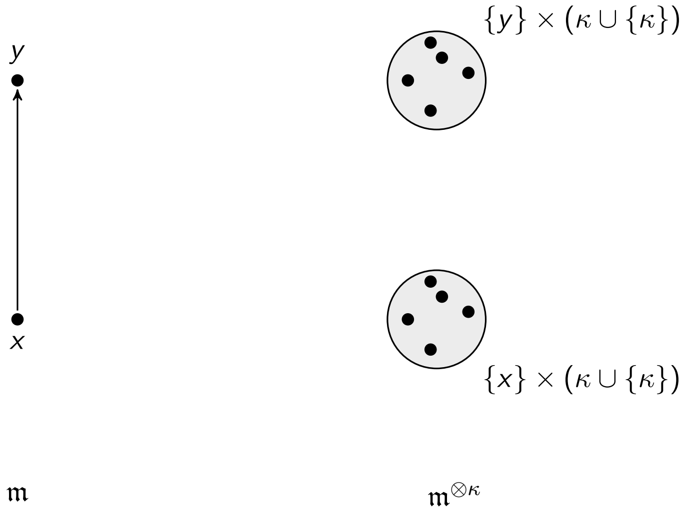

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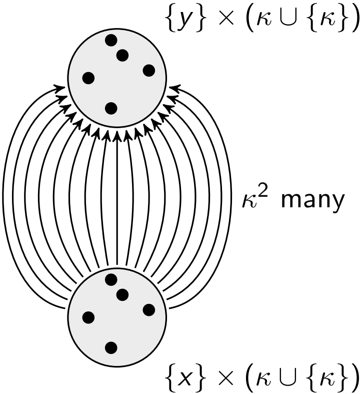

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Ehrenfeucht-Fraïssé

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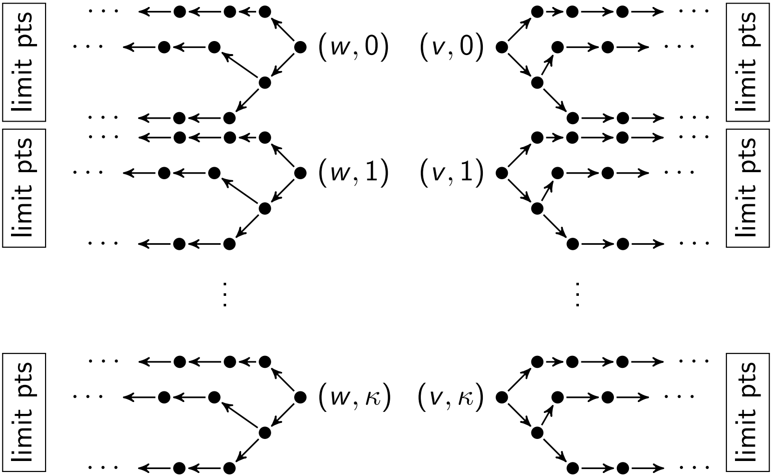

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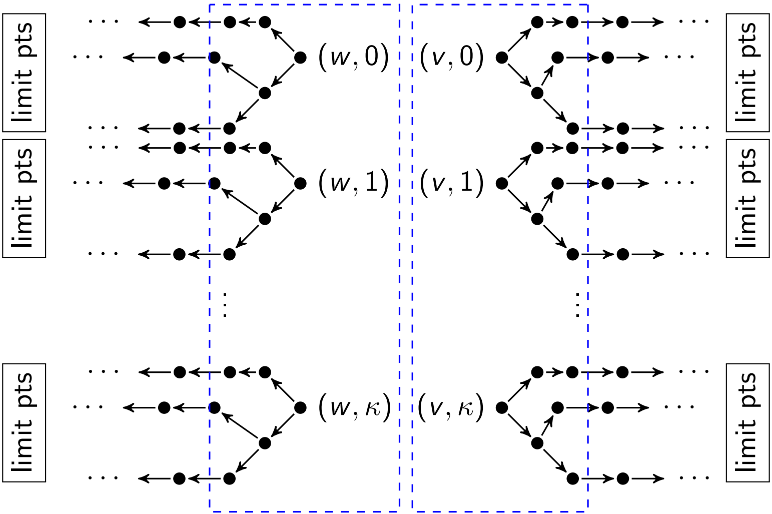

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Conclusions and future research

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Van Benthem Characterisation Theorem for descriptive modal

Failure of Compactness Theorem

Failure of upward Löwenheim-Skolem Theorem

Failure of Beth Definability Theorem

Failure of Craig Interpolation Theorem

Conclusions and future research

Future research

Janin-Walukiewicz for descriptive models

Coalgebraic-generalisations: Vietoris-like functors on **Stone**

Intuitionistic models: Esakia spaces

Neighbourhood semantics, following Hansen and Kupke (2004)

Weak version of upward Löwenheim-Skolem Theorem

Thank you for listening!

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