Projective unification in NExt(K4)

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Transitive modal logics and unifiers

- $\mathrm{Var} = \{x, x_1, x_2, \dots, y, y_1, y_2, \dots\}$ - the set of propositional variables,
- Fm - the set of modal formulas,
- $\mathrm{Var}(A)$ - the (finite) set of variables occurring in A.

By a (transitive) modal logic we mean any set of formulas that contains:

- all propositional tautologies,
- $\mathrm{K} : \Box(x \to y) \to (\Box x \to \Box y)$,
- $4 : \Box x \to \Box\Box x$,

which is closed under substitutions and

$$MP : \frac{A \to B, A}{B} \quad \text{and} \quad RN : \frac{A}{\Box A}.$$

$\Box^+ A = A \wedge \Box A$ (dually $\Diamond^+ A = A \vee \Diamond A$).
$Cons(\mathrm{L})$ - the set of all constants of L (modulo equivalence).

Transitive modal logics and unifiers

A unifier for a formula A in a modal logic L is a substitution σ such that $\vdash_{\mathsf{L}} A[\sigma]$. σ is said to be ground if $x[\sigma] \in Cons(\mathsf{L})$ for each $x \in \mathsf{Var}(A)$.

Lemma

Let A be a modal formula and L *be a modal logic. The following condition are equivalent:*

1. *A is unifiable in* L*,*
2. *there exists a ground unifier for A in* L*,*
3. *A is satisfiable in $\langle Cons(\mathsf{L}), \wedge, \neg, \top, \Box \rangle$.*

Transitive modal logics and unifiers

Lemma

If the formula $\mathsf{T}^{\Box}: \Box(\overbrace{\Box x \to x}^{\mathsf{T}})$ *is a theorem of a transitive modal logic* L*, then the following formulas are equivalent:*

$$\begin{aligned} .3: &\quad \Box(\Box^{+}x \to y) \lor \Box(\Box^{+}y \to x), \\ \mathrm{D1}: &\quad \Box(\Box x \to y) \lor \Box(\Box y \to x), \\ \mathrm{D1'}: &\quad \Box(\Box x \to \Box y) \lor \Box(\Box y \to \Box x). \end{aligned}$$

Corollary

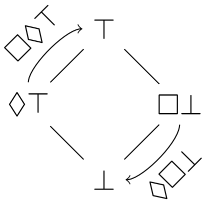
The following equality holds:

$$\mathrm{K4.3T}^{\Box} = \mathrm{K4D1} = \mathrm{K4D1'T}^{\Box}.$$

There are infinitely many constants in K4.3 and K4D1′.

Passive rules

The modal algebra $\langle Cons(\mathsf{K4T}^{\Box}), \wedge, \neg, \top, \Box \rangle$,



is isomorphic to the product of modal algebras

$\langle \{0, 1\}, \wedge, \neg, \top, \Box_1 \rangle$ in which $\Box_1 0 = 0$
$(\mathsf{Triv} = \mathsf{Log}\{\circ\} = \mathsf{K} + \Box x \leftrightarrow x)$

and

$\langle \{0, 1\}, \wedge, \neg, \top, \Box_2 \rangle$ in which $\Box_2 0 = 1$.
$(\mathsf{Verum} = \mathsf{Log}\{\bullet\} = \mathsf{K} + \Box \bot)$

Passive rules

Corollary

A formula is unifiable in $\mathsf{K4T}^{\Box}$ *if and only if it is unifiable in* Triv *and in* Verum.

Lemma

Let $\mathsf{K4T}^{\Box} \subseteq \mathsf{L}$ *be a modal logic with four constants. Let* A *be a non-unifiable formula in* L *such that* $\mathrm{Var}(A) \subseteq \{x_1, \ldots, x_n\}$. *Then,*

$$A \vdash_{\mathsf{L}} \Diamond\top$$

or

$$A \vdash_{\mathsf{L}} \Box\bot \vee (\Diamond^{+} x_1 \wedge \Diamond^{+}\neg x_1) \vee \ldots \vee (\Diamond^{+} x_n \wedge \Diamond^{+}\neg x_n).$$

Passive rules

The modal logic K4G is the smallest transitive modal logic containing the Gleach formula

$$\mathsf{G}\colon \Diamond\Box x \to \Box\Diamond x.$$

Lemma

The modal logic K4G *is characterized by the class* Fr_{K4G} *of all finite transitive rooted frames fulfilling the condition*

$$\forall w_1, w_2 \in W \setminus \{\rho\}(\exists w_3(w_1 R w_3 \wedge w_2 R w_3)).$$

Lemma

Let L *be a modal logic extending* K4G*. Then, for every n there exists a formula B such that*

$$\Box\bot \vee (\Diamond^+ x_1 \wedge \Diamond^+ \neg x_1) \vee \ldots \vee (\Diamond^+ x_n \wedge \Diamond^+ \neg x_n) \vdash_{\mathsf{L}} \Box\bot \vee (\Diamond B \wedge \Diamond \neg B).$$

Passive rules

Modal logics $\mathsf{K4T}^{\Box}$ and K4G are incomparable.

Corollary

Let L *be a modal logic extending* $\mathsf{K4GT}^{\Box}$ *such that* $Cons(\mathsf{L}) = \{\top, \bot, \Diamond\top, \Box\bot\}$. *If a formula* A *is not unifiable in* L*, then*

$$A \vdash_{\mathsf{L}} \Diamond\top$$

or there exists a formula B *such that*

$$A \vdash_{\mathsf{L}} \Box\bot \vee (\Diamond B \wedge \Diamond\neg B).$$

Passive rules

An inference rule A/B is:

- admissible in a logic L iff $\vdash_{\mathsf{L}} A[\varepsilon] \;\Rightarrow \vdash_{\mathsf{L}} B[\varepsilon]$,
- derivable iff $A \vdash_{\mathsf{L}} B$,
- passive iff A is not unifiable (passive $\Rightarrow$ admissible).

A modal logic is (almost) structurally complete if every (non-passive) admissible rule is also derivable.
An inference rule A/B is a consequence of a collection $\mathcal{B}$ of rules in a modal logic L if B is derivable in L from A using rules of $\mathcal{B}$. A collection $\mathcal{B}$ of rules is said to be a basis of a collection $\mathcal{R}$ if each rule of $\mathcal{R}$ is a consequence of $\mathcal{B}$.

Lemma

Let $\mathsf{K4GT}^{\Box} \subseteq \mathsf{L}$ *be a modal logic with four constants. Then each passive rule in* L *is a consequence of the rules*

$$\frac{\Diamond\top}{\bot} \quad \textit{and} \quad \frac{\Box\bot \vee (\Diamond A \wedge \Diamond\neg A)}{\bot}.$$

Passive rules

Lemma

Let $\mathsf{K4GT}^{\Box} \subseteq \mathsf{L}$ *be a modal logic with four constants. Then each passive rule in* L *is a consequence of the rules*

$$\frac{\Diamond\top}{\bot} \quad \textit{and} \quad \frac{\Box\bot \vee (\Diamond A \wedge \Diamond\neg A)}{\bot}.$$

For each $\mathsf{L} \in \mathit{NExt}(\mathsf{K4GT}^{\Box} + \Diamond\top)$ the second rule can be replaced with

$$\mathsf{P}_2 : \quad \frac{\Diamond A \wedge \Diamond\neg A}{\bot}.$$

The only non-unifiable formula in $\mathsf{Verum} = \mathsf{K} + \Box\bot$ is $\bot$. Verum is structurally complete.

Projective unification

A unifier σ for a formula A is said to be projective (in a modal logic L) if

$$A \vdash_{\mathsf{L}} x \leftrightarrow x[\sigma]$$

for each $x \in \mathsf{Var}$. A formula is projective (in L) iff there exists a projective unifier for the formula. If each unifiable formula is projective (in L), then we say that L has projective unification.

Projective unification

Lemma

If a transitive modal logic L *enjoys projective unification, then* $\vdash_{\mathsf{L}} D1$ *(i.e.* $K4D1 \subseteq L$*).*

Proof.

1. projectivity $\Rightarrow \vdash_{\mathsf{L}} \mathsf{T}^{\Box}$
2. $\vdash_{\mathsf{L}} \mathsf{T}^{\Box}$ and $\vdash_{\mathsf{L}} 4 \Rightarrow \underbrace{\vdash_{\mathsf{L}} \Box\Box A \leftrightarrow \Box A \text{ and } \vdash_{\mathsf{L}} \Box A \to \Box\Diamond A}_{*}$
3. * $\Rightarrow \underbrace{\Box x \vee \Box y =_{\mathsf{L}} \Box^{+}(\Box x \vee \Box y)}_{**}$
4. ** $,\vdash_{\mathsf{L}} \mathsf{T}^{\Box}$ and projectivity of $\Box x \vee \Box y \Rightarrow \vdash_{\mathsf{L}} D1$.

$\Box$

Projective unification

The formula G (and $\mathsf{T}^{\Box}$) is an instance of D1 in K4D1.

$$\sigma(z) = \begin{cases} \neg x & \text{for } z = y \\ z & \text{for } z \neq y \end{cases}$$

$$D1[\sigma] =_{\mathsf{K4D1}} \mathsf{G}.$$

Corollary

$\mathsf{K4GT}^{\Box} \subseteq \mathsf{K4D1}$ *and* $\mathit{Cons}(\mathsf{K4D1}) = \{\top, \bot, \Diamond\top, \Box\bot\}$.

Projective unification

A variant of a transitive Kripke model $\langle W, R, v\rangle$ is a model $\langle W, R, v'\rangle$ such that the equality $v(w) = v'(w)$ holds for each $w \in W \setminus cl(\rho)$.
A class $\mathcal{K}$ of Kripke models based on rooted L-frames is said to have the extension property iff for every Kripke model $\mathfrak{M}$ based on a rooted L-frame, if $\mathfrak{M}_w \in \mathcal{K}$ for each $w \notin cl(\rho)$, then there is a variant $\mathfrak{M}'$ of $\mathfrak{M}$ such that $\mathfrak{M}' \in \mathcal{K}$.

Theorem (Ghilardi)

Let L *be a transitive modal logic characterized by a class $\mathcal{C}$ of finite rooted frames. A formula A is projective in* L *if and only if the class*

$$\{\langle \mathfrak{F}, v\rangle \colon \mathfrak{F} \in \mathcal{C} \text{ and } \langle \mathfrak{F}, v\rangle \models A\}$$

has the extension property.

Projective unification

Lemma

The modal logic K4D1 *is characterized by the class* Fr_{K4D1} *of all finite transitive frames of the form* $\langle W, R, \rho\rangle$ *fulfilling the condition*

$$\forall w_1, w_2 \in W \setminus \{\rho\}(w_1 R w_2 \vee w_2 R w_1).$$

Theorem

A transitive modal logic L *has projective unification if and only if* $\mathsf{K4D1} \subseteq \mathsf{L}$.

Proof.

1. K4D1 enjoys projective unification
2. extension L of K4D1 with four constants enjoys projective unification,
3. $\mathsf{K4D1} + \Diamond\top$ enjoys projective unification,
4. extension $\mathsf{K4D1} + \Diamond\top$ enjoys projective unification,
5. Verum enjoys projective unification.

□

Projective unification

Theorem

Every modal logic containing K4D1 *is almost structurally complete.*

Proof.

A/B a non-passive admissible rule (A is unifiable). Let ε be a projective unifier for A.

$$\vdash_{\mathsf{L}} A[\varepsilon] \text{ and } A \vdash_{\mathsf{L}} B[\varepsilon] \leftrightarrow B,$$

$$\vdash_{\mathsf{L}} B[\varepsilon],$$

$$A \vdash_{\mathsf{L}} B.$$

□

Projective unification

Theorem

A modal logic L *extending* K4D1 *is structurally complete if and only if either* L = Verum *or* K4D1M ⊆ L *(*M: $\Box\Diamond x \to \Diamond\Box x$*).*

Proof.

1. each extension of K4D1 is almost structurally complete,
2. $Cons(\mathsf{L}) = \{\top, \bot, \Diamond\top, \Box\bot\}$
   $\Diamond\top/\bot$ is admissible, but not derivable,
3. Verum. $\bot$ is the only non-unifiable formula,
4. $\mathsf{K4D1} + \Diamond\top \subseteq \mathsf{L}$.
   ($\Rightarrow$) The rule $\frac{\Diamond A \wedge \Diamond\neg A}{\bot}$ is derivable. i.e. $\Diamond A \wedge \Diamond\neg A \vdash_{\mathsf{L}} \bot$.

$$\Box(\Diamond A \wedge \Diamond\neg A) \to \bot =_{\mathsf{K}} \Box\Diamond A \to \Diamond\Box A =_{\mathsf{K}} \mathsf{M}.$$

($\Leftarrow$) Assume that $\vdash_{\mathsf{L}} \mathsf{M}$ and A/C is a passive rule.

- there exist B such that $A \vdash_{\mathsf{L}} \underbrace{\Box(\Diamond B \wedge \Diamond\neg B)}_{\neg\mathsf{M}}$,
- $A \vdash_{\mathsf{L}} \bot$ and A/C is derivable.

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