Nonclassical first order logics: Semantics and proof theory

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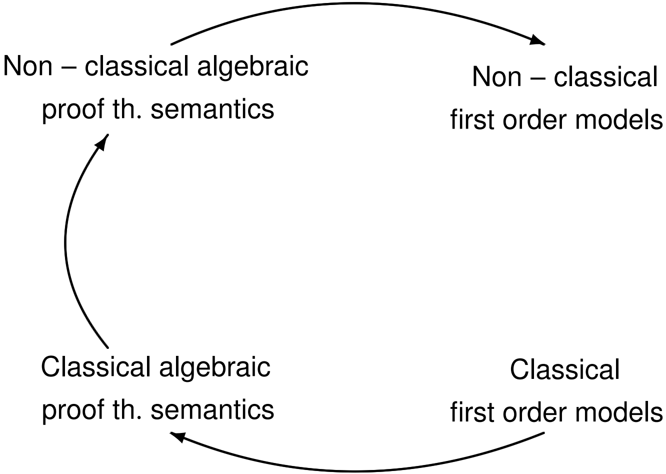
Starting point

Question

How can we define general relational semantics for arbitrary non-classical first-order logics?

- What are the models?
- What do quantifiers mean in those models?
- What does completeness mean?

Methodology: dual characterizations



A brief recap on classical first-order logic: Language

- Set of relation symbols $(R_i)_{i \in I}$ each of finite arity n_i.
- Set of function symbols $(f_j)_{j \in J}$ each of finite arity n_j.
- Set of constant symbols $(c_k)_{k \in K}$ (0-ary functions).
- Set of variables Var $= \{v_1, \ldots, v_n, \ldots\}$.

The first-order language $\mathcal{L} = ((R_i)_{i \in I}, (f_j)_{j \in J}, (c_k)_{k \in K})$ over Var is built up from **terms** defined recursively as follows:

$$\mathsf{Trm} \ni t ::= v_m \mid c_k \mid f_j(t, \ldots, t).$$

The **formulas** of first-order logic are defined recursively as follows:

$$\mathcal{L} \ni A ::= R_i(\bar{t}) \mid t_1 = t_2 \mid \top \mid \bot \mid A \wedge A \mid A \vee A \mid \neg A \mid \forall v_m A \mid \exists v_m A$$

A brief recap on classical first-order logic: Meaning

The models of a first-order logic $\mathcal{L}$ are tuples

$$M = (D, (R_i^D)_{i \in I}, (f_j^D)_{j \in J}, (c_k^D)_{k \in K})$$

where D is a non-empty set and R_i^D, f_j^D, c_k^D are concrete n_i-ary relations over D, n_j-ary functions on D and elements of D resp. interpreting the symbols of the language in the model M.

$$
\begin{array}{lcl}
M \models \top & & \text{Always} \\
M \models \bot & & \text{Never} \\
M \models A \wedge B & \Longleftrightarrow & M \models A \text{ and } M \models B \\
M \models A \vee B & \Longleftrightarrow & M \models A \text{ or } M \models A \\
M \models \neg A & \Longleftrightarrow & M \not\models A \\
M \models \forall x A(x) & \Longleftrightarrow & M \models A(d) \text{ for all } d \in D \\
M \models \exists x A(x) & \Longleftrightarrow & M \models A(d) \text{ for some } d \in D.
\end{array}
$$

For a set of sentences Σ, we write $\Sigma \models A$ if $M \models A$ for every model M such that $M \models B$ for all $B \in \Sigma$.

Display Calculi

- Natural generalization of Gentzen's sequent calculi;
- sequents $X \vdash Y$, where X and Y are **structures**:
  - formulas are **atomic structures**
  - built-up: **structural connectives** (generalizing meta-linguistic comma in sequents $\phi_1, \ldots, \phi_n \vdash \psi_1, \ldots, \psi_m$)
  - generation **trees** (generalizing sets, multisets, sequences)
- **Display property**:

$$\cfrac{\cfrac{\cfrac{Y \vdash X > Z}{X \,;\, Y \vdash Z}}{Y \,;\, X \vdash Z}}{X \vdash Y > Z}$$

display rules semantically justified by **adjunction/residuation**

- **Canonical proof of cut elimination (via metatheorem)**

Quantifiers as adjoints

Consider $\forall x : \wp(D \times D^n) \to \wp(D^n)$, $\exists x : \wp(D \times D^n) \to \wp(D^n)$ and $\pi^{-1} : \wp(D^n) \to \wp(D \times D^n)$ defined as:

- $\forall x(A) = \bigcap_{d_0 \in D} \{\overline{d} \in D \mid (d_0, \overline{d}) \in A\}$
- $\exists x(A) = \bigcup_{d_0 \in D} \{\overline{d} \in D^n \mid (d_0, \overline{d}) \in A\}$
- $\pi_x^{-1}(B) = D \times B$

We have:

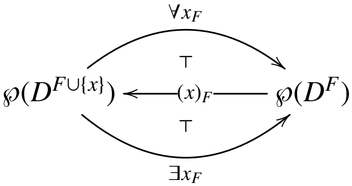
$$\pi_x^{-1}(B) \subseteq A \quad \Longleftrightarrow \quad B \subseteq \forall x(A)$$

$$\exists x(A) \subseteq B \quad \Longleftrightarrow \quad A \subseteq \pi_x^{-1}(B)$$

- Existential and universal quantification are the left and right adjoints respectively of the inverse projection map (Lawvere).

Algebraic properties of inverse projection maps

Let M be a model of $\mathcal{L}$. For every finite set $F \subsetneq \mathsf{Var}$ such that $x \notin F$ we define maps $(x)_F, \exists x_F, \forall x_F$ such that $(x)_F$ is the inverse projection map,



then the following properties hold:

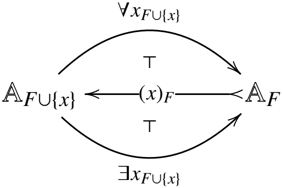
$$
\begin{array}{rclrcl}
(x)(A \cap B) & = & (x)(A) \cap (x)(B) & (x)(A \cup B) & = & (x)(A) \cup (x)(B) \\
(x)(D^F \setminus A) & = & D^{F \cup \{x\}} \setminus (x)(A) & (x)(y)(A) & = & (y)(x)(A) \\
(x)\forall y(A) & = & \forall y(x)(A) & (x)\exists y(A) & = & \exists y(x)(A)
\end{array}
$$

Algebraic semantics for first-order logic

An *heterogeneous $\mathcal{L}$-algebra* is a tuple $\mathbb{H} = (\mathcal{A}, \mathcal{Q})$, such that

- $\mathcal{A} = \{\mathbb{A}_F \mid F \in \mathcal{P}_\omega(\mathrm{Var})\}$;
- $\mathcal{Q} = \{(x)_F, \exists x_F, \forall x_F, \mid x \notin F \subsetneq \mathrm{Var}\}$;

where for every $F \in \mathcal{P}_\omega(\mathrm{Var})$, $\mathbb{A}_F$ is a complete Boolean algebra



such that $(x)_F$ is an order embedding and the following hold:

$$
\begin{aligned}
(x)(a \wedge b) &= (x)(a) \wedge (x)(b) & \qquad (x)(a \vee b) &= (x)(a) \vee (x)(b) \\
(x)(-a) &= -(x)(a) & (x)(y)(a) &= (y)(x)(a) \\
(x)\forall y(a) &= \forall y(x)(a) & (x)\exists y(a) &= \exists y(x)(a)
\end{aligned}
$$

Logical connectives and types

- Types will be named after the elements $F \in \wp_\omega(Var)$.
- A type $\mathcal{L}_F$ contains a formula A iff $\mathsf{FV}(\varphi) = F$.
- $A \in \mathcal{L}_{F \cup \{y\}} \iff \forall y A \in \mathcal{L}_F$
- $A \in \mathcal{L}_{F \setminus \{x\}} \iff (x) A \in \mathcal{L}_F$

- Symbols for quantifiers and cylindrification for each $x \in Var$:

Structural symbols	Q_x		$((x))$	
Operational symbols	$\exists x$	$\forall x$	(x)	(x)

Display Calculus

Introduction rules for quantifiers and their adjoint:

$$\exists_L \frac{\mathsf{Q}_x A \vdash_F X}{\exists x A \vdash_F X} \qquad \frac{X \vdash_F A}{\mathsf{Q} x X \vdash_{F\setminus\{x\}} \exists x A} \exists_R$$

$$\forall_L \frac{A \vdash_F X}{\forall x A \vdash_{F\setminus\{x\}} \mathsf{Q} x A} \qquad \frac{X \vdash_F \mathsf{Q}_x A}{X \vdash_F \forall x A} \forall_R$$

$$\circ_M \frac{X \vdash_{F\setminus\{x\}} Y}{((x))X \vdash_{F\cup\{x\}} ((x))Y}$$

$$\cdot_L \frac{((x))A \vdash_F X}{(x)A \vdash_F X} \qquad \frac{X \vdash_F ((x))A}{X \vdash_F (x)A} \cdot_R$$

Display Calculus

Display postulates for quantifiers and cylindrification:

$$\frac{\mathsf{Q}_x X \vdash_{F\setminus\{x\}} Y}{X \vdash_{F\cup\{x\}} ((x))Y} \qquad \frac{Y \vdash_{F\setminus\{x\}} \mathsf{Q}_x X}{((x))Y \vdash_{F\cup\{x\}} X}$$

Further adjunction rules:

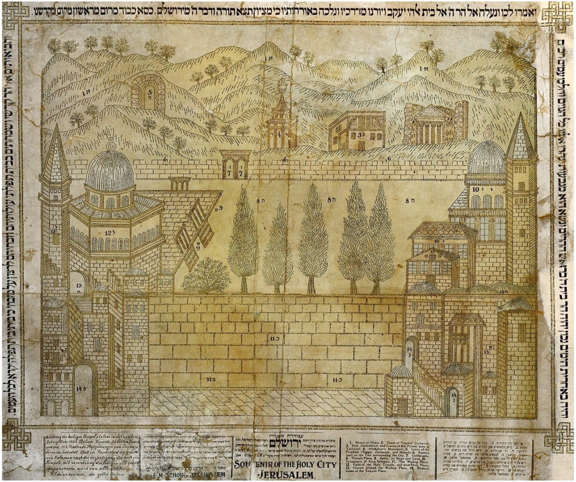
$$\frac{((x))\mathsf{Q}_x X \vdash_F Y}{X \vdash_F Y} \qquad \frac{X \vdash_F ((x))\mathsf{Q}_x Y}{X \vdash_F Y}$$

Interaction rules:

$$\frac{((x))X ; ((x))Y \vdash Z}{((x))(X ; Y) \vdash Z} \qquad \frac{Z \vdash ((x))X ; ((x))Y}{Z \vdash ((x))(X ; Y)}$$

$$\frac{((x))\mathsf{Q}_y X \vdash Y}{\mathsf{Q}_y((x))X \vdash Y} \qquad \frac{Y \vdash ((x))\mathsf{Q}_y X}{Y \vdash \mathsf{Q}_y((x))X}$$

Completeness (Canonical model)



If Σ is consistent then it has a model.

Σ can be extended to a set Σ' such that

- Σ' is a maximal consistent theory (an ultrafilter)
- If $\exists x A \in \Sigma'$ then $A(t) \in \Sigma'$ for some term t

Define $t \equiv s$ if and only if $t = s \in \Sigma'$:

- Let $D = \mathrm{Trm}/\equiv$
- Let $R^D(\bar{t})$ iff $R(\bar{t}) \in \Sigma'$
- Let $M = (D, R^D, f^D, c^D)$

Then $M \models A$ if and only if $A \in \Sigma'$.

Witnesses

Problem

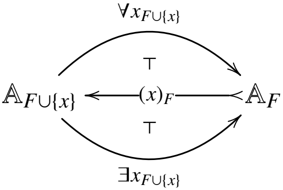
$\{\exists x P(x), \neg P(t_1,), \ldots, \neg P(t_n), \ldots\}$.

Solution(s)

- Add infinite constants in the language, and construct the ultrafilter by "carefully" using the constants for witnesses.
- Rename the variables in your set so that you have enough (infinite) variables unused.

Algebraic predicate semantics

A *heterogeneous $\mathcal{L}$-algebra* is a tuple $\mathbb{H} = (\mathcal{A}, \mathcal{Q})$ where for every $F \in \mathcal{P}_\omega(\mathrm{Var})$, $\mathbb{A}_F$ is a complete Heyting algebra/ distributive lattice/DLE/LE/etc. . .



such that $(x)_F$ is an order embedding and a Heyting algebra/distributive lattice/etc. . . homomorphism and the following hold:

$$(x)(y)(a) = (y)(x)(a)$$

$$(x)\forall y(a) = \forall y(x)(a) \qquad (x)\exists y(a) = \exists y(x)(a)$$

Completeness revisited

Classical completeness

If Σ is consistent then it is satisfiable.

Completeness in weaker logics

- If Σ is consistent then it is satisfiable
- If Δ is not provable then it is falsifiable
- If Σ does not imply Δ then there is a model that satisfies Σ and falsifies Δ.

Filter-ideal pairs

DL: Every disjoint filter ideal pair can be extended to a prime filter-ideal pair.

Lattices: Every disjoint filter ideal pair can be extended to a maximal one.

A case in point: Intuitionistic logic

- We let $M := (W, \leq)$, where each element of W is a classical first-order model
- $u \leq w$ implies that $f : D_w \to D_u$ is a homomorphism of models.
- $w \models \exists x A(x)$ if and only if there is some $d \in D_w$, $w \models A(d)$
- $w \models \forall x A(x)$ if and only if for all $w \in W$ such that $u \leq w$ and for all $d \in D_u$, $u \models A(d)$.

Categorically

A model is a functor from a poset W to the category of f.o. models and homomorphisms. The meaning of a formula (with one free variable) is a subobject of the model in this category of functors.

Canonical model: A story in multi-type

- $\mathbb{A}$ is the algebra of the logic excluding infinite free variables
- $\mathcal{F}$ is a prime filter of $\mathbb{A}$
- If $\exists x A \in \mathcal{F}$ then $A(t) \in \mathcal{F}$ for some term t

Define $t \equiv s$ if and only if $t = s \in \mathcal{F}$:

- Let $D = \mathrm{Trm}/\equiv$
- Let $R^D(\bar{t})$ iff $R(\bar{t}) \in \Sigma'$
- Let $M_{\mathcal{F}} = (D, R^D, f^D, c^D)$

Define $M = (W, \leq)$

- $W = \{M_{\mathcal{F}} \mid \mathcal{F}$ is a prime filter of some $\mathbb{A}\}$
- $M_{\mathcal{F}} \leq M_{\mathcal{G}}$ if and only if $\mathcal{G} \subseteq \mathcal{F}$

A curiosity?

$(x)(A \to B) = (x)A \to (x)B$ if and only if $(\exists x A(x)) \wedge B = \exists x (A(x) \wedge B)$

$(x)(A \setminus B) = (x)A \setminus (x)B$ if and only if $\forall x (A(x) \vee B) = (\forall x A(x)) \vee B$

Witnesses, counterexamples and completeness

Question

Why witnesses and not counterexamples? Why not require that if $\forall x A(x)$ is falsified then $A(t)$ is falsified for some t?

Lemma

1. Assume that $(\exists x A(x)) \wedge B = \exists x(A(x) \wedge B)$ and let (F, I) be a disjoint filter-ideal pair. Then (F, I) can be expanded to a filter ideal pair $(\mathcal{F}, \mathcal{I})$ such that for all $\exists x A(x) \notin \mathcal{I}$ there exists some $A(x_n) \notin \mathcal{I}$ for some n.
2. Assume that $\forall x(A(x) \vee B) = (\forall x A(x)) \vee B$ and let (F, I) be a disjoint filter-ideal pair. Then (F, I) can be expanded to a filter ideal pair $(\mathcal{F}, \mathcal{I})$ such that for all $\forall x A(x) \notin \mathcal{F}$ there exists some $A(x_n) \notin \mathcal{F}$ for some n.

Co-intuitionistic logic: Local counterexamples

- We let $M := (W, \leq)$, where each element of W is a classical first-order model
- $u \leq w$ implies that $R \subseteq D_u \times D_w$ is a co-homomorphic relation and $R^{-1} D_w = D_u$
- $u \models \exists x A(x)$ if and only if there is some w such that $u \leq w$ and some $d \in D_w$, $w \models A(d)$
- $w \models \forall x A(x)$ if and only if for all $d \in D_w$, $w \models A(d)$.

Categorically

A model is a functor from a poset W to the category of f.o. models and co-homomorphic relations. The meaning of a formula (with one free variable) is a subobject of the model in this category of functors.

Canonical model: A story in multi-type part 2

- $\mathbb{A}$ is the algebra of the logic excluding infinite free variables
- $\mathcal{I}$ is a prime ideal of $\mathbb{A}$
- If $\forall x A \in \mathcal{I}$ then $A(t) \in \mathcal{I}$ for some term t

Define $t \equiv s$ if and only if $t = s \notin \mathcal{I}$:

- Let $D = \text{Trm}/\equiv$
- Let $R^D(\bar{t})$ iff $R(\bar{t}) \notin \mathcal{I}$
- Let $M_{\mathcal{I}} = (D, R^D, f^D, c^D)$

Define $M = (W, \leq)$

- $W = \{M_{\mathcal{I}} \mid \mathcal{I}$ is a prime ideal of some $\mathbb{A}\}$
- $M_{\mathcal{I}} \leq M_{\mathcal{J}}$ if and only if $\mathcal{I} \subseteq \mathcal{J}$

Distributive lattices

- We let $M := (W, \leq_1, \leq_2)$, where each element of W is a classical first-order model
- $u \leq_1 w$ implies that $f : D_w \to D_u$ is a homomorphism of models.
- $u \leq_2 w$ implies that $R \subseteq D_u \times D_w$ is a co-homomorphic relation.
- $u \models \exists x A(x)$ if and only if there is some w such that $u \leq_2 w$ and some $d \in D_w$, $w \models A(d)$
- $w \models \forall x A(x)$ if and only if for all $w \in W$ such that $u \leq_1 w$ and for all $d \in D_u$, $u \models A(d)$.

Question

Are two separate relations needed?

Non-distributive logic

- No prime filters/ideals
- Logic as given should not "locally" provide witnesses and counterexamples

Witnesses and counterexamples

Let F be a filter of an lattice and I an ideal of $\mathbb{A}$, the lattice of the first-order logic excluding infinite free variables.

1. If $\exists x A(x) \notin I$ then I can be expanded to an ideal I' such that $A(x_n) \notin I'$
2. If $\forall x A(x) \notin F$ then F can be expanded to an ideal F' such that $A(x_n) \notin I'$

Non-distributive semantics

Let $\mathbb{P} = (\mathcal{M}, \mathcal{C}, \mathcal{N}, S)$, where $\mathcal{M}$ is a set of first-order $\mathcal{L}$-models, the *models*, $\mathcal{C}$ is a set of first-order $\mathcal{L}$-models, the *countermodels*, $\mathcal{N} \subseteq \mathcal{M} \times \mathcal{C}$ and $S : \bigcup \mathcal{M} \times \bigcup \mathcal{C}$ is a similarity relation between points of models and countermodels.
Subobjects:

- Let $X : \mathcal{M} \to \wp(\bigcup \mathcal{M}^n)$ and $Y : \mathcal{C} \to \wp(\bigcup \mathcal{C}^n)$
- $X(M) \subseteq M^n$ and $X(C) \subseteq C^n$

$$X^{\uparrow}(C) = \{\overline{b} \in C^n \mid \forall M \in \mathcal{M} \forall \overline{a} \in M^n((\overline{a} S \overline{b} \;\&\; \overline{a} \in X(M)) \Rightarrow M \mathcal{N} C)\}$$

$$Y^{\downarrow}(M) = \{\overline{a} \in M^n \mid \forall C \in \mathcal{C} \forall \overline{b} \in C^n((\overline{a} S \overline{b} \;\&\; \overline{b} \in Y(C)) \Rightarrow M \mathcal{N} C)\}.$$

- $(\cdot)^{\uparrow}$ and $(\cdot)^{\downarrow}$ form a Galois connection.
- A subojbect is a Galois-closed pair.

Satisfaction and refutation

Interpretation:

$M \Vdash \forall x A(\overline{a})$ iff for all $C \in \mathcal{C}, \overline{a} S \overline{b}$ and $b \in C(C \succ A(b, \overline{b}) \Rightarrow M \mathcal{N} C)$

$M \succ \exists x A(\overline{b})$ iff for all $M \in \mathcal{M}, \overline{a} S \overline{b}$ and $a \in M(M \Vdash A(a, \overline{a}) \Rightarrow M \mathcal{N} C)$

Canonical model: A story of algebra and co-algebra

- $\mathbb{A}$ is the algebra of the logic excluding infinite free variables
- $\mathcal{F}$ is a filter $\mathbb{A}$
- $\mathcal{I}$ is an ideal $\mathbb{A}$

Define $t \equiv s$ if and only if $t = s \in \mathcal{F}$:

- Let $D = \text{Trm}/\equiv$
- Let $R^D(\bar{t})$ iff $R(\bar{t}) \in \mathcal{F}$
- Let $M_{\mathcal{F}} = (D, R^D, f^D, c^D)$

Define $\neg t \equiv s$ if and only if $A(t) \in \mathcal{I}$ while $A(s) \notin \mathcal{I}$:

- Let $D_{\mathcal{F}} = \text{Trm}/\equiv$
- Let $R^D(\bar{t})$ iff $R(\bar{t}) \notin \mathcal{I}$
- Let $C_{\mathcal{I}} = (D, R^D, f^D, c^D)$

Define $\mathbb{P} = (\mathcal{M}, C, \mathcal{N}, S)$

- $M_{\mathcal{F}} \mathcal{N} C_{\mathcal{I}}$ if and only if $M_{\mathcal{F}} \cap C_{\mathcal{I}} \neq \emptyset$
- aSb if and only if $[a] \cap [b] \neq \emptyset$

Final thoughts

- Define (reasonable?) algebraic semantics for predicate logics encompassing already well-studied logics.
- Designe modular and general proof-systems for predicate logics.
- Provide understanding for semantics for non-classical logics.
- What is the categorical framework for non-distributive predicate logics?